

Mathematical Techniques

From the beginning of term, the mathematics modules will assume that you are 'fluent' in A-level techniques in differentiation, integration and trigonometry. Fluent means that you can use and apply them accurately without the help formulae books or calculators.

As a rough guide to what is expected, attached are some tests that have been used in the past to assess students familiarity with these techniques. The questions should help you back into doing maths again. We suggest that you attempt all the questions on the sheets to see how well you have mastered the techniques and mark them according to the attached mark schemes. If you do not find them straightforward, please revise the relevant sections of your A-level mathematics textbooks.

Some classic texts which also cover this material are:

(BC) K. Bostok and S. Chandler, Core Maths for A-level, Stanley Thornes 1994;

(ST) A.J. Stadler and D.W.S. Thorning, Understanding Pure Mathematics, Oxford 1987;

(MK) G. Mannal and M. Kenwood, Pure Mathematics 2, Heinemann 1995; and

(JO) J. Olive, Maths: A Students Survival Guide, Cambridge 2003

The Additional Techniques material contains further practice questions on differentiation, integration and trigonometry as well as questions on inequalities. You will be using inequalities a great deal, particularly in Analysis. This material will be covered in lectures, but it is a good idea to revise what is in A-level before getting here.

NdA

August 15, 2014.

Mathematical Techniques : Differentiation SAMPLE

The total number of marks is 30. The marks available for each question are indicated in parentheses. The pass mark is 24 or above. Calculators must not be used.

Section A. Differentiate the following expressions with respect to x .

$$1. \quad \frac{x^3}{4} + \frac{1}{x^2} \quad (2) \qquad 2. \quad \frac{1}{1 + \frac{1}{x}} \quad (2)$$

$$3. \quad x^2 e^x \cos(2x) \quad (3) \qquad 4. \quad \log(xe^x) \quad (2)$$

Section B.

5. Find the gradient of the curve $y = x^2 - 6x - 7$ at the points where the curve cuts the x -axis. (3)

6. Find the x -coordinates and nature of any stationary points on the curve $y = xe^{-x}$. (3)

7. Let $y = e^{ax} + b(x+1)^3$. When $x = 0$, suppose that $dy/dx = 0$ and $d^2y/dx^2 = 0$. Find the possible values of a and b . (3)

Section C. Find the derivatives of the following functions at the indicated point.

$$8. \quad \sin(\sin x), \quad x = \pi \quad (3) \qquad 9. \quad \frac{(3x-8)^3}{x}, \quad x = 3 \quad (3)$$

$$10. \quad \cos x \cdot \log(3x), \quad x = \pi \quad (3) \qquad 11. \quad \log\left(\frac{x^3}{\sqrt{x}}\right), \quad x = 5/2 \quad (3)$$

Differentiation: Solutions.

A) 1. $\frac{3}{4}x^2 - \frac{2}{x^3}$ 2. $\frac{1}{(1+x)^2}$
3. $2xe^x \cos(2x) + x^2 e^x \cos(2x) - 2x^2 e^x \sin(2x)$
4. $1 + \frac{1}{x} \left(\frac{(1+x)e^x}{xe^x} \right)$

B) 5. $\frac{dy}{dx} = 2x - 6$ curve cuts when $x^2 - 6x - 7 = 0$
at $x = 7$, $\frac{dy}{dx} = 8$
at $x = -1$, $\frac{dy}{dx} = -8$

6. $\frac{dy}{dx} = (1-x)e^{-x}$, $\frac{d^2y}{dx^2} = (x-2)e^{-x}$
 $\frac{dy}{dx} = 0$ when $x = 1$, at which point $\frac{d^2y}{dx^2} = -e^{-1} < 0$
so is a maximum.

7. $\frac{dy}{dx} = ae^{ax} + 3b(x+1)^2$, $\frac{d^2y}{dx^2} = a^2 e^{ax} + 6b(x+1)$
at $x = 0$ $\frac{dy}{dx} = a + 3b$, $\frac{d^2y}{dx^2} = a^2 + 6b$.
so $(a=0 \text{ and } b=0)$ or $(a=2 \text{ and } b = -\frac{2}{3})$

C) 8. -1 $(\cos(x) \times \cos(\sin(x)))$
9. $\frac{26}{9}$ $(\frac{9x(3x-8)^2 - (3x-8)^3}{x^2})$
10. $-\frac{1}{\pi}$ $(-\sin(x) \log(3x) + \frac{1}{x} \cos(x))$
11. 1 $(\frac{5}{2x})$

Mathematical Techniques : Integration SAMPLE

The total number of marks is 30. The marks available for each question are indicated in parentheses. The pass mark is 24 or above. Calculators must not be used.

Section A. Integrate the following expressions with respect to x .

$$1. \quad \frac{1}{4+3x} \quad (3) \qquad 2. \quad \sin x \cos x \quad (3)$$

$$3. \quad \sin^2(3x) + \cos^2(3x) \quad (3) \qquad 4. \quad (4x-2)^6 \quad (3)$$

Section B. Evaluate the following definite integrals.

$$5. \quad \int_0^1 x e^{x^2} dx \quad (3) \qquad 6. \quad \int_0^{\pi/2} \cos x \sin^5 x dx \quad (3)$$

$$7. \quad \int_0^{\pi/2} x \cos x dx \quad (3) \qquad 8. \quad \int_0^2 \frac{dx}{2x+5} \quad (3)$$

Section C.

