

Vectors

Geogebra Instructions

Geogebra is an online resource of maths activities and we'll be using quite a few here. It's best used on a laptop/desktop in my experience but there is an app available for smartphones and tablets. As some of the activities we're doing are quite complex (as far as Geogebra is concerned), the app can sometimes crash which is why we're recommending the laptop/desktop option with an up-to-date browser.

Coordinate Systems

[Watch this video clip](#) until 43mins (it should start at 37mins 30s).

Comprehension questions

1. How many coordinates do we most typically need to specify a point in space?
2. What is a cartesian coordinate system?
3. What is an origin?
4. Does it matter where we put the origin of a system?
5. Susskind says "The important questions that we're interested in should not depend on the convention of where we put the origin" - what does he mean by this?
6. Can we change the origin of our coordinate system?
7. Once we have an origin, what else do we need?
8. The origin is not just in terms of space, what else do we need an origin for?
9. What is the right-handed convention for coordinate systems?
10. How do we label a point in space?
11. If a point in space is labelled at $(2m, 4m, 0.5m)$ what does this mean?
12. There is such a thing as a 'best choice' for your coordinate system. Imagine a car is driving North down a straight, flat road.
 - 12.1. What would be a good choice of coordinate system (one that makes the maths easier)?
 - 12.2. What would be a bad choice of coordinate system (one that makes the maths hard)?

This activity emphasises that a point plotted in a 3D coordinate system is given by the three coordinates: x , y and z . Edit the sizes of x , y and z , remembering that they can be positive or negative and then slide the slider to plot the point that you've created (you may need to zoom out). Whilst trying out the tool, think about the following:

- A. Is the coordinate system a right-handed one?
- B. Does it matter in which order we move along the x , y and z axes to get to the final point?
- C. Are the positive and negative quadrants where you could expect?

Vectors

Watch this video clip until 49mins (it should start at 43mins 4s).

Comprehension questions

1. What is a vector?
2. Give three examples of vectors.
3. What is a scalar?
4. Give three examples of scalars.
5. Does a vector have a location?
6. What are some of the conventions for demonstrating that a value is a vector?
7. What is true about the length/magnitude of a vector?
8. What are *components* of a vector?
9. Susskind says that "*the magnitude of V is the square root of the sums of the components.*" He has made a mistake though when he says that (but not in what he's written it down). What is his mistake?
10. How does Susskind describe how the direction of a vector is formed?
11. "If you move a vector around, it doesn't change the vector"

Susskind discusses that a vector doesn't have to be tied to a specific direction - it is completely valid for a vector to be moved around and retain its identity. However, this doesn't mean that a vector *can't* have a location. There are certain scenarios in which a vector is definitely located at a specific point. For example, when I push open a door I apply the force to the handle. My force vector happens to be located at a specific point and this has consequences for how the door opens. If I push a door much closer to the hinge with the exact same force, it will open much slower. This is the idea of turning moments, and the location of the applied force is crucial to the physics. Whilst a vector does not *need* to have a location, it is more than valid for us to say that it has one.

Writing vectors

Vectors are often referred to in different ways according to different people's conventions - this is absolutely fine so long as you stay consistent. Some ways that you might see used to express that a quantity is a vector are:

- \vec{V} - a horizontal arrow over the quantity (typically in italics). This is the convention I will use in this sheet.
- \underline{V} - the quantity being underlined (typically in italics)
- \mathbf{V} - the quantity being shown in bold
- $\underline{\mathbf{V}}$ - you may see combinations such as bold and underlined.

Equally, when specifying the components of a vector, there are many different conventions about the types of brackets used. I've used the convention that a vector's components are specified in angled brackets so that $\vec{V} = \langle V_x, V_y, V_z \rangle$ whilst a point in space is specified in normal brackets - and isn't a vector, just an isolated location with respect to the origin - so that $P = (P_x, P_y, P_z)$ - note that P doesn't have a vector symbol.

The important point is to make it clear when a quantity is a vector and when it isn't, and then to preferably stick to that convention. You will encounter different methods in different books and with different lecturers, so it's important to be aware and to be adaptable.

Unit vectors

The link between vectors and the coordinate system in which they live is that each of a vector's components is actually specifying how far we move along each of our coordinate system's axes. Each axis of our coordinate system can be considered as a vector and we define *unit vectors* as a series of vectors with unit length (a length of 1) that lie along each of your axes. There will be the same number of unit vectors as there are dimensions in your system e.g. for a 2D system, we need two unit vectors.

