Thermal Physics II – Solutions for Problem Sheet 5

1. Partition Function for Combined Systems/Degrees of Freedom

- We start from the definition of the partition function and the fact that the two subsystems are independent

$$Z = \sum_j exp(-\beta E_j) \qquad \text{and} \qquad E_j = E_a + E_b$$

Thus, every energy is the sum of the possible energies in the subsystems.

- Now we have for the partition function of the total system

$$\begin{aligned} Z_{a+b} &= \sum_{a,s} exp[-\beta(E_a + E_b)] \\ &= \sum_{a,s} exp(-\beta E_a) exp(-\beta E_b) \\ &= \sum_a exp(-\beta E_a) \sum_b exp(-\beta E_b) \\ &= Z_a + A_b \,, \end{aligned}$$

where we have used the definition for the partition functions in the two subsystems.

- For the Helmholtz free energy, we obtain

$$\begin{aligned} F_{a+b} &= -k_B T \ln Z_{a+b} \\ &= -k_B T \ln(Z_a Z_b) \\ &= -k_B T \Big[\ln Z_a + \ln Z_b \Big] = F_a + F_b \end{aligned}$$

- Thus, we find the the total partition function is the product of the partial functions from the subsystems whilst the Helmholtz free energy is the sum of the contributions from both systems (as it should be as F is an extensive quantity).
WARNING: this is ONLY the case IF we have independent subsystems!

2. Magnet with Three Energies

- The probabilities are given by

$$\begin{aligned} p_1 &= \frac{1}{Z}\exp(-\beta E_1) = \frac{1}{Z}\exp(-\beta\mu B), \\ p_2 &= \frac{1}{Z}\exp(-\beta E_2) = \frac{1}{Z}, \\ p_3 &= \frac{1}{Z}\exp(-\beta E_3) = \frac{1}{Z}\exp(\beta\mu B)\,. \end{aligned}$$

- The partition functionis then given by

$$Z = \sum_j \exp(-\beta E_j) = \exp(-\beta\mu B) + 1 + \exp(\beta\mu B)\,.$$

- If the second state is four times degenerate, $g_2 = 4$, we have

$$p_2 = \frac{4}{Z} \qquad \text{and} \qquad Z = \exp(-\beta\mu B) + 4 + \exp(\beta\mu B)\,.$$

- The average magnetisation is

$$\langle\mu\rangle = \sum_j \mu_j P_j = \frac{\mu}{Z}\Big[\exp(-\beta\mu B) - \exp(\beta\mu B)\Big]$$

- If the magnetic field is very small, we have

$$\begin{aligned} p_1 = p_2 = p_3 \qquad &\text{for} \qquad g_2 = 1 \\ p_1 = \frac{1}{4}p_2 = p_3 \qquad &\text{for} \qquad g_2 = 4 \end{aligned}$$

  In both cases, the entropy is at its maximum: $S \to$ max for $B \to 0$.

- If the magnetic field is very large, we have

$$p_1 \ll p_2 \ll p_3 \qquad \text{regardless of the value of } \; g_2$$

  This means, we have almost perfect magnetisation (all atoms are pointing in the same direction) and the entropy is minimum: $S \to 0$ for $B \to \infty$.