Gravitational Waves from Compact Object Binaries

Andrew Levan
Astronomy and Astrophysics Group - Department of Physics - University of Warwick

Introduction

Outline:
Gravitational waves are tiny changes in the distance between two objects, created by the acceleration of other objects. They are caused by accelerating masses on all scales. This includes almost all movement on the Earth. However, they are also very weak, and their strongest sources are those in which the mass is large, and the size small. These conditions are most naturally met in neutron stars (which have the mass of the Sun but are 10 km across), or in black holes. Such gravitational waves have now been detected from two black holes merging in a distant galaxy. In this project you will understand why gravitational waves are so weak, get to grips on how they are detected and make predictions about the kind of gravitational wave signals that astronomers will see in the coming years. This is real science, and although there are some simplifications, these are the sorts of calculations that are being done around the world by astronomers and astrophysicists today.

At the end of this document are several appendices that describe how to approach the sheet, measuring astronomical distances, and important constants, you may find it useful to refer to these at several points through the sheet.

These real calculations are not meant to be easy. However, they have been broken down so that you can have a good attempt at them, please give them a go.

Gravitational waves are a key prediction of Einstein's general theory of relativity, first published in 1916. The principle underlying general relativity is that rather than viewing the effects of gravity as simply a consequence of mass (as we do in Newtonian gravity) we view gravity as the result of the curvature of spacetime due to the presence of mass.

In a Newtonian view, the mass of an object (M) is the thing that dictates the force (F) it creates, e.g.

$$F = -\frac{GMm}{r^2} \tag{1}$$

where m is the mass of a test mass and r the distance from the source, G here is the gravitational constant. This force then induces an acceleration on the test mass, using the standard form of Newton's second law.

$$F = ma \tag{2}$$

However, in general relativity, mass (or more precisely mass energy due to the equivalence of mass and energy via the famous $E = mc^2$) provides a curvature to space time and it is this curvature to space time that provides information to the test mass about how to move. The

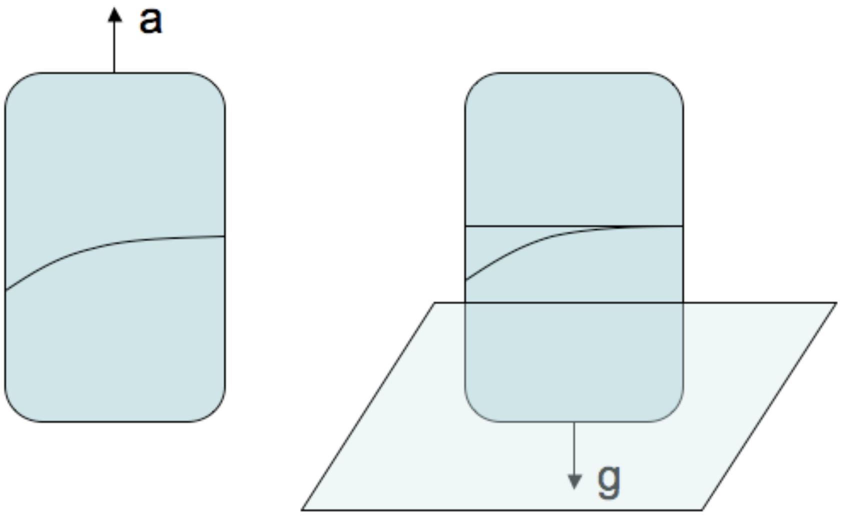

Figure 1: The equivalence principle in action. The left hand frame has uniform acceleration, while the right hand frame sits in a gravitational field. Since there must be no internal experiment in which one could determine which frame one was in, the results of firing a laser across each frame must be the same. Since in the left hand example the acceleration would mean the light hit the frame slightly below the point it was emitted, the same must happen in the gravitational field. This means that the light must not have travelled in a "straight" line. In fact it has (because light must), but to an observer it looks to have curved because the space-time through which it travels is bent by the mass.

