Adapting Meshfree Galerkin Schemes for Representing Highly Anisotropic Fields

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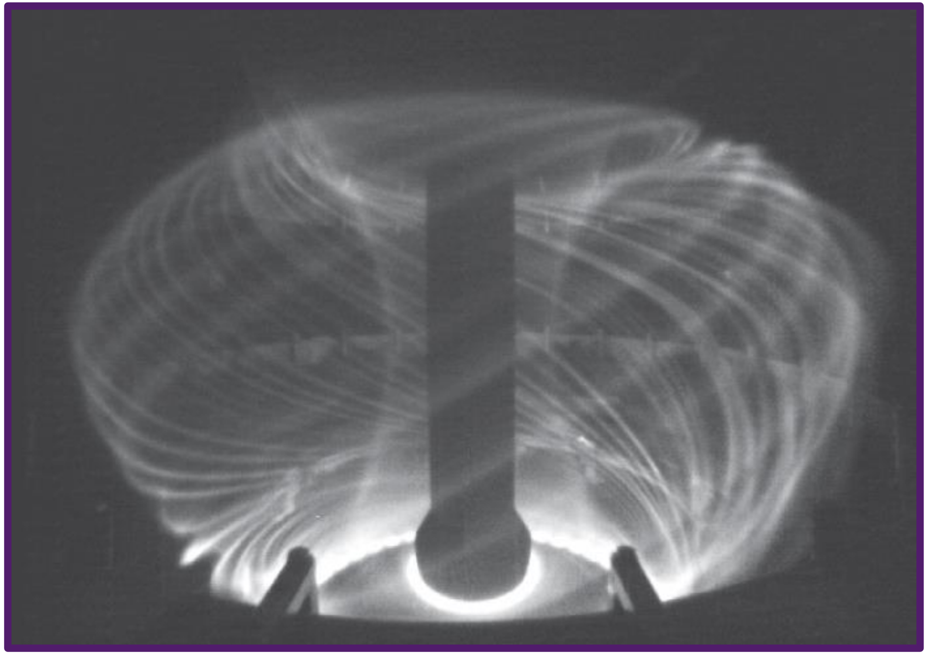


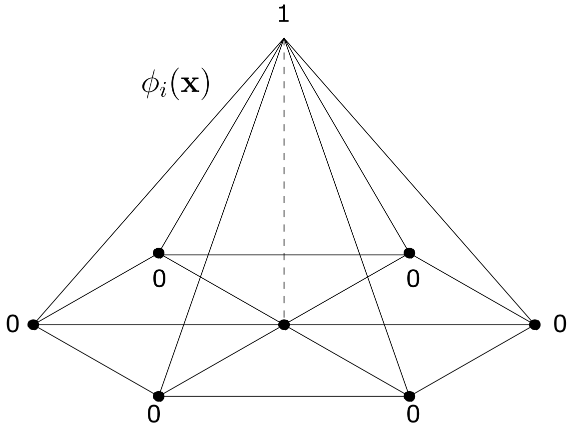
Motivating Problem

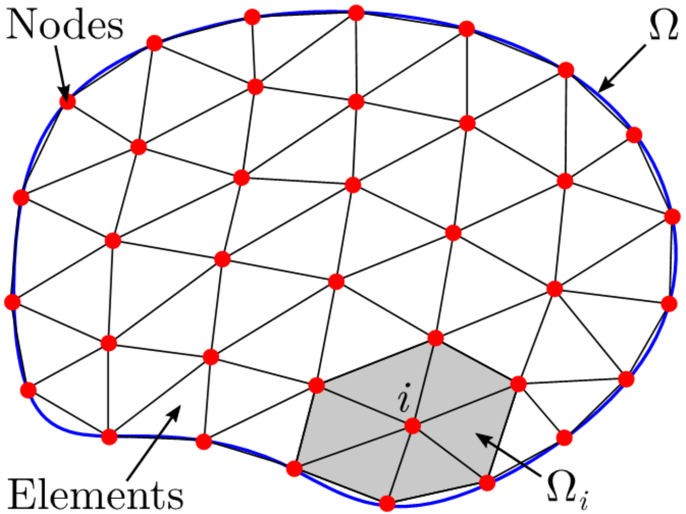
- Efficiently representing **highly anisotropic** fields
- Structures elongated along one direction
  → **smaller gradients/wavenumbers**
  - Require **fewer degrees of freedom** to represent variation
  - Generally **not aligned** with any physical coordinate direction
- How to **align numerical representation** with physical anisotropy?

