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Kepler's laws
Math 131 Multivariate Calculus
D Joyce, Spring 2014

Kepler's laws of planetary motion. Kepler (1571–1630) developed three laws of planetary motion. Although he did his work before the invention of calculus, we can more easily develop his theory, as Newton did, with multivariate calculus. His laws state:

1. The orbit of a planet is an ellipse with the sun at one focus of the ellipse.
2. During equal time intervals, a planet sweeps out equal areas, that is, the line from the sun to the planet covers equal areas in equal times.
3. The square of the period of an orbit is proportional to the cube of the length of the semimajor axis of the ellipse.

We'll derive the first law from Newton's laws of motion. Derivations of the other two laws are described in the text.

The path of a planet is planar. We'll use one of Newton's principles to show that the path of a planet lies in a plane. First, some notation. Let's put the sun at the origin $\mathbf{0} \in \mathbf{R}^3$, and let the position of the planet at time t be

$$\mathbf{x} = (x, y, z).$$

Throughout this discussion, t is the independent variable and we'll simplify the notation by leaving off the (t), so, for instance, we'll write $\mathbf{x}$ instead of $\mathbf{x}(t)$.

Let r be the distance from the planet to the sun

$$r = \|\mathbf{x}\| = \sqrt{x^2 + y^2 + z^2}.$$

Now, the derivative of the position $\mathbf{x}$ with respect to t is the velocity,

$$\mathbf{v} = \mathbf{x}' = (x', y', z'),$$

and its derivative is the acceleration,

$$\mathbf{a}(t) = \mathbf{v}' = \mathbf{x}'' = (x'', y'', z'').$$

Now, Kepler determined that the acceleration of the planet was toward the sun. Newton explained that in terms of gravitational force by saying, first, that the gravitation of the sun is an attractive force F on the planet in the direction of the sun. More precisely,

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{u}$$

where G is a gravitational constant, M is the mass of the sun, m is the mass of the planet, $r = \|\mathbf{x}\|$ is the distance of the planet from the sun, and $\mathbf{u}$ is the unit vector in the direction of the planet, that is,

$$\mathbf{u} = \frac{\mathbf{x}}{r}.$$

Second, force equals mass times acceleration. (That's more or less Newton's definition of force.)

$$\mathbf{F} = m\mathbf{a}$$

Therefore, $m\mathbf{a} = -\dfrac{GMm}{r^2}\mathbf{u}$. and, so,

$$\mathbf{a} = -\frac{GM}{r^2}\mathbf{u} = -\frac{GM}{r^3}\mathbf{x}.$$

Thus, the acceleration $\mathbf{a}$ is the scalar $-\dfrac{GM}{r^3}$ times the position vector $\mathbf{x}$ of the planet.

That's enough to allow us to conclude, as we will soon, that the path of the planet lies in a plane, and furthermore, the cross product $\mathbf{x} \times \mathbf{v}$ of the position and velocity of the planet is a constant vector. (Our argument isn't quite the same as Newton's since he didn't use the concept of cross product.)

We'll show it's a constant vector by showing that its derivative is $\mathbf{0}$. Here's how.

$$\begin{aligned}(\mathbf{x} \times \mathbf{v})' &= \mathbf{x}' \times \mathbf{v} + \mathbf{x} \times \mathbf{v}' \\ &= \mathbf{v} \times \mathbf{v} + \mathbf{x} \times \mathbf{a}\end{aligned}$$

But the cross product of any vector with itself is $\mathbf{0}$, so $\mathbf{v} \times \mathbf{v} = \mathbf{0}$. Furthermore, the cross product of any vector with a scalar multiple of itself is also $\mathbf{0}$, and since $\mathbf{a}$ is a scalar multiple of $\mathbf{x}$, therefore $\mathbf{x}\times\mathbf{a} = \mathbf{0}$. Thus, $(\mathbf{x}\times\mathbf{v})' = \mathbf{0}$. But if the derivative of $\mathbf{x}\times\mathbf{v}$ is $\mathbf{0}$, then $\mathbf{x}\times\mathbf{v}$ itself is constant, that is,

$$\mathbf{x}\times\mathbf{v} = \mathbf{c}$$

where $\mathbf{c}$ is a constant vector.

Since the cross product of two vectors is orthogonal to each, therefore $\mathbf{c}$ is orthogonal to $\mathbf{x}$. Hence, the position $\mathbf{x}$ of a planet at any given time is in the plane orthogonal to the constant vector $\mathbf{c}$. In other words, the orbit of the planet—which is the path $\mathbf{x}$—lies in the plane orthogonal to the constant vector $\mathbf{c}$.

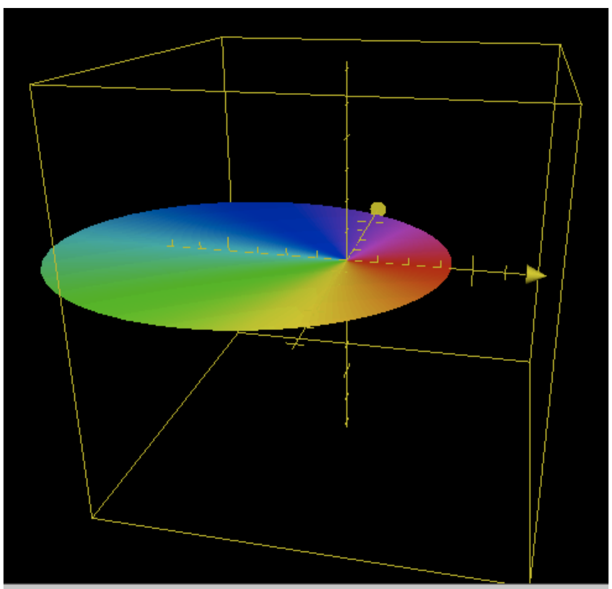

Figure 1: Orbit

Let's set the coordinate system of $\mathbf{R}^3$ so that the plane of the planet is the (x, y)-plane. Then $\mathbf{c}$ is on the z-axis, so $\mathbf{c} = (0, 0, c)$ where $c = \|\mathbf{c}\|$. In figure 1, the vector $\mathbf{c}$ is vertical, and the orbit lies in the xy-plane. We have yet to show that the orbit is an ellipse with the sun at one focus.