This gives us an additional way of specifying vectors - as multiples of unit vectors. In a Cartesian system (x, y, z) we typically give our units vectors the symbols $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. The hat (^) indicates that it's a unit vector, and I've put them in bold too so as to be especially clear that these are vectors. As with any vectors, different places use different conventions. So I can specify a vector's components explicitly as:

$$\vec{V} = \langle 1, 2, 3 \rangle = 1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

These representations are completely equivalent.

[This applet](#) is very simple and shows vectors being specified via their components and as multiples of unit vectors. Here, they use the convention that unit vectors are simply in italics.

Finding the magnitude of a vector

The length of a vector is given by the following formula:

$$|\vec{V}| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

To find the length/magnitude of a vector, you simply:

- a) square each of the individual components
- b) sum the squares
- c) take the square root of the sum

This video gives proof of this geometrically if you've not seen it before. This is simply Pythagoras's rule for right-angled triangles (in 3D) applied to the situation of vectors.

Comprehension questions

1. Can a vector have a negative magnitude? Why?
2. I have a vector $\vec{V}_1 = \langle V_x, V_y, V_z \rangle$, where V_x , V_y and V_z are numbers. I have a second vector $\vec{V}_2 = \langle -V_x, -V_y, -V_z \rangle$ whose components are the opposite sign. Are they the same length? Why?

Worked example

$\vec{V} = \langle 3, 4, 5 \rangle$, what is the magnitude of \vec{V} ?

$$|\vec{V}| = \sqrt{3^2 + 4^2 + 5^2}$$

$$|\vec{V}| = \sqrt{9 + 16 + 25}$$

$$|\vec{V}| = \sqrt{50}$$

$$|\vec{V}| = 7.07 \text{ (2d.p.)}$$

Complete these faded examples

$\vec{V}_1 = \langle 2, 1, 3 \rangle$, what is the magnitude of \vec{V}_1 ?	$\vec{V}_2 = \langle 4.0, -2.0, 2.5 \rangle$, what is the magnitude of \vec{V}_2 ?	$\vec{V}_3 = \langle 1.5, -1.0, 6.0 \rangle$, what is the magnitude of \vec{V}_3 ?
$ \vec{V}_1 = \sqrt{2^2 + 1^2 + 3^2}$	$ \vec{V}_2 = \sqrt{4.0^2 + (-2.0)^2 + 2.5^2}$	$ \vec{V}_3 =$
$ \vec{V}_1 = \sqrt{4 + 1 + 9}$	$ \vec{V}_2 =$	$ \vec{V}_3 =$
$ \vec{V}_1 =$	$ \vec{V}_2 =$	$ \vec{V}_3 =$
$ \vec{V}_1 =$	$ \vec{V}_2 =$	$ \vec{V}_3 =$

Try them yourself

- $\vec{V}_1 = \langle 3, 0, 4 \rangle$
- $\vec{V}_2 = \langle -3, 0, -4 \rangle$
- $\vec{V}_3 = \langle 2, -4, 4 \rangle$
- $\vec{V}_4 = \langle -5, 3, 2 \rangle$
- $\vec{V}_5 = \langle -2.5, 3.2, -1.8 \rangle$
- Vector \vec{V}_6 has a magnitude of 6. The ratio of its components is 1:1:1. What is vector \vec{V}_6 ?
There is more than one solution.
- Vector \vec{V}_7 has a magnitude of 28. The ratio of its components is 2:1:3. What is vector \vec{V}_7 ?
There is more than one solution.
- Vector \vec{V}_6 joins the points $P_1 = (2, 6, 3)$ and $P_2 = (3, 4, 5)$. What is \vec{V}_6 and what is its length?
- Vector \vec{V}_7 joins the points $P_1 = (-1.5, 2.4, 0.8)$ and $P_2 = (3.2, -4.1, -0.1)$. What is \vec{V}_7 and what is its length?
- Vector \vec{V}_{10} has a magnitude of 120. The ratio of its components is 5:1:2. What is vector \vec{V}_{10} ?
There is more than one solution.

Finding the direction of a vector

With a sensible choice of coordinate system, we can very often reduce ourselves to working in 2D. For example, a airport might have a runway that runs in a north-easterly direction. I could set up my coordinate system so that East was the x -axis, North was the y -axis and the vertical axis was the z -axis but this gives me three coordinates to keep track of. Instead, I could instead choose my x -axis to lie along the North-East direction that the runway follows and then I can track the motion of planes along the runway using just the x and z coordinates (assuming the pilots can travel adequately in a straight line along the runway). As another example, in a game of pool, I could set my x -axis to lie along the narrower side of the table, the y -axis to lie along the longer length of the table and the z -axis to lie vertically out from the table, but given that I'm going to assume the balls stay on the surface of the table (as I'm pretty good at pool) then I can forget about keeping track of the z -coordinate entirely and just use an x - y coordinate system.