famous, but nonetheless instructive analogy used for this is the example of a bowling ball on a rubber sheet, which stretches the material in the same way that mass stretches space time (albeit in three dimensions rather than two). It arises from a conceptual leap in the equating of two different kinds of mass. Looking at the equations above both Newton's second law, and the gravitational force have a mass term within them. This is important, the mass in Newton's second law is the mass that acts to inhibit the action of the force, it is the so called *inertial mass*, the mass in the gravitational force equation is the *gravitational mass*, it is a mass that determines how much gravitational force a source emits. It isn't obvious that these two should be the same. General relativity is based on this assumption and leads to the so called *equivalence principle*. This can be demonstrated through a thought experiment of the kind that that Einstein found very useful, and is shown in Figure 1.

Imagine that two laboratories are conducting experiments. One of these sits on the Earth, the other is on a spacecraft that is accelerating at 1g. If inertial mass and gravitational mass are the same then there is no way to distinguish these scenarios. However, what if the experiment being conducted involved firing a laser across the laboratory and looking at where it hits the opposite wall. In the accelerating case, because the laboratory will have moved between the emission of the light and it hitting the opposite side of the laboratory the light will appear to bend down. In principle this wouldn't happen to the Earth bound observer because they were not accelerating. However, if the equivalence principle is correct then you shouldn't be able to conduct an experiment in these two laboratories that gives a different result. Hence, the light on the Earth-bound laboratory must also bend. This is because space-time is bent around the mass of the Earth, and is a natural prediction of general relativity.

Gravitational Waves

However, now imagine that we have two objects close to each other on the hypothetical sheet described above. Both create a curvature to spacetime, as one object moves around the other (or more accurately they both move around their centre of mass) there is movement in the space time. However, this movement can't travel at an arbitrary speed as this would result in information being transferred at faster than the speed of light (something forbidden under special relativity). Therefore, the changes in the spacetime as the objects orbit one another manifest themselves as ripples. These ripples are gravitational waves, and since space time extends across the Universe the ripples can be detected at any point.

You can see this visually at https://www.youtube.com/watch?v=dw7U3BYMs4U

These ripples can only be created if there is a change in the space time. For example if a source simply gets bigger or smaller it doesn't change the curvature of space time and so doesn't create waves. In practice they are created only from sources that are accelerating in a way which is not spherically symmetric.

In many ways gravitational waves act like light and can be treated in a similar way, but there are central differences between gravitational waves and electromagnetic waves. In particular.

- EM waves consist of light in two *polarisations*. These are actually the separate electric and magnetic bits of the wave, and are oriented at right angles to each other. Gravitational waves also consist of two polarisations, often called h_+ and $h_\times$, but are oriented at 45 degrees to each other. The total strength (or amplitude) of the wave is then the sum in quadrature of the two polarisations (essentially just Pythagoras), so that the total amplitude $h = \sqrt{(h_+^2 + h_\times^2)}$.
- The intensity (or brightness) of an electromagnetic wave falls off as $1/d^2$, where d is the distance to the source. This is because we measure a brightness per square metre, if you move a square twice as far away, each side of the square becomes half the size it was originally, and so its area decreases by a factor of 4 (i.e. $1/d^2$). However, with gravitational waves we care about the amplitude of the wave. This varies linearly with distance, or $1/d$.

Your task here is to look at the expected amplitudes of gravitational waves from merging binaries containing neutron stars and black holes to determine the distances that they can be observed with the current generation of gravitational wave detectors. From this you will then calculate the relative rate that different kinds of mergers will be seen at with these detectors.