Kepler's first law. Next, we'll derive an equation for the orbit of a planet. There's an awful lot of work to this step. Eventually, we'll get the equation

$$r = \frac{c^2}{GM + d\cos\theta},$$

where r and θ are the polar coordinates of the moving planet, G is the gravitational constant, M is the mass of the sun, and c and d are parameters that describe the shape of the ellipse. This equation is the polar form of the equation of an ellipse with one focus at the origin.

First, the position $\mathbf{x}$ of the planet is the product $r\mathbf{u}$ of its distance r to the sun times the unit direction $\mathbf{u}$ of the planet. We'll differentiate the equation $\mathbf{x} = r\mathbf{u}$ with respect to time to get another expression for the velocity $\mathbf{v}$:

$$\mathbf{v} = (r\mathbf{u})' = r\mathbf{u}' + r'\mathbf{u}.$$

Now, put that in the equation $\mathbf{x}\times\mathbf{v} = \mathbf{c}$ to get

$$\begin{aligned}
\mathbf{c} &= \mathbf{x}\times\mathbf{v} \\
&= (r\mathbf{u})\times(r\mathbf{u}' + r'\mathbf{u}) \\
&= r^2(\mathbf{u}\times\mathbf{u}') + rr'(\mathbf{u}\times\mathbf{u}) \\
&= r^2(\mathbf{u}\times\mathbf{u}')
\end{aligned}$$

From the discussion above on force and acceleration, we have

$$\mathbf{a} = -\frac{GM}{r^2}\mathbf{u}.$$

Now, putting together the two equations for $\mathbf{a}$ and $\mathbf{c}$, we get

$$\begin{aligned}
\mathbf{a}\times\mathbf{c} &= \left(-\frac{GM}{r^2}\mathbf{u}\right)\times r^2(\mathbf{u}\times\mathbf{u}') \\
&= -GM\ \mathbf{u}\times(\mathbf{u}\times\mathbf{u}') \\
&= GM\ (\mathbf{u}\times\mathbf{u}')\times\mathbf{u}
\end{aligned}$$

At this point, we can use an identity on cross products to evaluate this expression. In general, if $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are three vectors, then

$$(\mathbf{x}\times\mathbf{y})\times\mathbf{z} = (\mathbf{x}\cdot\mathbf{z})\mathbf{y} - (\mathbf{y}\cdot\mathbf{z})\mathbf{x}$$

In our case, we get

$$\mathbf{a}\times\mathbf{c} = GM\ \left((\mathbf{u}\cdot\mathbf{u})\mathbf{u}' - (\mathbf{u}\cdot\mathbf{u}')\mathbf{u}\right)$$

But $\mathbf{u}$ is a unit vector, so $\mathbf{u} \cdot \mathbf{u} = 1$. Also, as we saw earlier, $\mathbf{u}$ is orthogonal to $\mathbf{u}'$, that is, $\mathbf{u} \cdot \mathbf{u}' = 0$. Then the last equation simplifies to

$$\mathbf{a} \times \mathbf{c} = GM\mathbf{u}'.$$

Also, $\mathbf{a} \times \mathbf{c}$ is the derivative of $\mathbf{v} \times \mathbf{c}$ since $\mathbf{c}$ is a constant vector. Now, since

$$(\mathbf{v} \times \mathbf{c})' = GM\mathbf{u}',$$

therefore $\mathbf{v} \times \mathbf{c}$ and $GM\mathbf{u}$ have the same derivative. Hence, they differ by a constant vector $\mathbf{d}$:

$$\mathbf{v} \times \mathbf{c} = GM\mathbf{u} + \mathbf{d}.$$

Furthermore, since both $\mathbf{v} \times \mathbf{c}$ and $\mathbf{u}$ lie in the (x, y)-plane, so does $\mathbf{d}$.

At this point, we may adjust the coordinate system so that $\mathbf{d}$ lies on the x-axis. Then $\mathbf{d} = (d, 0, 0)$, where $d = \|\mathbf{d}\|$.

We now have the constants c and d necessary to describe the equation of the elliptical orbit. But we still have to derive that equation, and when we do, it will be in polar coordinates r and θ.

Let θ be the angle of $\mathbf{x} = (x, y, 0)$. Then $x = r\cos\theta$ and $y = r\sin\theta$, as usual. Also, $\mathbf{u} = (\cos\theta, \sin\theta, 0)$.

Now,

$$\begin{aligned}
c^2 &= \|\mathbf{c}\|^2 = \mathbf{c} \cdot \mathbf{c} \\
&= (\mathbf{x} \times \mathbf{v}) \cdot \mathbf{c} \qquad \text{(triple scalar product)} \\
&= \mathbf{x} \cdot (\mathbf{v} \times \mathbf{c}) \\
&= r\mathbf{u} \cdot (GM\mathbf{u} + \mathbf{d}) \\
&= GMr + r\,\mathbf{u} \cdot \mathbf{d} \\
&= GMr + rd\cos\theta
\end{aligned}$$

which gives us the polar form for the equation of an ellipse

$$r = \frac{c^2}{GM + d\cos\theta}.$$

Math 131 Home Page at
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