To show the *direction* of a vector with a 2D coordinate system, I just need to specify an angle relative to one of the axes (in a 3D coordinate system, I need to specify two angles to show the direction). This, combined with the magnitude of the vector describes it entirely.

It is useful to be able to move between specifying a vector through its components and specifying it through a magnitude and an angle. This is where triangles, and in particular right-angles triangles, are the key. We're going to assume that you can find any side length or any angle for a right-angled triangle using a combination of the [Pythagoras theorem](#) and [SOHCAHTOA](#). If you need to test yourselves on the finding the angles within a right-angled triangle, have a go at [this applet](#). [This applet](#) is also useful as a refresher - the top slider changes what is specified about the triangle, you can alter the settings and then test yourself to ensure you can calculate all of the sides and angles within the triangle (ticking 'Show Other Measures' gives the answers).

Comprehension questions

1. Why does it not matter how I set up my coordinate system?
2. What is the benefit of being able to choose your coordinate system?
3. Do vectors change if I change the coordinate system?
4. Do scalars change if I change the coordinate system?
5. In my example of a game of pool, I set my coordinate system to ignore the z -axis. Why is this actually a bad idea in a real game of pool?
6. Draw a right angled triangle and label the side lengths and angles.
 - 6.1. Write the relationship between the lengths of the sides.
 - 6.2. Write the relationship between the angles and the side lengths.
7. In 2D, to specify the direction of a vector I need to use one angle. In 3D, I need to specify two angles. Explain why this is.
8. A scalar is not a one dimensional vector. Explain why this is.

How do right-handed triangles link to vectors? The axes of our coordinate system, remember, are perpendicular and so when we specify a vector through its components, these components are at ninety degrees to one another. This means that our vector is actually the hypotenuse of a right angled triangle formed from its components. To swap between the component representation of a vector and the magnitude and direction representation, then we do one of the following:

To go from <i>components</i> to <i>magnitude and direction</i> :	To go from <i>magnitude and direction</i> to <i>components</i> :
1. Our x and y components form the adjacent and opposite sides of our triangle respectively.	1. The magnitude of our vectors forms the length of the hypotenuse of our triangle and the angle is taken to be the angle between the vector and the x axis (by convention).
2. We find the <i>magnitude</i> of the vector by using the Pythagoras theorem (square both components, sum them, square root).	2. We can find the x component of our vector as the adjacent length using $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ so that $\text{adjacent} = \text{hypotenuse} \times \cos \theta$
3. We find the angle between the vector and the x axis, θ , (we always use this angle as a convention) by using $\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$	3. We can find the y component using the two sides we have and the Pythagoras theorem or we can use trigonometry, knowing that the y component forms the opposite side of our triangle and $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$ so that $\text{opposite} = \text{hypotenuse} \times \sin \theta$

Use [this link](#) to alter the magnitude and direction of the vector \vec{A} and see how the resulting triangle of components, A_x and A_y , changes. Check that you can calculate one set of numbers from the other using the methods in the table above.

We're going to use [this applet](#) to practice. x will be the horizontal axis of the graph paper and y will be the vertical axis. The origin will be the 'Initial Point'. You can drag the 'Terminal point' (the end of the vector) around to create vectors.

We're going to start by calculating the magnitude and direction of vectors from their components (the left hand method above). We can then plot these vectors and check them with the Geogebra applet.

Worked example

If $\vec{u} = \langle 3, 2 \rangle$, what is $|\vec{u}|$ and the angle θ between \vec{u} and the x axis?

