Physics of fusion power

Lecture 2: Lawson criterion / some plasma physics

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Quasi-neutrality

- Using the Poisson equation

$$\nabla \cdot \mathbf{E} = -\nabla^2 \phi = \frac{\rho}{\epsilon_0}$$

- And a Boltzmann relation for the densities

$$n_e = n_0 \exp\left[\frac{e\phi}{T}\right] \qquad n_i = n_0 \exp\left[-\frac{e\phi}{T}\right]$$

- One arrives at an equation for the potential

$$-\nabla^2 \phi = \frac{1}{\epsilon_0}\left[Q\delta(\mathbf{r}) - \frac{2e^2 n_0}{T}\phi\right]$$

Positive added charge

Response of the plasma

Solution

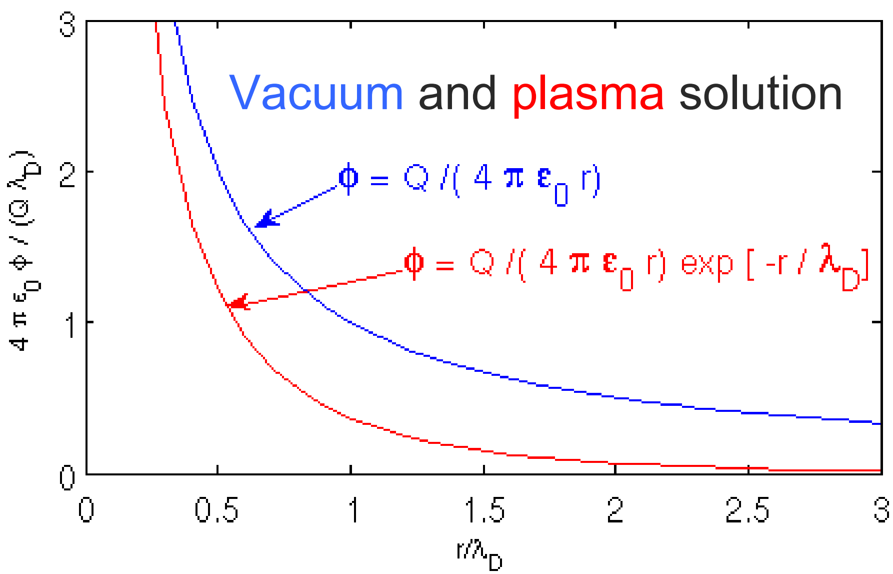
- The solution of the Poisson equation is

$$\phi = \frac{Q}{4\pi r \epsilon_0} \exp\left[-\frac{r}{\lambda_D} \right] \qquad \lambda_D = \sqrt{\frac{\epsilon_0 T}{2 e^2 n_0}}$$

Potential in vacuum

Shielding due to the charge screening

The length scale for shielding is the Debye length which depends on both Temperature as well as density. It is around 10^{-5} m for a fusion plasma



Quasi-neutrality

- For length scales larger than the Debye length the charge separation is close to zero. One can use the approximation of quasi-neutrality

$$\sum_i Z_i n_i = 0 \qquad n_i = n_e$$

- Note that this does not mean that there is no electric field in the plasma
- Under the quasi-neutrality approximation the Poisson equation can no longer be used to calculate the electric field

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \quad \rightarrow \quad \rho = 0$$

Divergence free current

- Using the continuity of charge

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

- Where J is the current density

$$\mathbf{J} = e n_i \mathbf{u}_i - e n_e \mathbf{u}_e$$

- One directly obtains that the current density must be divergence free

$$\nabla \cdot \mathbf{J} = 0.$$

Also the displacement current must be neglected

- From the Maxwell equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

- Taking the divergence and using that the current is divergence free one obtains

$$\frac{\partial \nabla \cdot \mathbf{E}}{\partial t} = 0.$$

- The displacement current must therefore be neglected, and the relevant equation is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Quasi-neutrality

- The charge density is assumed zero (but a finite electric field does exist)
- One can not use the Poisson equation to calculate this electric field (since it would give a zero field)
- Length scales of the phenomena are larger than the Debye length
- The current is divergence free
- The displacement current is negligible

Lawson criterion

- Derives the condition under which efficient production of fusion energy is possible
- Essentially it compares the generated fusion power with any additional power required
- The reaction rate of one particle B due to many particles A was derived

$$\frac{\mathrm{d}N}{\mathrm{d}t} = n_A \langle \sigma v \rangle$$

- In the case of more than one particle B one obtains

$$\frac{\mathrm{d}N}{\mathrm{d}t} = N_B n_A \langle \sigma v \rangle = n_A n_B \langle \sigma v \rangle V$$