Gravitational Wave Strengths

The maximum strength of the two polarisations in gravitational waves for a binary star are given by

$$h_+ = \frac{1}{d}\frac{G^2}{c^4}\frac{2m_1m_2}{r} \tag{3}$$

and

$$h_\times = \frac{1}{d}\frac{G^2}{c^4}\frac{4m_1 m_2}{r} \tag{4}$$

such that the maximal strain for a given system at a distance d and with a separation r can be calculated[1] Inspecting these equation it is clear that the gravitational wave strength is at a maximum when the mass is large and the distance between the two objects are small (there is also the dependence on the distance of the source, d, as discussed above). Unfortunately, for most astronomical bodies this is problematic, because they are too big to get very close. For example, the Sun has a radius of about 700,000 km, and so it cannot get very close to another star. This means the most likely systems to find gravitational waves from are compact object binaries, that is binaries that contain stars much smaller than the Sun, but of comparable mass.

Task: Gravitational waves are weak because gravity is weak. Consider an atom, consisting of a proton and an electron. How does the electrostatic force, compare to the gravitational one. You may use the law for electrostatic force,

$$F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}, \tag{5}$$

where F is the force q_1 and q_2 the two changes and $\epsilon_0 = 8.85 \times 10^{-12}$ C V^{-1} m^{-1}, is the electric permativity of the vacuum. You can assume the electron and proton are separated by a Bohr radius $= 5.29 \times 10^{-11}$ m

Task: Here we want to understand why we see gravitational waves from black holes merging, but not from other systems, for example the Earth orbiting the Sun. Calculate the maximum gravitational wave amplitude for the Earth-Sun system when viewed from α-Centauri at 1 pc distance, and for two black holes, each 10 times the mass of the sun, positioned 50 km apart, viewed from 100 Mpc distance.

Compact object binaries

Compact objects are exotic stars that are much denser than the matter that makes up stars and planets. For stars and planets the collapse of the star due to gravity is held up by electromagnetic forces, either in the form of direct electrostatic repulsion (you can't put your hand through a table because the electrons in your hand repel those in the table), or by the transfer of heat. As objects become heavier their gravitational forces increase, and if they also become smaller at the same time then the gravity can overwhelm these forces, leading to a gravitational collapse. This collapse may be halted either at the stage of a white dwarf or a neutron star, or may not be halted, resulting in the formation of a black hole. A white dwarf has a mass comparable to the Sun, but is about the size of the Earth (about 100 times smaller than the Sun). On the other hand, a neutron star has the mass of the Sun,

[1] Note, that as these are waves they oscillate up and down at twice the frequency of the binary star. There is also an inclination factor such that $h_+ \propto 1 + \cos^2\theta$ and $h_\times \propto \cos\theta$ (where θ is the angle of inclination of the observer to the binary star).

but is only $\sim$10 km across. Black holes have no measurable surface, but are defined by their Schwarschild radius (see below).

Because these objects are massive they can be strong sources of gravitational waves, but because they are also small they can be found close together. Getting them so close is not trivial and involves quite complicated interactions between the stars as they evolve from being stars rather more massive (and larger) than the Sun, to these compact objects. In a lot of cases it is not possible to create such systems, and this means they are *very rare*. Typically once this complex evolution has run its course you can have two objects which are less than the radius of the Sun apart, but which both have masses comparable (or even exceeding the Sun).

Gravitational wave emission and mergers

It is only at this stage in their evolution that gravitational wave emission becomes significant. As shown above, the gravitational wave emission from the Sun– Earth system is tiny. But once the compact object binary is formed the gravitational waves start to extract measurable angular momentum and energy from the orbit and it begins to decay. We have directly observed this in a famous astronomical system called the double pulsar, in which two neutron stars orbit each other. As we observe over time the orbital period actually gets very gradually smaller, and this is because of the emission of gravitational waves (see Figure 2).

This gravitational wave emission is actually extremely well constrained and theoretically described. The change in the orbital separation as a function of time is given by the equation

$$\frac{dr}{dt} = -\frac{64}{5}\frac{G^3}{c^5}\frac{(m_1 m_2)(m_1 + m_2)}{r^3} \tag{6}$$

so that the rate of decay is a strong function of both the binary separation and the total mass of the binary.