We start by using the Pythagoras theorem to find the magnitude of \vec{u}

$$|\vec{u}| = \sqrt{3^2 + 2^2}$$

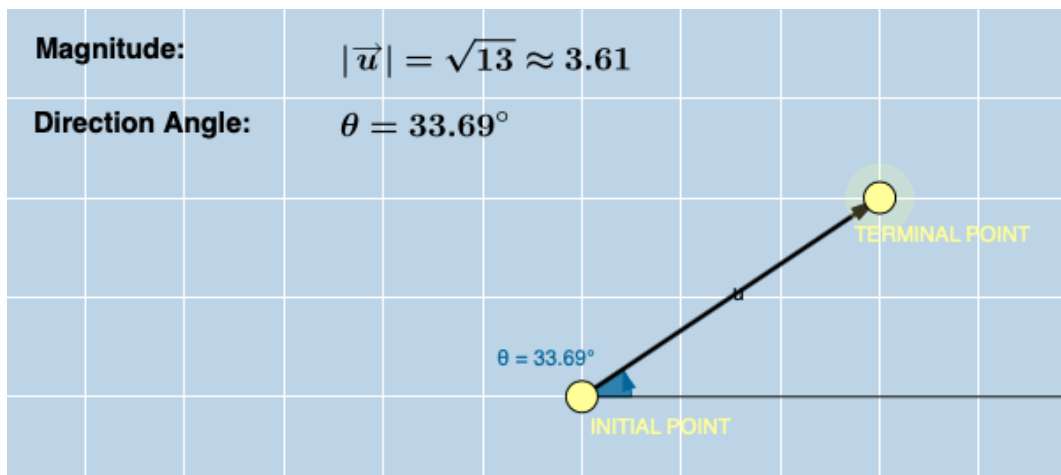
$$|\vec{u}| = \sqrt{13} \approx 3.6$$

To find the direction, we use trigonometry. The length opposite θ is the y value and the length adjacent to θ is the x value.

$$\tan \theta = \frac{2}{3}$$

$$\theta = \tan^{-1} \frac{2}{3} \approx 33.7^\circ$$

We can check this in Geogebra and should find something like the image below:



Try them yourself and use Geogebra to check as you go

1. If $\vec{u} = \langle 2, 3 \rangle$, what is $|\vec{u}|$ and the angle θ between \vec{u} and the positive x axis?
2. If $\vec{u} = \langle 5, 1 \rangle$, what is $|\vec{u}|$ and the angle θ between \vec{u} and the positive x axis?
3. If $\vec{u} = \langle 0, 2 \rangle$, what is $|\vec{u}|$ and the angle θ between \vec{u} and the positive x axis?
4. If $\vec{u} = \langle 4, 0 \rangle$, what is $|\vec{u}|$ and the angle θ between \vec{u} and the positive x axis?
5. If $\vec{u} = \langle 5, 5 \rangle$, what is $|\vec{u}|$ and the angle θ between \vec{u} and the positive x axis?

By convention, we measure our angles relative to the positive x axis, and, more specifically, we measure *anticlockwise* from this positive x direction. This means we need to be especially careful when we're not working in the positive quadrant of the graph i.e. whenever the angle θ is greater than 90° .

- If we're working in the top left quadrant (x negative and y positive), then the angle we calculate using the method above needs to be subtracted from 180° to get our true angle.
- If we're working in the bottom left quadrant (x negative and y negative), then the angle we calculate using the method above needs to be added to 180° to get our true angle.
- If we're working in the bottom right quadrant (x positive and y negative), then the angle we calculate using the method above needs to be subtracted from 360° to get our true angle.

Try them yourself and use Geogebra to check as you go

1. If $\vec{u} = \langle -1, 3 \rangle$, what is $|\vec{u}|$ and the angle θ between \vec{u} and the positive x axis?
2. If $\vec{u} = \langle -3, 2 \rangle$, what is $|\vec{u}|$ and the angle θ between \vec{u} and the positive x axis?
3. If $\vec{u} = \langle -3, -2 \rangle$, what is $|\vec{u}|$ and the angle θ between \vec{u} and the positive x axis?
4. If $\vec{u} = \langle -2, -5 \rangle$, what is $|\vec{u}|$ and the angle θ between \vec{u} and the positive x axis?
5. If $\vec{u} = \langle 6, -3 \rangle$, what is $|\vec{u}|$ and the angle θ between \vec{u} and the positive x axis?
6. If $\vec{u} = \langle 1, -4 \rangle$, what is $|\vec{u}|$ and the angle θ between \vec{u} and the positive x axis?
7. If $\vec{u} = \langle -2, -3 \rangle$, what is $|\vec{u}|$ and the angle θ between \vec{u} and the positive x axis?
8. If $\vec{u} = \langle 1, -1 \rangle$, what is $|\vec{u}|$ and the angle θ between \vec{u} and the positive x axis?
9. If $\vec{u} = \langle -3, 6 \rangle$, what is $|\vec{u}|$ and the angle θ between \vec{u} and the positive x axis?
10. If $\vec{u} = \langle 2, 4 \rangle$, what is $|\vec{u}|$ and the angle θ between \vec{u} and the positive x axis?

