

# An Introduction to Complex Numbers

## - A Complex Solution to a Simple Problem

("If  $i$  didn't exist, it would be necessary invent me.")

**Our Problem.** The rules for multiplying real numbers tell us that the product of two negative numbers is positive, for instance  $(-1)(-1) = 1$ . Thus the square of any real number, positive or negative, is never negative. This fact can be summarised in symbols as follows:

$$\text{If } x \in \mathbb{R}, \text{ then } x^2 \geq 0.$$

In particular, there is no real number  $x$  such that  $x^2 = -1$ , or equivalently, such that

$$x^2 + 1 = 0.$$

This harmless-looking quadratic equation has no solutions; at least, there are no real numbers which satisfy it. This was a problem for mathematicians right up to the 17th century, and it is now our problem in these notes.

### Historical Reflections

**An Ancient Problem.** The mathematicians of classical Greece, who lived over 2000 years ago, had a similar difficulty with the so-called 'real' numbers. They accepted fractions as numbers — after all they could represent them as lengths constructed with straight-edge and compass — but they had trouble with a number like  $\sqrt{2}$  which they knew was not a fraction<sup>1</sup>. They also knew from Pythagoras's Theorem that  $\sqrt{2}$  was the length of the diagonal of a unit square (a square with sides of length 1), and so if numbers are to represent length, there has to be a number whose square is 2, a number which cannot be expressed as a fraction. (Numbers expressible as positive or negative fractions are called *rational* numbers, and the set of all rational numbers is denoted by the special symbol  $\mathbb{Q}$ . Our proof in the footnote below shows that  $\sqrt{2}$  is irrational.)

**An Imaginary Problem.** We now know how to approximate  $\sqrt{2}$  as closely as we please because there are algorithms which generate its decimal expansion to as many places as we like (pocket calculators use such an algorithm). In fact, the real numbers were created in logically satisfactory ways by Cauchy and Dedekind over 100 years ago. We shall say a little more about their construction later in the course, and you will learn to exploit their properties. The problem mathematicians had with the square root of minus one was a psychological one; they had a complex about numbers being 'real', they worried a lot about whether they existed. Nowadays we realise that even the so-called 'real' numbers are a human invention, a product of our imagination, and are no more real than angels dancing on the head of a pin. They make a very good model for handling length in geometry, but they are only a model, an abstract representation. The sense of their being real is a trick of the brain and comes from identifying the numbers with the perceived reality of length. Of course,  $\sqrt{-1}$  does not represent a length, but then neither does  $-1$  itself and yet no one denies the usefulness of negative numbers — this point was made forcefully as long ago as 1673 by the British mathematician Wallis. In fact, we shall see below that the complex numbers provide a good model for 2-dimensional space (i. e. the plane). Therefore let us ignore the question of the *existence* of  $\sqrt{-1}$  and base our judgement on whether the concept is useful. Utility is a more important property than 'reality' (whatever we mean by that).

**A Solution, hedged about with doubts.** Diophantus (c.275 AD) attempted to solve the plausible problem of finding a right-angled triangle with perimeter 12 and area 7 and derived a quadratic equation for the length of a side which had no real roots (so no such triangle exists). The Italian mathematician Pacioli stated in 1494 that the equation  $x^2 + c = bx$  cannot be solved

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<sup>1</sup>If  $\sqrt{2}$  could be written as a fraction  $a/b$ , we could also suppose that  $a$  and  $b$  are not both even (otherwise we could cancel a common factor of 2 from  $a$  and  $b$ ). Observe that the square of an odd number is always odd. If the square of  $a/b$  were 2, it would follow that  $a^2 = 2b^2$ , so  $a$  could not be odd. But then  $a$ , being even, is twice some whole number  $c$  whence  $a^2 = (2c)^2 = 4c^2$ , which means that  $4c^2 = 2b^2$  and therefore  $b^2 = 2c^2$ . Now by the same reasoning  $b$ , like  $a$ , must be even, which we supposed not to be the case. This contradiction means that  $\sqrt{2}$  is not a fraction. (This is an example of 'proof by contradiction'.)

unless  $b^2 \geq 4c$ . Another Italian mathematician Girolamo Cardano described the equation  $x^4 + 12 = 6x^2$  as “impossible”, referring to the roots as “fictitious”. However, he was willing to use square roots of negative numbers to divide 10 into two parts whose product is 40 thus:  $10 = (5 + \sqrt{-15}) + (5 - \sqrt{-15})$ . Karl Friedrich Gauss was the first to call expressions of this kind “complex numbers”. Complex numbers gradually established themselves as a valuable extension to the real number system and although doubts about their existence slowly disappeared, the legacy of the old terminology ‘real’ and ‘imaginary’ still survives. By the middle of the last century the theory of complex variables was a thriving and central branch of mathematics. Now let’s go to it!