Image: Ham et al. 2020 doi.org/10.1038/s42254-019-0144-1

FEM

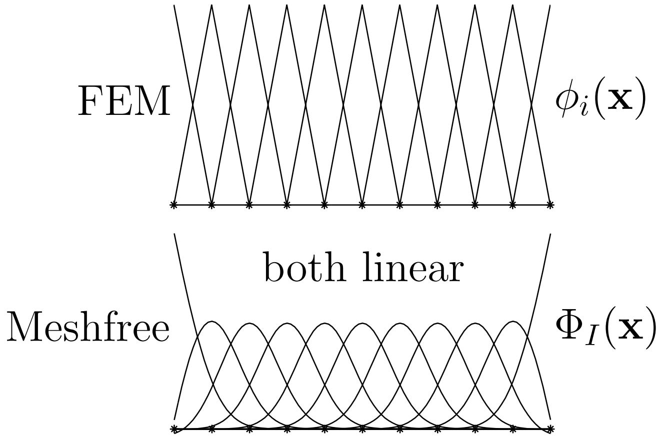
$$u(\mathbf{x},t) \approx u_h(\mathbf{x},t) = \sum_{i=1}^{N_n} u_i(t)\phi_i(\mathbf{x})$$





Meshfree (EFG)

$$u(\mathbf{x},t) \approx u_h(\mathbf{x},t) = \sum_{I=1}^{N_n} u_I(t)\Phi_I(\mathbf{x})$$



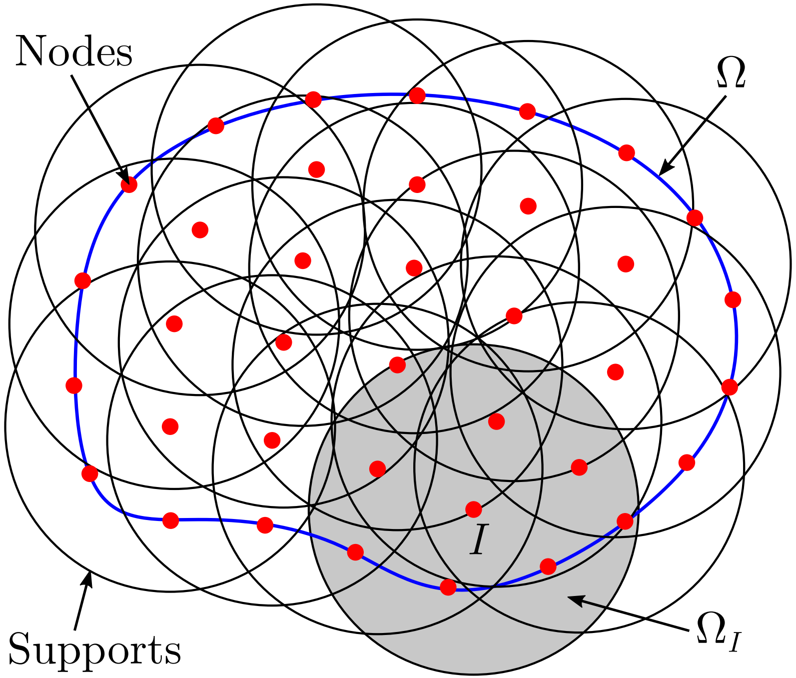


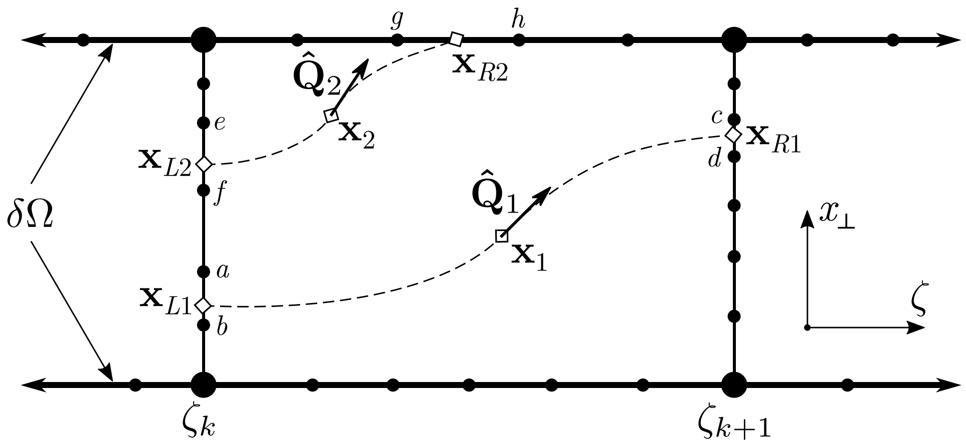
Image (left): A. Huerta et al. 2004 doi.org/10.1002/0470091355.ecm005