Basic vector operations

Watch [this video clip](#) until 55mins (it should start at 49mins). We can summarise simple vector algebra as “you can add ‘em, you can subtract ‘em, you can multiply ‘em by numbers”.

Comprehension questions

1. If you multiply a vector by a positive, scalar value, what happens?
2. If you multiply a vector by a negative, scalar value, what happens?
3. How must we line two vectors up to add them together?
4. Once two vectors are lined up to be added, how do we see the result?
5. When discussing the addition of vectors, Susskind says that we may end up with a ‘degenerate triangle’ - what does he mean here?
6. Is vector addition commutative? Does $\vec{A} + \vec{B} = \vec{B} + \vec{A}$?
7. How do you add vectors \vec{A} and \vec{B} by their components?
8. Subtracting vectors is simply a combination of vector addition and multiplying vectors by (negative) numbers. Explain this idea.

Multiplying a vector by a number

You might call this scaling of vectors and [this activity](#) lets you alter the magnitude and direction of an initial vector, v (by moving the circular end points), change the scale factor, k (although only through integer values) and then create your scaled vector by sliding the ‘Slide Me!’ bar. Use this tool to figure out what happens to vectors under the following conditions:

- A. What happens if you multiply a vector by a number that is greater than 1?
- B. What happens if you multiply a vector by a number that is less than 1 (but greater than 0)?
- C. What happens if you multiply vector by zero?
- D. What happens if you multiply a vector by a number less than zero but greater than -1?
- E. What happens if you multiply a vector by -1?
- F. What happens if you multiply a vector by a number less than -1?

Worked example

If $\vec{V} = \langle 3, 4, 5 \rangle$, what is $2\vec{V}$?

$$2\vec{V} = 2\langle 3, 4, 5 \rangle$$

$$2\vec{V} = \langle 2 \times 3, 2 \times 4, 2 \times 5 \rangle$$

$$2\vec{V} = \langle 6, 8, 10 \rangle$$

Complete these faded examples

If $\vec{V}_1 = \langle 0.5, 3.0, 2.4 \rangle$, what is $3\vec{V}_1$?

$$3\vec{V}_1 = 3\langle 0.5, 3.0, 2.4 \rangle$$

$$3\vec{V}_1 = \langle 3 \times 0.5, 3 \times 3.0, 3 \times 2.4 \rangle$$

$$3\vec{V}_1 =$$

If $\vec{V}_2 = \langle 4.4, 12.0, 2.0 \rangle$, what is $0.5\vec{V}_2$?

$$0.5\vec{V}_2 = 0.5\langle 4.4, 12.0, 2.0 \rangle$$

$$0.5\vec{V}_2 =$$

$$0.5\vec{V}_2 =$$

If $\vec{V}_3 = \langle 1.1, 1.3, -2.1 \rangle$, what is $-1.0\vec{V}_3$?

$$-1.0\vec{V}_3 =$$

$$-1.0\vec{V}_3 =$$

$$-1.0\vec{V}_3 =$$

Try them yourself

1. If $\vec{V} = \langle 3, 7, 1 \rangle$, what is $5\vec{V}$?
2. If $\vec{V} = \langle -2, 4, -6 \rangle$, what is $3\vec{V}$?
3. If $\vec{V} = \langle 2.2, 1.5, -3.0 \rangle$, what is $-4\vec{V}$?
4. If $\vec{V} = \langle 0.35, -1.39, -7.21 \rangle$, what is $2\vec{V}$?
5. If $\vec{V} = \langle -3, 5, 4 \rangle$, what is $2.5\vec{V}$? What is $|\vec{V}|$?
6. If $\vec{V} = \langle -1, 8, -6 \rangle$, what is $-1.2\vec{V}$? What is $|\vec{V}|$?
7. If $\vec{V} = \langle 2.6, 3.5, -2.7 \rangle$, what is $3.5\vec{V}$? What is $|\vec{V}|$?
8. If $\vec{V} = \langle -0.24, 1.79, -5.87 \rangle$, what is $-3.4\vec{V}$? What is $|\vec{V}|$?
9. If $\vec{V} = \langle 0.12, 0.45 \rangle$, what is $2.1\vec{V}$? What is $|\vec{V}|$? What is the angle θ between \vec{V} and the positive x axis?
10. If $\vec{V} = \langle -1.4, 2.8 \rangle$, what is $-0.5\vec{V}$? What is $|\vec{V}|$? What is the angle θ between \vec{V} and the positive x axis?

