Numerical study of the small scale dynamics of two-dimensional magnetohydrodynamic turbulence

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Abstract: Recent improvements in the scale and accuracy of direct numerical simulations of isotropic magnetohydrodynamic (MHD) turbulence enable many of its fundamental properties to be investigated anew. Here we report progress on questions regarding the small scale dynamics of compressible two-dimensional magnetohydrodynamic turbulence.

1. MHD turbulence phenomenology

Scaling exponents (ζ_p, τ_p) give a statistical description of MHD turbulence

$$S_l^{(\pm)p} = \left\langle \partial z_l^{(\pm)p} \right\rangle = \left\langle \left| z^{(\pm)}(x+l)\cdot\frac{l}{|l|} - z^{(\pm)}(x)\cdot\frac{l}{|l|} \right|^p \right\rangle \sim l^{\zeta_p^{\pm}}$$

$$\left\langle \varepsilon_l^p \right\rangle = \left\langle \left(\frac{\nu}{l^D}\int_{l'} \varepsilon(x+l')dl'^3 \right)^p \right\rangle \sim l^{\tau_p}$$

Where $z^{\pm} = v \pm B/\sqrt{\rho\mu_0}$ are the Elsasser field variables, l is a differencing length, ν is a viscosity, ε is the local rate of dissipation and D is the number of spatial dimensions

Models:

Random eddy scrambling (Kolmogorov)	Alfvenic collisions (Iroshnokov / Kraichnan)
Favoured model of 3D MHD turbulence	Favoured model for 2D MHD turbulence
$S_l^{(\pm)p} \sim \left\langle \varepsilon_l^{p/3} \right\rangle l^{p/3}$	$S_l^{(\pm)p} \sim \left\langle \varepsilon_l^{p/4} \right\rangle l^{p/4}$

2. Dissipation structure functions

•1D measure based on Elsasser field gradient

$$\left\langle \varepsilon_l^{(\pm)p} \right\rangle = \left\langle \left(\frac{1}{l}\int_0^l \left(\partial_i z_i^{(\pm)}(x+l_1,t)\right)^2 dl_1 \right)^p \right\rangle$$

•2D measure, Ohmic plus viscous dissipation

$$\left\langle \varepsilon_l^p \right\rangle = \left\langle \left(\frac{1}{l^2}\int_0^l\int_0^l \left\{ \frac{\nu}{2}\left(\partial_i v_j + \partial_j v_i\right)^2 - \frac{\nu 2}{3}|\nabla\bullet v|^2 + \frac{\eta}{\rho}(\nabla\times B)^2 \right\} dl_1 dl_2 \right)^p \right\rangle$$

Where B, v and ρ are evaluated at $(x+l_1, y+l_2, t)$

3. Extended Self-Similarity (ESS)

•Direct Numerical Simulation (DNS) must fully resolve the dissipation range to prevent the 'pile up' of energy at small scales

•Most resolution is used for this purpose thus inertial ranges are small

•Use Extended Self-Similarity (ESS) which extends the scaling laws into the dissipation range such that:

$$S_l^{(\pm)p} \sim l^{\zeta_p} \qquad \left\langle \varepsilon_l^p \right\rangle \sim l^{\tau_p}$$

In the inertial range

$$S_l^{(\pm)p} \sim S_l^{(\pm)q\left(\zeta_p/\zeta_q\right)} \qquad \left\langle \varepsilon_l^p \right\rangle \sim \left\langle \varepsilon_l^q \right\rangle^{\left(\tau_p/\tau_q\right)}$$

In the inertial and dissipation range

•Models used to explain these scaling laws often use the refined similarity hypothesis. Two versions of this are shown below in ESS compatible form.

Random eddy scrambling	Alfvenic collisions
Kolmogorov 1941 (K41)	Iroshnokov / Kraichnan (IK)
$S_l^{(\pm)3p} \sim \left\langle \varepsilon_l^p \right\rangle \left(S_l^3\right)^p$	$S_l^{(\pm)4p} \sim \left\langle \varepsilon_l^p \right\rangle \left(S_l^4\right)^p$

•Directly testing these relations is an important consistency check for currently favored models of turbulence.

4. Numerical Scheme

•We have developed high order finite difference simulations of the two dimensional isothermal equations of MHD.

•Spatial derivatives are calculated to sixth order

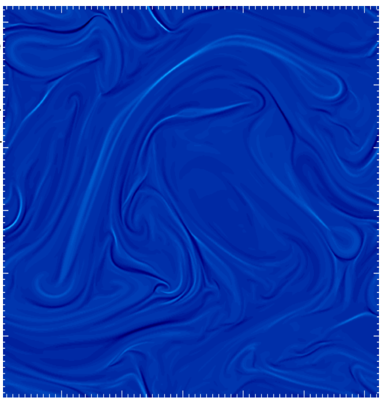
•Time iteration is by third or fourth order Runge-Kutta scheme.

•The fourth order scheme remains stable for steeper gradients than for third order.

•Higher Reynolds numbers can be achieved at fourth order

•However, the fourth order scheme has less numerical diffusion so is more susceptible to the "chequer board" instability in density

•The figure (right) shows an example of the current density obtained from 1024^2 driven turbulence run.