### The Answer to our Problem

#### **Definition.**

A *complex number* is an expression of the form

$$x + iy,$$

where  $x$  and  $y$  are real numbers and  $i$  is an algebraic symbol satisfying  $i^2 = -1$ . The set of all complex numbers is denoted by the special symbol  $\mathbb{C}$ .

#### **Remarks and Notation.**

1. When  $y = 0$  we write simply  $x$  rather than  $x + i0$ ; in particular,  $0 + i0$  is denoted by  $0$ . The subset of  $\mathbb{C}$  consisting of all  $x + iy$  with  $y = 0$  is a copy of the set of real numbers sitting inside  $\mathbb{C}$ . If we denote the set of real numbers by the special letter  $\mathbb{R}$ , we can express this symbolically by  $\mathbb{R} \subseteq \mathbb{C}$ . (Here the notation ‘ $\subseteq$ ’ means ‘is a subset of’.) From this viewpoint, a real number is a special case of a complex number.
2. One of the structural rules of the complex numbers is that

$$x + iy = 0 \quad \text{if and only if} \quad x = 0 \quad \text{and} \quad y = 0.$$

(*Aside:* In the language of Linear Algebra, the complex numbers  $1$  and  $i$  are *linearly independent* over  $\mathbb{R}$ .)

3. Two more structural rules in  $\mathbb{C}$  are:  $iy = yi$  and  $(-y)i = -(yi)$ . Putting them together, we can write a complex number like  $5 + i(-2)$  as  $5 - 2i$  instead.
4. It is often convenient to use a single letter, such as  $z$ , to stand for a typical complex number  $x + iy$ .

#### **Two Remarkable Facts**

**I.** In  $\mathbb{C}$  we can (i) add, (ii) subtract, (iii) multiply, and (iv) divide by non-zero complex numbers in such a way that the familiar rules of arithmetic are satisfied, for example the *Distributive Law*, which states that  $u(v + w)$  is the same number as  $uv + uw$  for all choices of complex numbers  $u$ ,  $v$ , and  $w$ . (A set which admits these four basic algebraic operations is called a *field*. Thus  $\mathbb{Q}$  (the rational numbers),  $\mathbb{R}$  (the real numbers), and  $\mathbb{C}$  are all examples of fields.)

**II.** *Every* polynomial equation (not just  $x^2 + 1 = 0$ ) now has a root in  $\mathbb{C}$ , even when the coefficients of the polynomial are themselves complex numbers. This fact, together with the remainder theorem, implies that a polynomial of degree  $n$  has exactly  $n$  roots in  $\mathbb{C}$  (counting repetitions). The second remarkable fact is too deep to prove at this stage, but the first is straightforward, as we now see.

#### *Proof of Remarkable Fact I*

Addition and subtraction of complex numbers is easy. Let  $z = x + iy$  and  $z' = x' + iy'$ , and define the following operations.

**Sum:**  $z + z' = (x + iy) + (x' + iy') = (x + x') + i(y + y')$

**Difference:**  $z - z' = (x + iy) - (x' + iy') = (x - x') + i(y - y')$

Note that in each case the right-hand side has the form of a real number plus  $i$  times another real number. Thus the sum and difference of two complex numbers is another complex number. We say that the set  $\mathbb{C}$  is *closed* under the operations of addition and subtraction.

Multiplication is not difficult, provided we remember to replace  $i^2$  by  $-1$  whenever we can. We will denote the product of two complex numbers  $z$  and  $z'$  by  $zz'$  rather than by  $z \times z'$ . Juxtaposition (writing two symbols next to each other) is often used by mathematicians to denote some kind of product or multiplication. The usual rules for expanding brackets give:

**Product:** 
$$\begin{aligned} zz' &= (x + iy)(x' + iy') = xx' + xiy' + iyx' + iyiiy' \\ &= xx' + i(xy' + yx') + i^2yy' = (xx' - yy') + i(xy' + yx') \end{aligned}$$

Since  $x, x', y,$  and  $y'$  are all real numbers, the two numbers  $(xx' - yy')$  and  $(xy' + yx')$  are also real. Thus the product we have just defined (the final expression in the above equations) is again a complex number. This shows that  $\mathbb{C}$  is closed under multiplication.