Adding vectors geometrically

Activity I

This brief demo should remind you what it looks like to add vectors geometrically. Simply slide the slider to add the vectors \vec{u} and \vec{v} together. It illustrates the relationship between the triangle and the parallelogram methods. This app also show the angles and components of the vectors and allows you to alter them continuously.

Comprehension questions

1. Is vector addition commutative? Explain your answer.
2. Explain how to add any two vectors geometrically.

Activity II

This activity gets you to add two-dimensional vectors and allows you to see the relationship between the geometrical method of adding and the component method. Do the following:

- set the sizes of two-dimensional vectors \vec{u} and \vec{v} through their horizontal and vertical components (for simplicity, only working with integers here)
- once set, you can see their numerical components displayed in vector notation on the left as $\vec{u} = \langle x, y \rangle$
- by sliding the slider, the two vectors will be plotted (\vec{u} starting from the white circle, \vec{v} starting from the yellow circle), showing their horizontal component in purple and their vertical component in green
- using the components on the left, calculate what the result of $\vec{u} + \vec{v}$ will be by doing $\langle u_x + v_x, u_y + v_y \rangle$
- check whether the geometrical method gives the same answer by dragging the start of \vec{v} (the yellow circle) to the end of \vec{u} (the orange circle). This can be a bit fiddly, but they should eventually snap together (maybe zoom out and in or reset if having trouble). A new slider will then construct $\vec{u} + \vec{v}$.
- count the horizontal and vertical components of this new vector to check your earlier addition via component addition

Worked example (this is the default example of the Geogebra applet in Activity II)

If $\vec{u} = \langle 9, -4 \rangle$ and $\vec{v} = \langle -5, -6 \rangle$, what is $\vec{u} + \vec{v}$?

$$\vec{u} + \vec{v} = \langle 9, -4 \rangle + \langle -5, -6 \rangle$$

$\vec{u} + \vec{v} = \langle 9 + (-5), -4 + (-6) \rangle$ Here we simply add the x and y components individually together. Take care with the negative signs - I've used brackets here to make it clear that we're definitely *adding* them together.

$$\vec{u} + \vec{v} = \langle 4, -10 \rangle$$

Complete these faded examples - use the Geogebra applet in Activity II to plot them and check your answers are consistent

If $\vec{u} = \langle 3, 1 \rangle$ and $\vec{v} = \langle 2, 7 \rangle$, what is $\vec{u} + \vec{v}$?

$$\vec{u} + \vec{v} = \langle 3, 1 \rangle + \langle 2, 7 \rangle$$

$$\vec{u} + \vec{v} = \langle 3 + 2, 1 + 7 \rangle$$

$$\vec{u} + \vec{v} =$$

If $\vec{u} = \langle -4, 5 \rangle$ and $\vec{v} = \langle 1, 2 \rangle$, what is $\vec{u} + \vec{v}$?

$$\vec{u} + \vec{v} = \langle -4, 5 \rangle + \langle 1, 2 \rangle$$

$$\vec{u} + \vec{v} =$$

$$\vec{u} + \vec{v} =$$

If $\vec{u} = \langle 0, -7 \rangle$ and $\vec{v} = \langle 0, 2 \rangle$, what is $\vec{u} + \vec{v}$?

$$\vec{u} + \vec{v} =$$

$$\vec{u} + \vec{v} =$$

$$\vec{u} + \vec{v} =$$

Try them yourself

1. If $\vec{u} = \langle -3, -2 \rangle$ and $\vec{v} = \langle -4, 2 \rangle$, what is $\vec{u} + \vec{v}$?
2. If $\vec{u} = \langle -5, -3 \rangle$ and $\vec{v} = \langle 5, 3 \rangle$, what is $\vec{u} + \vec{v}$?
3. If $\vec{u} = \langle -1.5, 2.0 \rangle$ and $\vec{v} = \langle -0.5, 0.5 \rangle$, what is $\vec{u} + 2\vec{v}$? What is $|\vec{u} + 2\vec{v}|$?
4. If $\vec{u} = \langle 0.0, 0.1 \rangle$ and $\vec{v} = \langle -0.6, 0.2 \rangle$, what is $2\vec{u} + 3\vec{v}$? What is $|2\vec{u} + 3\vec{v}|$?
5. If $\vec{u} = \langle 3.26, 1.84 \rangle$ and $\vec{v} = \langle -0.56, -1.52 \rangle$, what is $\vec{u} + 0.5\vec{v}$? What is $|\vec{u} + 0.5\vec{v}|$? What is the angle θ between $\vec{u} + 0.5\vec{v}$ and the positive x axis?