Meshfree Challenges

Meshfree schemes are very flexible, but also have drawbacks

- Computing shape functions and derivatives is much **more expensive**
  - Requires inversion of a small **moment matrix** at every evaluation point
  - Must **search for nodes** with non-zero support
- The stability and accuracy of the solution are **strongly-dependent on parameters** chosen
  - **support size and shape**, node placements, weight function, etc.
  - Necessitates knowledge and **intervention by user**
- MLS shape functions don't have the **delta property**
  - Complicates imposition of **Dirichlet BCs**
- Only **inexact quadrature** possible

Partially Meshfree Scheme (FCIFEM)



$$\phi_i(\mathbf{x}) = \rho_i(\mathbf{x})\phi_{i,\mathrm{FEM}}(\mathbf{x}_{\mathrm{map}})$$

$$\rho_i(\mathbf{x}) = \frac{\zeta - \zeta_t}{\zeta_o - \zeta_t}$$

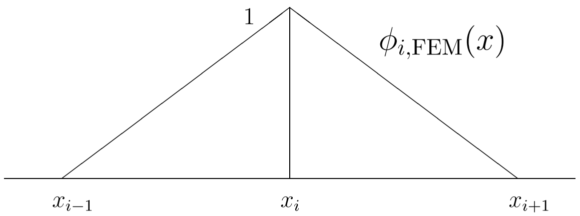
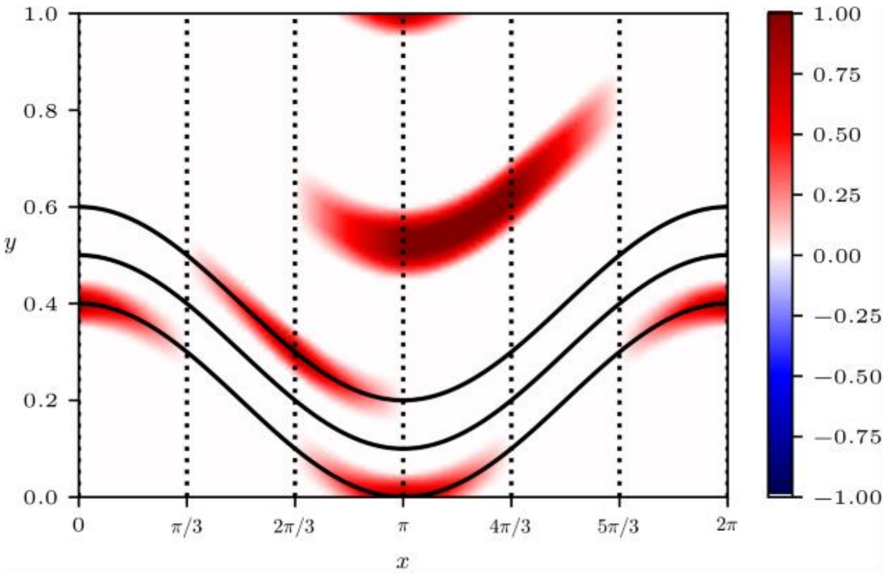