Through this decay the loss to gravitational waves becomes larger, since the amplitude of the gravitational waves increases as the separation of the source decrease. This means the process becomes increasingly faster in order to conserve energy.

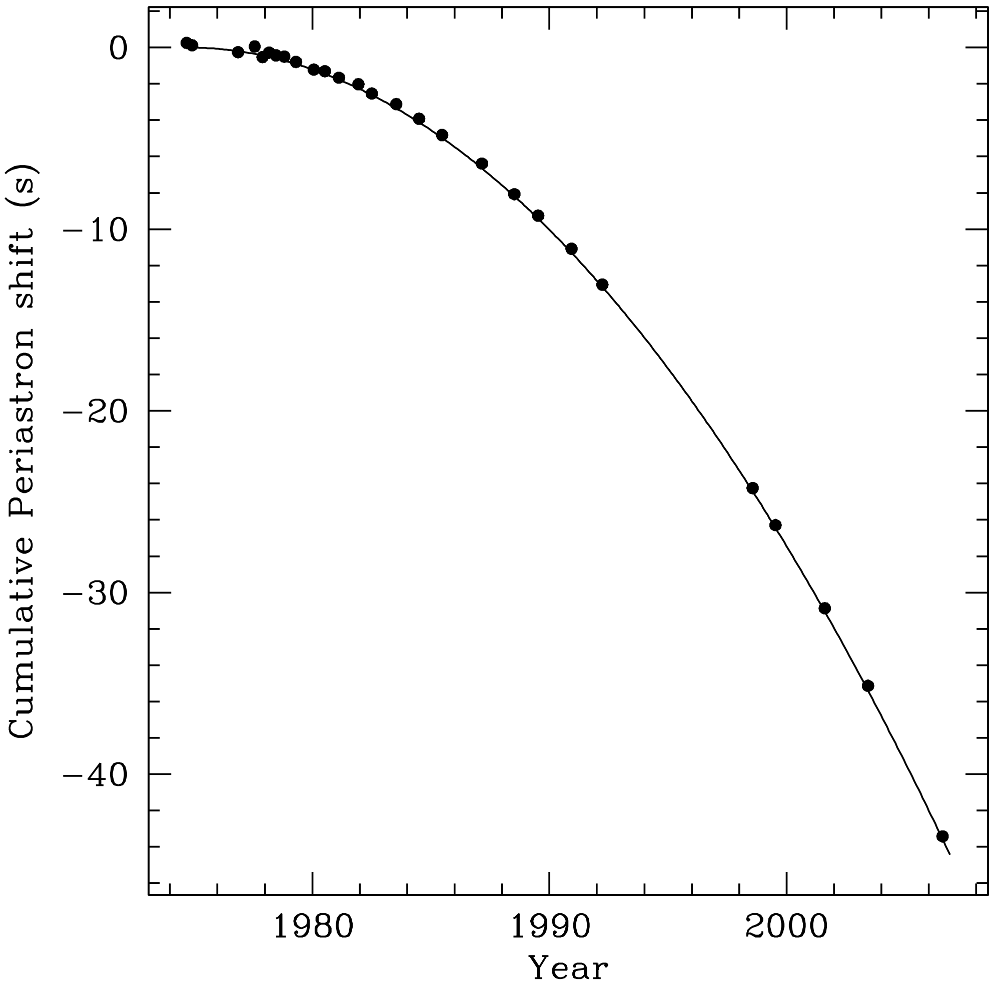

Figure 2: The evolution in the time of closest approach of two neutron stars in the system PSR B1913+16. The change in this time is because the orbit is shrinking. The line through the data points shows the expectation of general relativity, and is a remarkable match.

Task: For a system contain two neutron stars with a separation of one solar radius calculate the time it will take for gravitational wave radiation to drive the system to merger?