Finally, we want to divide by a non-zero complex number  $z = x + iy$ . To say that  $z$  is non-zero means that at least one of  $x$  and  $y$  is non-zero. To divide by  $z$  is the same as multiplying by its *inverse*  $1/z$ , and since we already know how to multiply, it will be enough to identify a complex number  $w$  equal to  $1/z$ . Now if  $w = 1/z$ , then  $zw = 1$ . The secret of finding such a  $w$ , is to notice that if  $z (= x + iy)$  is multiplied by its so-called *complex conjugate*  $\bar{z} = x - iy$ , the answer is the positive real number  $x^2 + y^2$  (why is it positive?). To see this, apply the product rule above with  $x' = x$  and  $y' = -y$ :

$$z\bar{z} = (x + iy)(x - iy) = (xx - y(-y)) + i(xy + (-y)x) = (x^2 + y^2) + i0 = x^2 + y^2.$$

If we now replace  $\bar{z}$  by

$$w = \left( \frac{1}{x^2 + y^2} \right) \bar{z} = \left( \frac{x}{x^2 + y^2} \right) - i \left( \frac{y}{x^2 + y^2} \right)$$

and carry through the same calculation, we obtain  $zw = 1$ . This yields the desired inverse  $1/z$  (also denoted by  $z^{-1}$ ) of a *non-zero* complex number  $z = x + iy$ :

**Inverse:** 
$$z^{-1} = \frac{1}{z} = \left( \frac{x}{x^2 + y^2} \right) - i \left( \frac{y}{x^2 + y^2} \right).$$

This completes our proof that in  $\mathbb{C}$  we can add, subtract, multiply pairs of complex numbers, and also divide by non-zero complex numbers, thus showing that  $\mathbb{C}$  is a field. We mentioned earlier that, just as the real numbers can be used to represent points on a line, the complex numbers are a useful model for studying points in the plane.

### **A Geometrical View of the Complex Numbers**

The underlying idea is simple. We identify the complex number  $z = x + iy$  with the point in the plane whose coordinates are  $(x, y)$ . We then interpret geometrically the four arithmetical operations defined above. If  $P$  is the point represented by the complex number  $z = x + iy$ , its coordinates are  $(x, y)$ , and the distance  $r$  of  $P$  from the origin  $O$  is  $\sqrt{x^2 + y^2}$  by Pythagoras's theorem. If the Greek letter  $\theta$  (theta) denotes the angle between the  $x$ -axis and  $OP$  (measured anticlockwise), we have  $x = r \cos \theta$  and  $y = r \sin \theta$ , and therefore

$$z = r(\cos \theta + i \sin \theta), \quad \text{where } r = \sqrt{x^2 + y^2}.$$

Therefore the point with polar coordinates  $(r, \theta)$  is represented by the complex number  $r(\cos \theta + i \sin \theta)$ . There are special names given to  $r$  and  $\theta$  in this situation.

**Definitions.**

(a) The *modulus* of a complex number  $z$ , denoted by  $|z|$ , is defined thus:

$$|z| = \sqrt{x^2 + y^2}.$$

(b) The *argument* of a complex number  $z$  is the angle  $\theta$  between the  $x$ -axis and  $OP$  measured positively in an anti-clockwise direction. By convention we choose the so-called *principal value* of the argument lying in the range  $0 \leq \theta < 2\pi$ . Thus

$$\arg z = \tan^{-1}(y/x) \quad \text{with } 0 \leq \arg z < 2\pi$$

The **sum** of two complex numbers has a well-known geometrical meaning. If  $P$  and  $P'$  are points represented by  $z = x + iy$  and  $z' = x' + iy'$ , the coordinates of the point represented by the sum