5.Decaying turbulence

•Turbulence decaying so structure functions from different times normalised to total energy

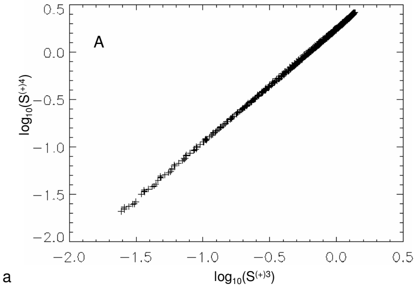
•Effect of normalisation and ESS procedure shown on plot A

•Initial conditions as specified in [1]

•Resolution is 1024^2

•Values of recovered exponents are consistent with those found for incompressible flows in [1] (plot B)

•Scaling exponents seem to be robust for a range of sub-sonic Mach numbers



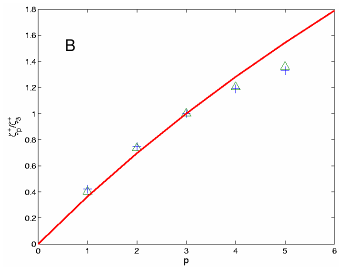


Figure A. shows Extended self-similarity (ESS) in Elsasser field structure functions (see section 2) for order 4 against order 3. Points are calculated from eight snapshots spanning a total time of 4 initial eddy turnover times. Normalisation by total energy shows the points lie on the same slope despite the decay process.

Figure B. shows the ratio of scaling exponents recovered by ESS (triangles). Those recovered for incompressible MHD decaying turbulence by Biskamp and Schwartz [1] are shown as crosses. The bold line shows exponents predicted by an IK based She-Leveque model. Neither set of exponents can be said to fit this model. Further investigation is needed.

[1] Biskamp and Schwartz, Phys. of Plasmas, 8(7)2001

6.Driven turbulence

•Turbulence driven by maintaining the energy of all Fourier modes with $0<k<2.5$

•Velocity magnetic field correlation is constrained at ≈ 15% throughout the simulation

•Statistics are calculated from ≈ 1200 snapshots to obtain statistical accuracy for high order moments. Each snap shot is separated by 3-4 large eddy turnover times.

•Resolution is 512^2

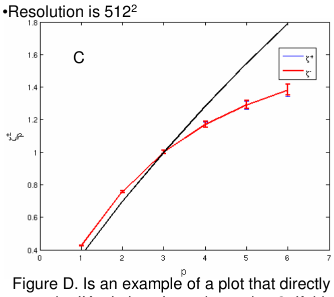


Figure C. shows the ratio of scaling exponents recovered by ESS for both Elsasser field variables (see legend). The black line shows exponents predicted by an IK based She-Leveque model. Errors are calculated as the maximum gradient variation allowed by the error of each point on a loglog plot. This error estimation combined with long runtimes lead to scaling exponents that agree to within errors for both Elsasser fields.

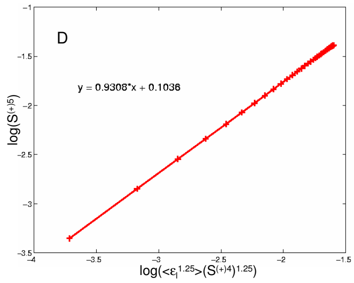


Figure D. Is an example of a plot that directly tests the IK relation shown in section 3. If this relation were true, the gradient of this graph would be 1.0. $\langle\varepsilon^p_l\rangle$ is calculated using the 1D measure shown in section 3. The observed deviation of the gradient from the ideal value of 1 becomes more pronounced as the order of the measure p increases. This could indicate that a correction is needs to be added to this relation or that the 1D simplification used to calculate $\langle\varepsilon^p_l\rangle$ is not adequate for compressible MHD.

7. Conclusions and further work

•Extended Self-Similarity has been found in the Elsasser field variables for both driven and decaying isothermal compressible MHD turbulence.

•Driven simulations have been performed with long runtimes (1200 snapshots each separated by a few large eddy turn over times).

•In the driven case it is found that exponents calculated from the $z^{(+)}$ and $z^{(-)}$ fields agree to within errors providing statistics are harvested from a large enough quantity of snap shots.

•The good quality statistics obtainable from driven runs with long runtimes enable the refined similarity hypothesis to be investigated (see plot D of section 6)

-Although good power-law scaling is observed, there are deviations from the ideal value of one.

-This may suggest a correction needs to be made to the IK relation in section 3.

-This may also suggest that the 1D measure used to calculate $\langle\varepsilon^p_l\rangle$ does not capture the scaling properties of dissipation for compressible MHD turbulence. This 1D measure has previously been applied to incompressible simulations and the solar wind [2],[3].

-Further investigation is needed in this area including an evaluation of the effect of varying the sound speed (and hence compressibility) and using the full 2D measure for $\langle\varepsilon^p_l\rangle$ shown in section 2.

[2] Bershadskii, Phys. Plasmas 10(12) 2003. [3] Merrifield et. al, Phys.`Plasmas 12 022301 2005

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