Make a plot that shows how the merger time varies as a function of the initial separation, r of the binary.

Task: Gravitational waves increase in amplitude as the binary system gets closer. In principle as $r \to 0$ the gravitational wave amplitude becomes infinite, but this can't happen in practice because either the systems merge at the size of a neutron star (10km) or the Schwarschild radius of a black hole

$$r_s = \frac{2GM}{c^2} \tag{7}$$

The Schwarschild radius of a black hole is the radius at which the escape velocity of the black hole is greater than the speed of light. Derive this expression by comparing the kinetic energy of an escaping object to the gravitational potential energy binding it to the massive source. How does the Schwarschild radius vary with the mass of the black hole?

Lets look at the frequency at which the gravitational waves are emitted. This frequency of the waves is actually twice the frequency of the binary star orbit, so as the orbit gets smaller, it takes less time for the stars to orbit and so the frequency increases. The frequency can be calculated from Kepler's laws.

Task: Kepler's laws come from equating the gravitational acceleration with a centripetal acceleration, so

$$\frac{mv^2}{r} = \frac{GMm}{r^2} \tag{8}$$

From this derive a relationship between the separation of a binary (r) and the period of the binary (T). You may find it useful to note that the time to complete a circular orbit is just the velocity divided by the distance around the edge of the circle. When you do this should find that

$$T^2 = \frac{4\pi^2}{GM} r^3, \tag{9}$$

and of course the frequency is just $1/T$.

Task: From Kepler's law that you have just calculated, determine the frequency at which detectors should be sensitive in order to measure the final merger of two black holes with a mass of 10 $M_\odot$, you can assume that the merger happens when the black holes reach a separation equal to their Schwarschild radius. NOTE: when dealing with two objects of similar mass you should replace the M in Kepler's law with $(M_1 + M_2)$.

Gravitational Wave detection

We can now see that gravitational wave amplitudes are very small, and that the frequencies for many objects of interest are in the range of a few hundred Hz. We can now think about how these might best be detected.

Essentially the task of a gravitational wave detector is to measure the changes in the distance between two points due to the motion of the gravitational wave. The problem is that these changes are extremely small. The gravitational wave amplitudes are of order 10^{-22} or smaller as seen above. This is extreme sensitivity, but is normally reached by the use of interferometry.

The gravitational waves induce a change in the relative distance between two points that depends on the amplitude of the wave. This change, ΔL is some very small fraction of the original length, L. It is common to describe this as a *gravitational wave strain*. This is essentially the amplitude of the wave, so you can use h as a measurement of of the strain[2], normally given as $\Delta L/L$.

The changes that are searched for are small, to put this in context, it is like measuring the distance to alpha centauri to the width of a human hair. Indeed, it is only a tiny fraction of the size of a proton, and so very difficult to measure. Interferometers are very sensitive however, even though random motions of their mirrors are much bigger than this. This is because these motions average out, and so using high powered lasers it is possible to measure the "average" length to extremely high precision.

Interferometers work by taking light and splitting it into two components that run down arms at right angles to each other. The light then comes back as is combined. However, if coherent light (i.e. a laser) is used then it is possible to interfere the light when it is combined. If destructive interference is perfect then the result is that there is no visible light. However, if the mirrors move because of the passage of a gravitational wave then this cancellation will cease to be perfect and some light will be allowed through. Gravitational wave detectors therefore work by destructively interfering light, and then waiting for flashes of light that can be observed as gravitational waves pass through.

There have been several generations of gravitational wave detector that have gradually built sensitivity over the past 30 years. The most recent, the Advanced-LIGO detectors have finally reached the sensitivity where detections are possible, and indeed the first detection was made at the end of 2015. This is shown in Figure 3, while the layout of the detectors is shown in Figure 4.

[2]Formally there is a factor of 2 so that $h = 2\Delta L/L$ because of the stretching and squeezing happening simultaneously (you don't need to worry about this factor of two in this sheet).