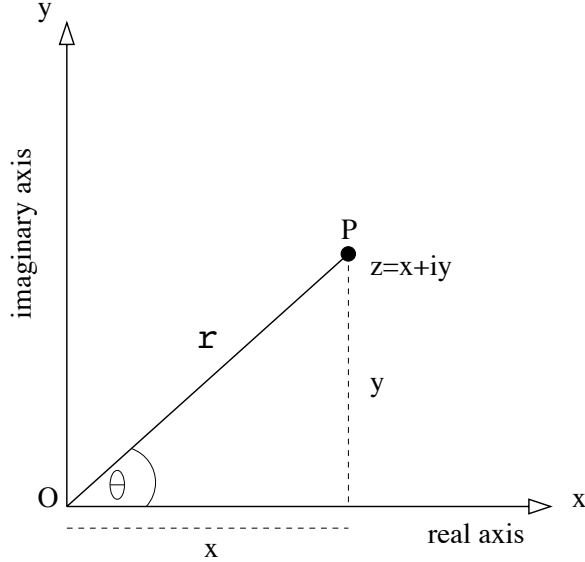


Figure 1: THE ARGAND DIAGRAM

$z + z' = (x + x') + i(y + y')$  are  $(x + x', y + y')$ , and these are the coordinates of the vector sum  $\overrightarrow{OP} + \overrightarrow{OP}'$ . Hence the sum of two complex numbers represents the *vector* sum of their points in the plane. Similarly the **difference**  $z - z'$  corresponds to the difference of their corresponding vectors. The sum and difference of two complex numbers can therefore be visualised by means of the pictures shown in figures 2 and 3.

The **product** of two complex numbers is best expressed in terms of their polar coordinates.

**Theorem.** Let  $z$  and  $z'$  be complex numbers with moduli  $r$  and  $r'$  and arguments  $\theta$  and  $\theta'$  respectively. In other words, suppose that  $z = r(\cos \theta + i \sin \theta)$  and  $z' = r'(\cos \theta' + i \sin \theta')$ . Then

$$zz' = rr'(\cos(\theta + \theta') + i \sin(\theta + \theta')). \quad (1)$$

Hence to calculate the product of two complex numbers, multiply their moduli and add their arguments. In symbols, this means:

$$|zz'| = |z||z'| \quad \text{and} \quad \arg zz' = \arg z + \arg z' \quad (\text{minus } 2\pi \text{ when necessary}).$$

*Proof* Using the product rule and the familiar formulas for  $\sin(A + B)$  and  $\cos(A + B)$ , we get

$$\begin{aligned} (\cos \theta + i \sin \theta)(\cos \theta' + i \sin \theta') &= (\cos \theta \cos \theta' - \sin \theta \sin \theta') + i(\sin \theta \cos \theta' + \cos \theta \sin \theta') \\ &= \cos(\theta + \theta') + i \sin(\theta + \theta') \end{aligned}$$

Equation 1 now follows, and shows that the product  $zz'$  can be expressed in the form of a complex number whose modulus is  $rr' = |z||z'|$  and whose argument is  $\theta + \theta'$ , provided we subtract  $2\pi$  to obtain the principle value when  $\theta + \theta'$  falls in the range  $2\pi$  to  $4\pi$ .

The rule for the product stated in the above Theorem makes it very easy to describe the **inverse**. If we multiply the complex number  $z$  corresponding to  $(r, \theta)$  by the complex number  $z'$  corresponding to  $(r^{-1}, -\theta)$ , we obtain the complex number whose modulus is  $r \times r^{-1} = 1$  and whose argument is  $\theta + (-\theta) = 0$ , and this is just the number  $1 + i0 = 1$ . In other words,  $zz' = 1$  and hence  $z' = z^{-1}$ . We have therefore shown that

$$\text{if } z = r(\cos \theta + i \sin \theta), \quad \text{then } z^{-1} = r^{-1}(\cos \theta - i \sin \theta)$$