Activity III

To test your vector addition in 2D, try [this activity](#). Manipulate the thicker arrow to be the result of adding the vectors \vec{u} and \vec{v} . You might do this by counting the components of \vec{u} and \vec{v} and then adding them, or by eye to manipulate \vec{v} to start at the end of \vec{u} . Remember for this that your vector doesn't have to be tied to a specific position, so provided the length and direction of your vector is correct then it can exist anywhere on the page.

Comprehension questions

1. Explain how to add any two vectors via their components.
2. Is it possible for the addition of two vectors to result in the zero vector? If so, what must be true about the two vectors?

Subtracting vectors

Vector subtraction is simply the combination of vector multiplication and vector addition. To subtract \vec{v} from \vec{u} all I need to do is multiply \vec{v} by -1 and then add it to \vec{u} . [This applet](#) illustrates this exact process.

Comprehension questions

1. Prove with an example that vector subtraction is not commutative.

Worked example

If $\vec{u} = \langle 2, 5, 3 \rangle$ and $\vec{v} = \langle 1, 5, 2 \rangle$, what is $\vec{u} - \vec{v}$?

$$-1\vec{v} = \langle -1, -5, -2 \rangle$$

$$\vec{u} - \vec{v} = \langle 2, 5, 3 \rangle + \langle -1, -5, -2 \rangle$$

$$\vec{u} - \vec{v} = \langle 2 + (-1), 5 + (-5), 3 + (-2) \rangle$$

$$\vec{u} - \vec{v} = \langle 1, 0, 1 \rangle$$

Complete these faded examples - use the Geogebra applet in Activity II to plot them and check your answers are consistent

If $\vec{u} = \langle -3, 2, -1 \rangle$ and $\vec{v} = \langle 1, 3, 2 \rangle$, what is $\vec{u} - 2\vec{v}$?	If $\vec{u} = \langle 1, -1, -2 \rangle$ and $\vec{v} = \langle -4, 2, 3 \rangle$, what is $2\vec{u} - \vec{v}$?	If $\vec{u} = \langle 0.3, 2.1, -1.4 \rangle$ and $\vec{v} = \langle 0.7, 1.9, -3.6 \rangle$, what is $2\vec{u} - 3\vec{v}$?
$-2\vec{v} = \langle -2, -6, -4 \rangle$ $\vec{u} - 2\vec{v} = \langle -3 - 2, 2 - 6, -1 - 4 \rangle$ $\vec{u} - 2\vec{v} =$	$-\vec{v} = \langle 4, -2, -3 \rangle$ $2\vec{u} - \vec{v} =$ $2\vec{u} - \vec{v} =$	$-3\vec{v} =$ $2\vec{u} - 3\vec{v} =$ $2\vec{u} - 3\vec{v} =$

Try them yourself

1. If $\vec{u} = \langle -9, -5, -3 \rangle$ and $\vec{v} = \langle -2, 3, -4 \rangle$, what is $2\vec{u} - 2\vec{v}$?
2. If $\vec{u} = \langle 3, 6, 1 \rangle$ and $\vec{v} = \langle -5, -1, 8 \rangle$, what is $\vec{u} - \vec{v}$?
3. If $\vec{u} = \langle 0, 3, 2 \rangle$ and $\vec{v} = \langle 3, 0, 9 \rangle$, what is $\vec{u} - 2\vec{v}$?
4. If $\vec{u} = \langle 3.0, 4.4, 6.2 \rangle$ and $\vec{v} = \langle 2.5, -3.3, -0.6 \rangle$, what is $0.5\vec{u} - 1.5\vec{v}$?
5. If $\vec{u} = \langle 0.0, -2.0, 4.7 \rangle$ and $\vec{v} = \langle 13.4, 12.2, 5.6 \rangle$, what is $1.2\vec{u} - 2.5\vec{v}$? What is the length of this vector?
6. If $\vec{u} = \langle 2.3, 1.8, -8.5 \rangle$ and $\vec{v} = \langle -6.2, -2.7, 4.0 \rangle$, what is $2\vec{u} - \vec{v}$? What is the length of this vector?
7. If $\vec{u} = \langle 2.5, 1.0, -3.5 \rangle$, $\vec{v} = \langle -1.6, 7.3, 2.5 \rangle$ and $\vec{w} = \langle -4.4, -3.2, 2.8 \rangle$, what is $4\vec{u} - \vec{v} - 0.5\vec{w}$? What is the length of this vector?