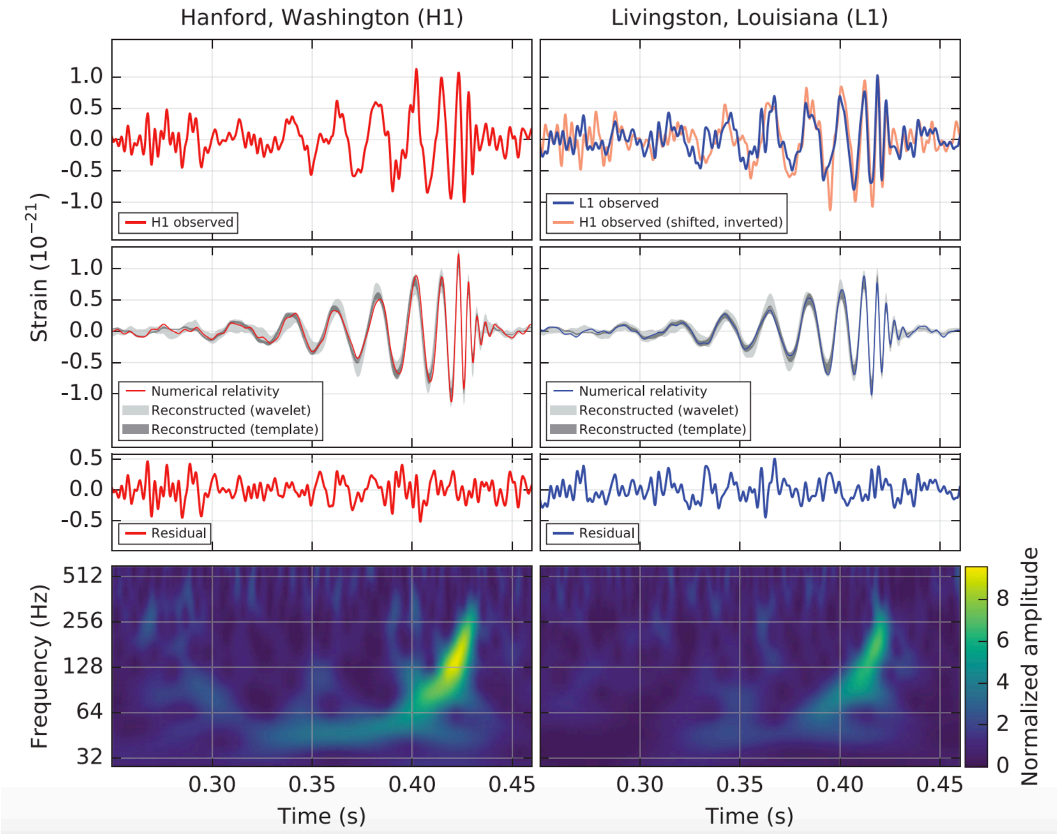

Figure 3: The detection of the first gravitational wave sources in two separate detectors in the continental US (Washington State and Louisiana). The top panel shows the data from the detectors, the centre panel a comparison of these results to those from numerical relativity, while the bottom panel shows the so called chirp signal. From Abbott et al. 2016 Physical Review Letters 116 1102.

Task: Now consider different sources of astrophysical gravitational waves. In particular, astronomers think we should detect three different types of merging star system; neutron star – neutron star, neutron star – black hole and black hole – black hole. Almost all neutron stars have a mass of 1.4 $M_{\odot}$, but black holes could have a much wider range of masses, perhaps up to 100 $M_{\odot}$.

Calculate the gravitational wave strength for a range of binaries including neutron star – neutron star, neutron star – black hole and black hole – black hole binaries. For simplicity you can assume that they merger at Schwarzschild radius for a black hole of the appropriate mass. You can assume any distance for the source that you like, but should note what it is. You should also choose a sensible range of compact object masses to investigate, and may find it useful to use loops in python to do this.