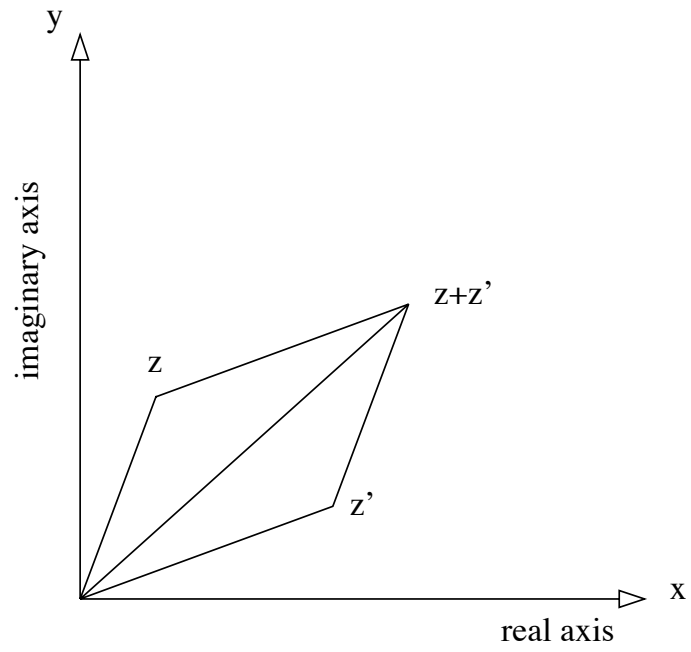


Figure 2: The *sum* of two complex numbers

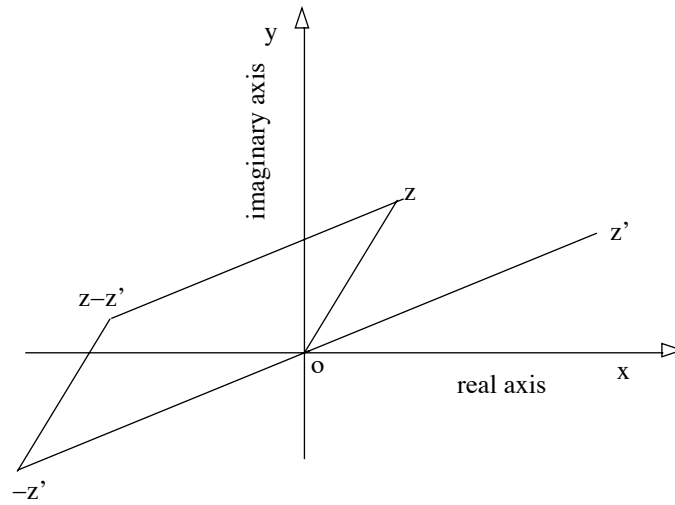
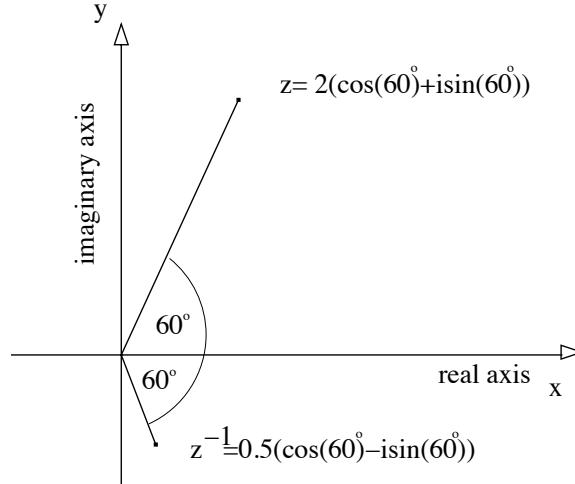


Figure 3: The *difference* of two complex numbers



since  $\sin(-\theta) = -\sin \theta$ . The above diagram shows the inverse  $z^{-1}$  of the following complex number:

$$z = 2\left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right) = 1 + i\sqrt{3}$$

### de Moivre's Theorem

Using the multiplication rule described above repeatedly for complex numbers we can show that for a complex number  $z = r(\cos \theta + i \sin \theta)$ ,

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

This result is known as **de Moivre's Theorem**.

For example  $z^2 = r^2(\cos(\theta + \theta) + i \sin(\theta + \theta))$  and  $z^3 = z z^2 = r^3(\cos(2\theta + \theta) + i \sin(2\theta + \theta)) = r^3(\cos 3\theta + i \sin 3\theta)$  etc.

de Moivre's Theorem is valid for any integer or rational  $n$ .