Image (right): E. Süli 2019 people.maths.ox.ac.uk/suli/fem.pdf

Example FCIFEM Basis Functions



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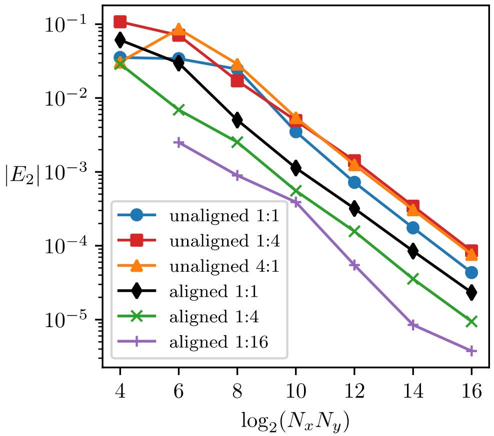
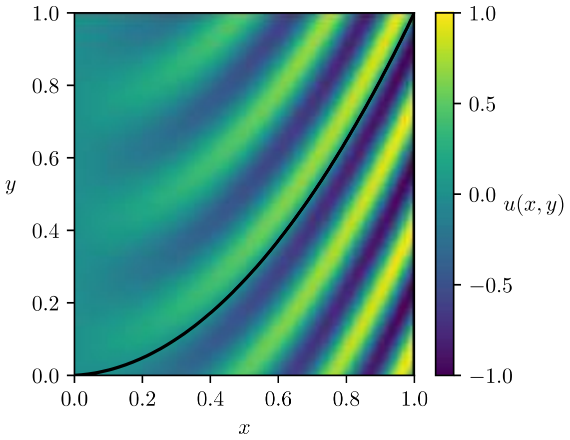
Poisson's Equation with Dirichlet Boundaries

$$\nabla^2 u = f, \qquad f = \nabla^2 x \sin(2\pi n[y - ax^2 - bx])$$





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Local Flux Conservation

Hughes et al. showed the standard continuous Galerkin method is locally conservative w.r.t. point-wise fluxes at the nodes; however, this relies on the exactness of quadrature in standard FEM

- Can prove similar conservation at quadrature points for any Galerkin scheme
  - Requires test functions form a partition of unity
  - Operator must be evaluated using integration by parts

$$\sum_{i=1}^{N_n} \phi_i = 1 \quad \Longrightarrow \quad \sum_{i=1}^{N_n} \nabla\phi_i = 0$$

T. J. R. Hughes et al. 2000 doi.org/10.1006/jcph.2000.6577

Inexact Quadrature

- Chen et al. derived **variationally consistent integration** constraints the quadrature scheme must fulfill to achieve **optimal convergence**
- Also suggested an **assumed strain method** to decouple corrections by **modifying the test functions**
    - This makes them **no longer a partition of unity** and negates conservation
- We propose adding **corrections to quadrature weights** instead of test functions
    - Means **the corrections are coupled** → need to solve linear system(s)
    - Seems to work very well!

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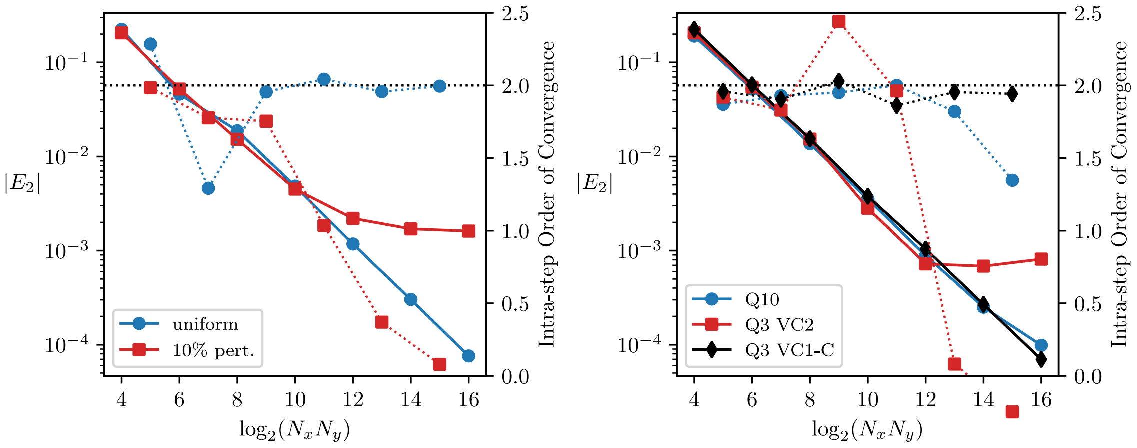
Variationally Consistent Integration