Task: Having calculated the measured GW strength in the previous task, you can now estimate how far away the current generation of gravitational wave detectors can observe the sources. If the strain limit for Advanced-LIGO is 10^{-22} then, for each type of binary considered above, calculate the distance that it could be observed to.

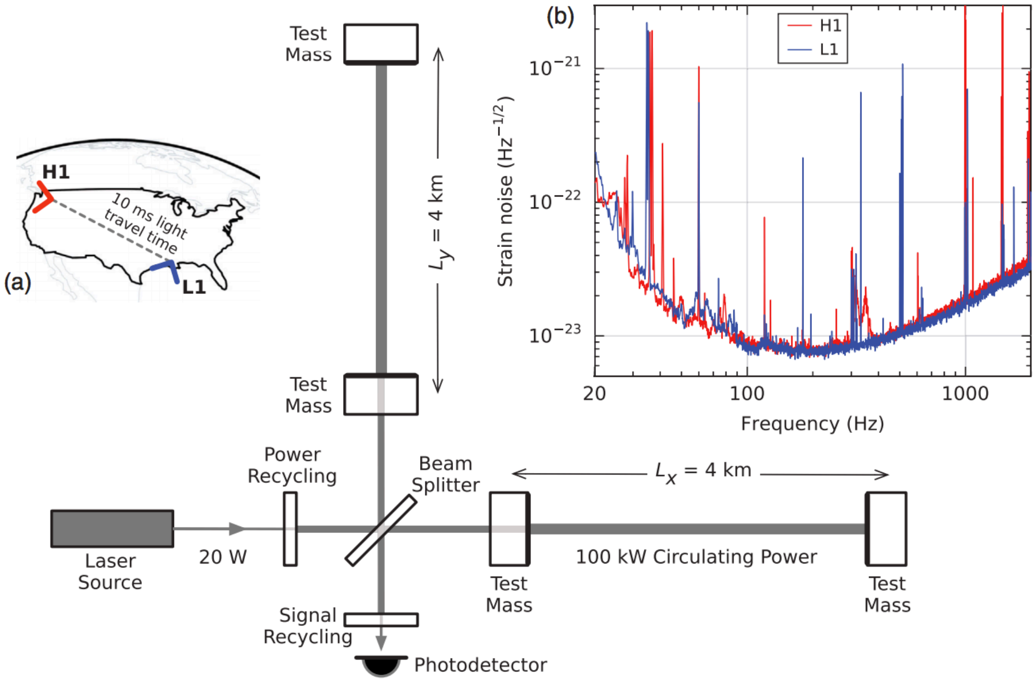

Figure 4: The set-up to detect the first gravitational waves. The locations of the two detectors are shown in panel (a), while panel (b) shows the sensitivity of the detector, measured as a strain sensitivity $\Delta L/L$. The main panel shows the set-up of an interferometer. From Abbott et al. 2016 Physical Review Letters 116 1102.

Task: We are now in a position to make the final calculation of the estimated *rate* of observations. That is how many of each type of event we expect to observe. To do this we can use estimates of the rate based on theoretical models of binary stars, and based on measured systems that have not yet merged (and will not for several billion years) in the Milky Way. These rates are 600, 2 and 5 Gpc^{-3} yr^{-1} for NS-NS, NS-BH and BH-BH. Calculate the observed rates that LIGO will see given the sensitivity of 10^{-23}.

Appendix 1: How to approach this worksheet

In this worksheet you are going to be asked to do some relatively complex calculations. When you do these it is easy to get things wrong because you put the incorrect information into your calculator at some point. Instead, it is much easier to put all of the information into a machine readable form, and then perform the calculations once you are sure that it is right. You can do this using either spreadsheets or programming languages. For example, lets look at calculating the gravitational force between the Sun and the Earth as a simple example. We have the law of gravitational attraction.

$$F = -\frac{GMm}{r^2} \tag{10}$$

so to calculate the force we have 4 things we need to know $(G, M, m \ \& \ r)$.