### The exponential form of a complex number

In our geometrical view of complex numbers we see that the combination  $\cos \theta + i \sin \theta$  involving the angle  $\theta$  keeps cropping up. It is convenient and very useful to find a more concise expression for this combination. Let's consider differentiating  $z = \cos \theta + i \sin \theta$  with respect to  $\theta$ ,

$\frac{dz}{d\theta} = -\sin \theta + i \cos \theta$  and again  $\frac{d^2 z}{d\theta^2} = -\cos \theta - i \sin \theta$  which is of course  $-z$ . Now we will repeat this process with the complex number  $z_1 = e^{i\theta}$ .  $\frac{dz_1}{d\theta} = i e^{i\theta}$  and  $\frac{d^2 z_1}{d\theta^2} = i^2 e^{i\theta}$  which is  $-z_1$ . This suggests that

$$\cos \theta + i \sin \theta \equiv e^{i\theta}$$

(This result can be shown formally by using power series expansions.  $\cos \theta$  can be expressed as a series

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

and similarly for  $\sin \theta$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

for any real value of  $\theta$ . (The factorial  $n! = 1.2.3 \dots n$ , e.g.  $4! = 1.2.3.4$ )  
The exponential function  $e^x$  can be expressed for any real  $x$  as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

and if we define  $e^z$  as

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

where  $z$  is a complex number, it is easy to show that

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots$$

which is equivalent to  $\cos \theta + i \sin \theta$  when  $i^2 = -1$  has been used to tidy up this expression.)

Thus a concise way of writing any complex number  $z = r(\cos \theta + i \sin \theta)$  is  $z = re^{i\theta}$ .

The complex conjugate  $\bar{z} = re^{-i\theta}$  and if we revisit de Moivre's Theorem

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta)$$

The relations

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

are also handy and often used.

**Example**

Find three different complex numbers which satisfy the equation  $z^3 = 1$ .

Express  $z$  as  $re^{i\theta}$  and find values of  $r$  and  $\theta$  that satisfy the equation.

$$(re^{i\theta})^3 = 1$$

becomes

$$r^3 e^{3i\theta} = 1$$

Equating real and imaginary parts gives

$$r^3 \cos 3\theta = 1$$

$$r^3 \sin 3\theta = 0$$

We find  $r = 1$  from squaring and adding the equations and  $\cos 3\theta = 1$  and  $\sin 3\theta = 0$ . This means that  $3\theta = 0, 2\pi, 4\pi, \dots$  and  $\theta = 0, 2\pi/3, 4\pi/3, 2\pi, 2\pi/3 + 2\pi, 4\pi/3 + 2\pi, \dots$ . The three complex numbers which satisfy  $z^3 = 1$  are thus

$$z_1 = 1$$

$$z_2 = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + i \frac{\sqrt{3}}{2}$$

$$z_3 = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - i \frac{\sqrt{3}}{2}$$

## Exercises

ATTEMPT THESE QUESTIONS AND HAND IN YOUR ANSWERS TO QUESTIONS 1 TO 8 TO YOUR PERSONAL TUTOR BY THE END OF THE FIRST WEEK OF TERM.

1. Write the following complex numbers in the form  $x + iy$ , where  $x$  and  $y$  are real:

(i)  $(-1 - i)(3 - 4i)$ ; (ii)  $(2 + i)^2 - (2 - i)^2$ ; (iii)  $(1 + i)/(1 - i)$ ; (iv)  $(9 + 3i)/(6i)$ .

[10 marks]

2. For what real value of  $a$  is the equation  $(1 + 3i)(5 - ai) = 50$  satisfied?

[10 marks]

3. On the Argand diagram mark the points represented by the following complex numbers:

(i)  $-4i$ ; (ii)  $1 - 2i$ ; (iii)  $(1 + i)3$ .

[10 marks]

4. Find the modulus and argument of each of the following complex numbers:

(i)  $\sqrt{3} - i$ ; (ii)  $-6 + 8i$ ; (iii)  $-2i$ .

[10 marks]

5. Mark the following points on the Argand diagram and verify the product rule given in the theorem on page 4: (a)  $3 + 2i$ ; (b)  $2 + i$ ; (c) their product.

[10 marks]

6. Show that the following four points form the vertices of a square in the Argand diagram:

$4 + 3i$ ,  $-3 + 4i$ ,  $-4 - 3i$ ,  $3 - 4i$ .

[10 marks]

7. Work out

$$\left(\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}\right)^4,$$

in other words, express this fourth power in the standard form  $x + iy$ .

Find four different complex numbers  $z$  satisfying  $z^4 + 1 = 0$ .

[20 marks]

8 Find two complex numbers which satisfy the equation  $\frac{(3+2i)}{z^2} = 1$  and mark their positions on the Argand diagram.

[20 marks]

\*(Optional) Give a sensible meaning to the symbol  $i^i$ , that is to say,  $i$  raised to the power of  $i$ .