8. If $\vec{u} = \langle 0.0, -1.5, 2.5 \rangle$, $\vec{v} = \langle 9.8, 4.6, -2.0 \rangle$ and $\vec{w} = \langle 1.6, -4.2, 8.0 \rangle$, what is $-0.2\vec{u} + \vec{v} - 2.5\vec{w}$? What is the length of this vector?
9. If $\vec{u} = \langle -1.5, 3.0 \rangle$, $\vec{v} = \langle 2.2, 8.4 \rangle$ and $\vec{w} = \langle -3.3, -12.4 \rangle$, what is $2.0\vec{u} - \vec{v} - 0.2\vec{w}$? What is the length of this vector? What is the angle θ between $2.0\vec{u} - \vec{v} - 0.2\vec{w}$ and the positive x axis?
10. If $\vec{u} = \langle 7.3, 6.5 \rangle$, $\vec{v} = \langle 8.6, -2.2 \rangle$ and $\vec{w} = \langle 0.1, -0.3 \rangle$, what is $3.0\vec{u} + 2.5\vec{v} - 8.0\vec{w}$? What is the length of this vector? What is the angle θ between $3.0\vec{u} + 2.5\vec{v} - 8.0\vec{w}$ and the positive x axis?

This app gives you the functionality to both add and subtract vectors, and allows you to adjust the magnitude and direction of the initial vectors continuously.

Vector problems

1. I'm on a football pitch and I walk a distance of 30m from the corner flag at an angle of 30° to the side of the pitch (which acts as my x axis). What is the vector that illustrates my displacement?
2. An proton experiences forces of $\vec{F}_1 = \langle 1.2, -7.4, 3.6 \rangle$ nN and $\vec{F}_2 = \langle -0.3, 2.9, 9.2 \rangle$ nN.
 - 2.1. What is the resultant force on the proton?
 - 2.2. What is the magnitude of the resultant force?
3. A ball is thrown with an initial velocity of $\vec{v} = \langle 2.1, 1.5 \rangle$ m s⁻¹, but travels through the air which itself is moving (due to wind) with a velocity of $\vec{u} = \langle 0.2, -0.3 \rangle$ m s⁻¹. What is the velocity of the ball as seen from the ground?
4. In a vast desert, I walk 100km North-East and then 50km East. What is the vector that illustrates my displacement? Assume that North represents my y axis and East represents by x axis.
5. A 5m ladder is propped up against a wall at an angle of 5° , what is the vector that describes the displacement of the very tip of the ladder compared to the foot of the ladder.
6. An 10kg object is acted on by three forces $\vec{F}_1 = \langle 500, 300, 100 \rangle$ N, $\vec{F}_2 = \langle -0.5, 0.2, -0.1 \rangle$ kN and $\vec{F}_3 = \langle 20, 50, 30 \rangle$ N.
 - 6.1. What is the vector that represents the total force?
 - 6.2. What is the magnitude of the vector?
 - 6.3. What is the magnitude of the acceleration of the object?
 - 6.4. What is the vector that represents the acceleration of the object?
7. A bird is flying with a velocity of $\vec{v} = \langle 2.5, 1.3 \rangle$ m s⁻¹, into a wind that has a magnitude of 0.6 m s⁻¹, and is at an angle of 320° to the x axis. What is the velocity of the bird as seen from the ground?

8. Below is an image of a road system in Milton Keynes (famed for its right-angled roads). Given the details in the image, what is the vector that describes the displacement between point B and point A?



9. A stationary shopping trolley is pushed through a car park by two children, each with a force of $\vec{F}_{child} = \langle 250, 300 \rangle$ N. Friction acts on the trolley, in the opposite direction to its motion and this force has a magnitude of 120 N. What is the overall force on the trolley?
10. A chimpanzee enclosure includes a series of vertical posts linked by taut rope. A first rope is tied to the ground (acting as our origin). The rope link at Post A is displaced from this origin by $\langle 2.7, 3.4, 1.4 \rangle$ m. The rope link at Post B is displaced from that at Post A by $\langle -3.1, 6.3, 0.8 \rangle$ m. The rope link at Post C is displaced from that at Post B by $\langle -5.1, -1.6, 1.1 \rangle$ m. A chimp climbs from the origin to Post A, then Post B, then half way to Post C.
- 10.1. What is their displacement vector from the origin?
- 10.2. What is the magnitude of their displacement?
- 10.3. What is the distance that the chimpanzee has travelled?