Imagine we wanted to calculate in a more automated way. We could do this using a coding language, like python, or with a spreadsheet. It is up to you which you would prefer.

Python

```
>>> G = 6.67e-11
>>> m1=2e30
>>> m2=6e24
>>> r=1.5e11
>>> F = G * m1* m2 / r**2
>>> print(F)
```

The outcome of which will be `3.55733333333e+22`. This is fine, but you have to change all the variables $(M, m \ \& \ r)$ each time. Alternatively, you can write a python script, that contains a function. You then just execute the script to calculate the data. For example, if you make a file that contains the following, and call it say `grav.py`

```
#we can write something called a function, which calculates
#the gravitational force for any set of masses or separations.
def grav_force(m1,m2,r):
    G = 6.67e-11
    F = G * m1* m2 / r**2
    return F

print grav_force(2e30,6e24,1.5e11)
```

When you execute it by typing `python grav.py`, you will get the same answer as above. In this case, notice the colon after the definition of the function, and the indents on the lines underneath it. You will need both of these for your code to work.

Sometimes you might also find it useful to do loops this can be done straightforwardly. For example,

```
msol=2e30
m2=10*msol
for m1 in range(1,100,100):
 m1=m1*msol
 F = G * m1* m2 / r**2
 print F
```

would allow you to calculate the gravitational attraction for a range of masses between 1 and 100. You can also nest loops inside loops if you want to, and it is possible to save the output to an array if you need (i.e. so you have a variable that has multiple values).

Spreadsheet

Alternatively, you could put all of this information into a spreadsheet, so you might fill cell A1,A2 and A3 with M, m & r, and then you could in cell A4 enter

```
= 6.67e-11 * A1*A2 / A3^2
```

(don't forget the equals sign). If you do this then the cell A4 will show the correct answer. At first sight this looks much simpler than writing a bit of code to do this. And in many of the examples in this worksheet it is. On the other hand, as things get more complicated, or there is more to do, the flexibility of the code can be valuable, and you may start to see this as you go along.

You can do effective loops in excel as well, by selecting multiple cells with relevant parameters on them, and then performing the same operation on all of them, populating a new set of cells.

Appendix 2: Astronomical data and constants

Throughout this worksheet there are various constants used, as well as normal astronomical notation. This may be unfamiliar, and so constants are given below, along with commonly used symbols.

Appendix 3: Astronomical distances and units

The normal SI units of distance (i.e. metres) are not very convenient when dealing with objects that are astronomical distances away. For example the astronomical unit, that defines the distances between the Sun and the Earth is 1.5×10^{11}m, and the closest stars are hundreds of thousands of astronomical units away. Instead, astronomers use distances such as light years, the distance that light travels in a year, or more commonly geometrically defined distances such as the parsec. One parsec is 3.26 light years or 3.08×10^{16}m.

Symbol	Description	value
$M_\odot$	solar mass	2×10^{30} kg
$R_\odot$	solar radius	7×10^{8} m
pc	parsec	3.08×10^{16} m
c	speed of light	3.0×10^{8} m s^{-1}
r_{Bohr}	Bohr radius	5.29×10^{-11} m

Table 1: Constants used in this worksheet. Note that parsecs can also be kpc or Mpc (kilo or Mega respectively).

Other SI units are also often not convenient, and so it is quite common to utilise different unit systems when dealing with astronomical figures. You will often see equations expressed in solar masses, and distances in solar radii. This is fine, and indeed you are free to scale units in various ways as long as your answers remain dimensionally correct (and of course you have the conversions correct in the equation). For most of what you are doing here you can keep to SI units if you want to, but if you look for information elsewhere (e.g. online or in textbooks) you may see quite different expressions.

Acknowledgements

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