Fusion power

$$\frac{\mathrm{d}N}{\mathrm{d}t} = N_B n_A \langle \sigma v \rangle = n_A n_B \langle \sigma v \rangle V$$

- The total fusion power then is

$$P_{\mathrm{Fusion}} = n_A n_B \langle \sigma v \rangle E_{\mathrm{fusion}} V$$

- Using quasi-neutrality

$$n_D + n_T = n_e = n$$

- For a 50-50% mixture of Deuterium and Tritium

$$P_{\mathrm{fusion}} = \frac{1}{4} n^2 \langle \sigma v \rangle_{DT} E_{\mathrm{fus}}$$

Fusion power

- To proceed one needs to specify the average of the cross section. In the relevant temperature range 6-20 keV

$$\langle \sigma v \rangle_{DT} \approx 1.1 \cdot 10^{-24} T_k^2 \ \ \mathrm{m}^3/\mathrm{s}$$

- The fusion power can then be expressed as

$$P_{\mathrm{Fusion}} = 7.7 n_{20}^2 T_k^2 V \ \ \mathrm{kW}$$

The power loss

- The fusion power must be compared with the power loss from the plasma
- For this we introduce the energy confinement time τ_E

$$\frac{\partial W}{\partial t} = -\frac{W}{\tau_E} + P_{\text{heat}}$$

- Where W is the stored energy

$$W = \left[\frac{3}{2} n_D T_D + \frac{3}{2} n_T T_T + \frac{3}{2} n_e T_e\right] V \approx 3nTV$$

Ratio of fusion power to heating power

- If the plasma is stationary

$$W = 3nTV = P_{\text{heat}}\tau_E \qquad nT = \frac{1}{3V}P_{\text{heat}}\tau_E$$

- Combine this with the fusion power

$$P_{\text{Fusion}} = 7.7 n_{20}^2 T_k^2 V \ \text{kW}$$

- One can derive the so called n-T-tau product

$$\frac{P_{\text{Fusion}}}{P_{\text{heat}}} = 0.16 n_{20} T_k \tau_E$$

Break-even

- The break-even condition is defined as the state in which the total fusion power is equal to the heating power

$$\frac{P_{\text{Fusion}}}{P_{\text{heat}}} = 0.16 n_{20} T_k \tau_E$$

$$n_{20} T_k \tau_E > 6 \ \ \text{Break} - \text{even}$$

- Note that this does not imply that all the heating power is generated by the fusion reactions

Ignition condition

- Ignition is defined as the state in which the energy produced by the fusion reactions is sufficient to heat the plasma.
- Only the He atoms are confined (neutrons escape the magnetic field) and therefore only 20% of the total fusion power is available for plasma heating

$$\frac{P_{\text{Fusion}}}{P_{\text{heat}}} = 0.16 n_{20} T_k \tau_E$$

$$n_{20} T_k \tau_E > 30 \ \text{ Ignition}$$

n-T-tau

- Difference between inertial confinement and magnetic confinement: Inertial short tE but large density. Magnetic confinement the other way around

$$\frac{P_{\text{Fusion}}}{P_{\text{heat}}} = 0.16 n_{20} T_k \tau_E$$

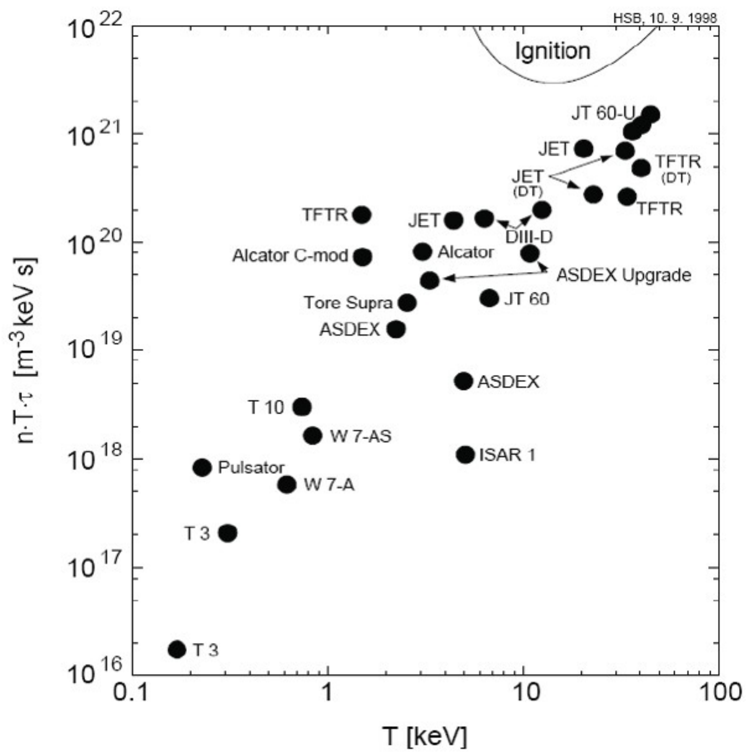
- Magnetic confinement: Confinement time is around 3 seconds