$$\nabla^2 u = \sin(2\pi x)\sin(2\pi y)$$

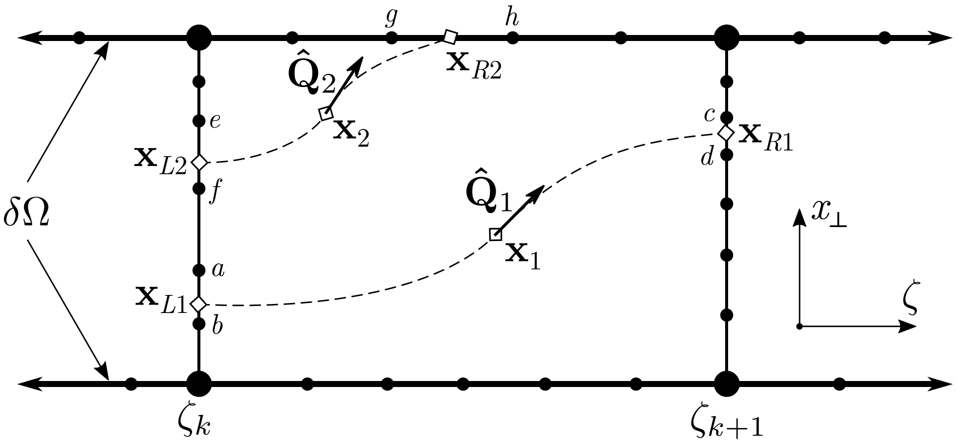


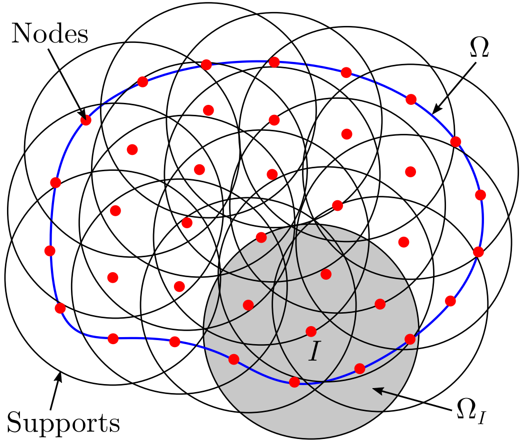
J. S. Chen et al. 2013 doi.org/10.1002/nme.4512

Higher Order Bases (ongoing/future work...)

Not obvious how to directly extend to higher orders, and boundaries still a bit of a pain to implement.

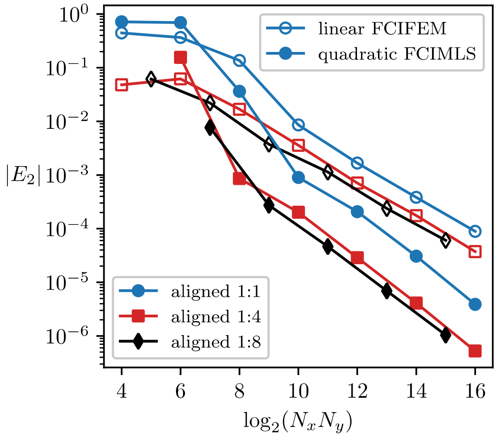
Trying a "fully" meshfree approach instead, still using similar arrangement of nodes.

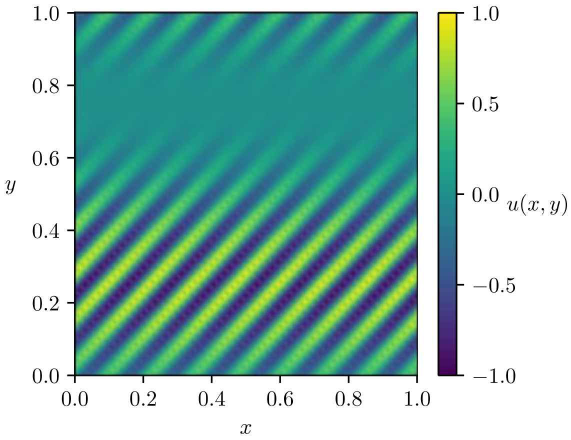




Higher Order Bases (preliminary results)

$$\nabla^2 u = \frac{1}{2}\sin(2\pi n[y - x])[1 + \sin(2\pi y)]$$





Paper and Acknowledgement

A partially mesh-free scheme for representing anisotropic spatial variations along field lines: Conservation, quadrature, and the delta property

S. A. Maloney, B. F. McMillan

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Questions?

Thanks for listening!

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