9. Calculate the area of the finite region enclosed by the curve $y = x^2 - 3x + 2$ and the x -axis. (3)

10. Calculate the area of the finite region enclosed by the curves $y = \frac{2|x|}{\pi}$ and $y = \sin x$. (3)

Integration: solutions

- (A)
1. $\frac{1}{3} \log(4+3x) + C$
 2. $\frac{1}{2} \sin^2 x + C$ or $-\frac{1}{2} \cos^2 x + C$
 3. $x + C$ (NB $\sin^2(3x) + \cos^2(3x) = 1!$)
 4. $\frac{1}{28} (4x-2)^7 + C$

(B) 5. $[\frac{1}{2} e^{x^2}]_0^1 = \frac{1}{2} (e^1 - 1)$

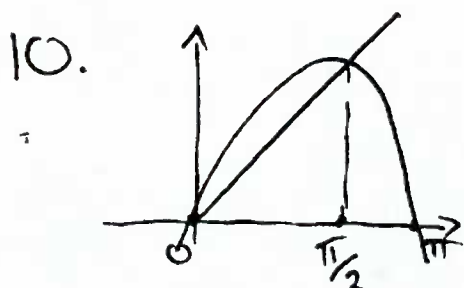
6. $[\frac{1}{6} \sin^6 x]_0^{\frac{\pi}{2}} = \frac{1}{6}$

7. $[x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx = \frac{\pi}{2} - 1$

8. $[\frac{1}{2} \log(2x+5)]_0^2 = \frac{1}{2} (\log 9 - \log 5)$
 $= \frac{1}{2} \log\left(\frac{9}{5}\right)$
 $= \log\left(\sqrt[3]{\frac{9}{5}}\right)$

(C) 9. Require $-\int_1^2 (x^3 - 3x + 2) \, dx = \underline{\underline{\frac{1}{6}}}$

(minus sign since area is under the x-axis).



(not to scale!!)

require $\int_0^{\frac{\pi}{2}} \sin x - \frac{2x}{\pi} \, dx$
 $= \underline{\underline{1 - \frac{\pi}{4}}}$

Mathematical Techniques : Trigonometry SAMPLE

The total number of marks is 30. The marks available for each question are indicated in parentheses. The pass mark is 24 or above. Calculators must not be used.

Section A. Determine the exact value of the following numbers.

1. $\cos \pi$ (2)
2. $\sin(5\pi/6)$ (2)
3. $\cos(5\pi/4)$ (2)
4. $\tan(3\pi/4)$ (2)
5. $\sin(7\pi/2)$ (2)
6. $\sin(11\pi/3)$ (2)

Section B.

7. Find all values of x in the range $\pi \leq x \leq 2\pi$ for which $\sin(x - 5\pi/6) = 1/2$. (3)
8. Find all values of x in the range $0 \leq x \leq 2\pi$ for which $4 \sin^2 x = 3$. (3)
9. Find all values of x in the range $\pi \leq x \leq 2\pi$ for which $2 \sin 2x \cos 2x + 1 = 0$ (3)

Section C.

10. Find all values of x in the range $-\pi \leq x \leq \pi$ for which $\sin x = \cos 2x$. (3)
11. If $\cos\left(x + \frac{\pi}{6}\right) = \sin x$, find the exact value of $\tan x$. (3)
12. Express $\sqrt{3} \sin \theta + \cos \theta$ in the form $R \cos(\theta - \alpha)$, where $R > 0$ and $0 \leq \alpha \leq \pi/2$. (3)

Trigonometry: solutions

(A) 1. -1 2. $\frac{1}{2}$ 3. $-\frac{1}{\sqrt{2}}$
4. -1 5. -1 6. $-\frac{\sqrt{3}}{2}$

(B) 7. $\sin(x - \frac{5\pi}{6}) = \frac{1}{2}$ for $\pi \leq x \leq 2\pi$

means $x - \frac{5\pi}{6} = \frac{\pi}{6} + 2n\pi$ so $x = \pi$

or $x - \frac{5\pi}{6} = \frac{5\pi}{6} + 2n\pi$ so $x = \frac{5\pi}{3}$

8. $4\sin^2 x = 3$ for $0 \leq x \leq 2\pi$ means $\sin x = \pm \frac{\sqrt{3}}{2}$

so $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

9. $2\sin 2x \cos 2x + 1 = 0$ for $\pi \leq x \leq 2\pi$

means $\sin 4x + 1 = 0$ so $x = \frac{11\pi}{8}, \frac{15\pi}{8}$

(C) 10. $\sin x = \cos 2x$ for $-\pi \leq x \leq \pi$

means $\sin x = 1 - 2\sin^2 x$ so $2\sin^2 x + \sin x - 1 = 0$

$2u^2 + u - 1 = 0 \Rightarrow u = -1$ or $\frac{1}{2}$

$\sin x = -1 \Rightarrow x = -\frac{\pi}{2}$ $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

11. $\cos(x + \frac{\pi}{6}) = \sin x \Rightarrow$

$\sin x = \cos x \cos \frac{\pi}{6} - \sin x \sin \frac{\pi}{6}$

i.e. $\sin x = \frac{\sqrt{3}}{2} \cos x - \frac{1}{2} \sin x$

i.e. $\tan x = \frac{1}{\sqrt{3}}$

PTO

$$12. \sqrt{3} \sin \theta + \cos \theta = R \cos(\theta - \alpha) = R \cos \theta \cos \alpha + R \sin \theta \sin \alpha$$

$$\text{so } R \cos \alpha = 1, \quad R \sin \alpha = \sqrt{3}$$

$$\text{so } R^2 = 4 \Rightarrow \underline{R = 2}$$

$$\tan \alpha = \sqrt{3} \Rightarrow \alpha = \underline{\underline{\frac{\pi}{3}}}$$