$$n_{20} T_k \tau_E > 30 \ \ \text{Ignition}$$

- Note that the electrons move over a distance of 200.000 km in this time

n-T-tau is a measure of progress

- Over the years the n-T-tau product shows an exponential increase
- Current experiments are close to break-even
- The next step ITER is expected to operate well above break-even but still somewhat below ignition



Force on the plasma

- The force on an individual particle due to the electro-magnetic field (s is species index)

$$\mathbf{F}_i = Z_s e[\mathbf{E} + \mathbf{v}_i \times \mathbf{B}]$$

- Assume a small volume such that

$$N_s = n_s V$$

- Then the force per unit of volume is

$$\mathbf{F}_s = \frac{1}{V}\sum_{i=1}^{N_s}\mathbf{F}_i = \frac{1}{V}\sum_{i=1}^{N_s} Z_s e[\mathbf{E} + \mathbf{v}_i \times \mathbf{B}]$$

Force on the plasma

- For the electric field

$$\frac{1}{V}\sum_{i=1}^{N_s} Z_s e\mathbf{E} = \frac{N_s}{V} Z_s e\mathbf{E} = Z_s e n_s \mathbf{E}$$

- Define an average velocity

$$\mathbf{u}_s = \frac{1}{N_s}\sum_{i=1}^{N_s} \mathbf{v}_i$$

- Then for the magnetic field

$$\frac{1}{V}\sum_{i=1}^{N_s} Z_s e\mathbf{v}_i \times \mathbf{B} = Z_s e\frac{N_s}{V}\left(\frac{1}{N_s}\sum_{i=1}^{N_s} \mathbf{v}_i\right) \times \mathbf{B} = Z_s e n_s \mathbf{u}_s \times \mathbf{B}$$

Force on the plasma

- Averaged over all particles

$$\mathbf{F}_s = Z_s e n_s [\mathbf{E} + \mathbf{u}_s \times \mathbf{B}]$$

- Now sum over all species

$$\sum_{s=1}^{p} Z_s e n_s \mathbf{E} = e\mathbf{E} \sum_{s=1}^{p} Z_s n_s = 0 \qquad \sum_{s=1}^{p} Z_s e n_s \mathbf{u}_s = \mathbf{J}$$

- The total force density therefore is

$$\mathbf{F} = \sum_{s=1}^{p} \mathbf{F}_s = \mathbf{J} \times \mathbf{B}$$

Force on the plasma

- This force contains only the electro-magnetic part. For a fluid with a finite temperature one has to add the pressure force

$$\mathbf{F} = -\nabla p + \mathbf{J} \times \mathbf{B}$$

$$p = \sum_{s=1}^{p} n_s T_s$$

Reformulating the Lorentz force

- Using $$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$
- The force can be written as

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B}$$

- Then using the vector identity

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\mathbf{a} = \mathbf{B} \qquad \mathbf{b} = \mathbf{B}$$

$$\nabla(B^2) = 2(\mathbf{B}\cdot\nabla)\mathbf{B} + 2\mathbf{B}\times(\nabla\times\mathbf{B})$$

Force on the plasma

- One obtains

$$-\nabla p + \mathbf{J} \times \mathbf{B} = -\nabla \left(p + \underbrace{\frac{B^2}{2\mu_0}}_{\text{Magnetic field pressure}} \right) + \underbrace{\frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B}}_{\text{Magnetic field tension}}$$

- Important parameter (also efficiency parameter) the plasma-beta

$$\beta = \frac{p}{B^2/2\mu_0}$$