Physics of Fusion power

Lecture3 : Force on the plasma / Virial theorem

Break-even

- The break-even condition is defined as the state in which the total fusion power is equal to the heating power

$$\frac{P_{\mathrm{Fusion}}}{P_{\mathrm{heat}}} = 0.16 n_{20} T_k \tau_E$$

$$n_{20} T_k \tau_E > 6 \ \ \mathrm{Break-even}$$

- Note that this does not imply that all the heating power is generated by the fusion reactions

Ignition condition

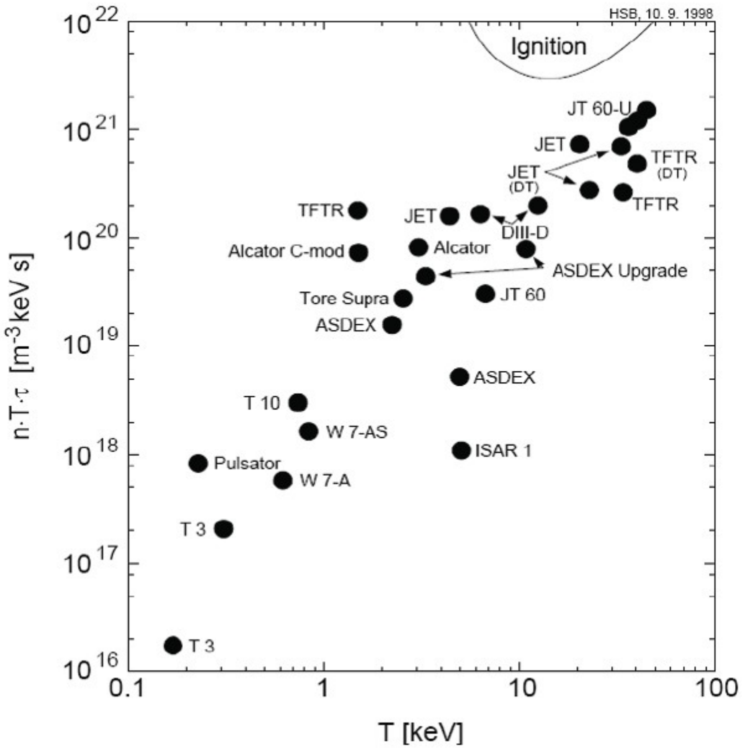
- Ignition is defined as the state in which the energy produced by the fusion reactions is sufficient to heat the plasma.
- Only the He atoms are confined (neutrons escape the magnetic field) and therefore only 20% of the total fusion power is available for plasma heating

$$\frac{P_{\text{Fusion}}}{P_{\text{heat}}} = 0.16 n_{20} T_k \tau_E$$

$$n_{20} T_k \tau_E > 30 \ \text{Ignition}$$

n-T-tau is a measure of progress

- Over the years the n-T-tau product shows an exponential increase
- Current experiments are close to break-even
- The next step ITER is expected to operate well above break-even but still somewhat below ignition



Force on the plasma

- The force on an individual particle due to the electro-magnetic field (s is species index)

$$\mathbf{F}_i = Z_s e[\mathbf{E} + \mathbf{v}_i \times \mathbf{B}]$$

- Assume a small volume such that

$$N_s = n_s V$$

- Then the force per unit of volume is

$$\mathbf{F}_s = \frac{1}{V}\sum_{i=1}^{N_s}\mathbf{F}_i = \frac{1}{V}\sum_{i=1}^{N_s} Z_s e[\mathbf{E} + \mathbf{v}_i \times \mathbf{B}]$$

Force on the plasma

- For the electric field

$$\frac{1}{V}\sum_{i=1}^{N_s} Z_s e\mathbf{E} = \frac{N_s}{V} Z_s e\mathbf{E} = Z_s e n_s \mathbf{E}$$

- Define an average velocity

$$\mathbf{u}_s = \frac{1}{N_s}\sum_{i=1}^{N_s} \mathbf{v}_i$$

- Then for the magnetic field

$$\frac{1}{V}\sum_{i=1}^{N_s} Z_s e\mathbf{v}_i \times \mathbf{B} = Z_s e \frac{N_s}{V}\left(\frac{1}{N_s}\sum_{i=1}^{N_s} \mathbf{v}_i\right) \times \mathbf{B} = Z_s e n_s \mathbf{u}_s \times \mathbf{B}$$

Force on the plasma

- Averaged over all particles

$$\mathbf{F}_s = Z_s e n_s [\mathbf{E} + \mathbf{u}_s \times \mathbf{B}]$$

- Now sum over all species

$$\sum_{s=1}^{p} Z_s e n_s \mathbf{E} = e\mathbf{E} \sum_{s=1}^{p} Z_s n_s = 0 \qquad \sum_{s=1}^{p} Z_s e n_s \mathbf{u}_s = \mathbf{J}$$

- The total force density therefore is

$$\mathbf{F} = \sum_{s=1}^{p} \mathbf{F}_s = \mathbf{J} \times \mathbf{B}$$

Force on the plasma

- This force contains only the electro-magnetic part. For a fluid with a finite temperature one has to add the pressure force

$$\mathbf{F} = -\nabla p + \mathbf{J} \times \mathbf{B}$$

$$p = \sum_{s=1}^{p} n_s T_s$$

Reformulating the Lorentz force

- Using $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$
- The force can be written as

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B}$$

- Then using the vector identity

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\mathbf{a} = \mathbf{B} \qquad \mathbf{b} = \mathbf{B}$$

$$\nabla(B^2) = 2(\mathbf{B}\cdot\nabla)\mathbf{B} + 2\mathbf{B}\times(\nabla\times\mathbf{B})$$

Force on the plasma

- One obtains

$$-\nabla p + \mathbf{J} \times \mathbf{B} = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B}$$

Magnetic field pressure   Magnetic field tension

- Important parameter (also efficiency parameter) the plasma-beta

$$\beta = \frac{p}{B^2/2\mu_0}$$

Writing the force in tensor notation

- It is convenient to write the force in a tensor notation

$$(\mathbf{B}\cdot\nabla)\mathbf{B} = B_\alpha \frac{\partial B_\beta}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}[B_\alpha B_\beta] - B_\beta \frac{\partial B_\alpha}{\partial x_\alpha}$$

Einstein summation, every index that appears twice is assumed to be summed over

$$B_\alpha \frac{\partial B_\beta}{\partial x_\alpha} \equiv \sum_{\alpha=1}^{3} B_\alpha \frac{\partial B_\beta}{\partial x_\alpha} = B_1 \frac{\partial B_\beta}{\partial x_1} + B_2 \frac{\partial B_\beta}{\partial x_2} + B_3 \frac{\partial B_\beta}{\partial x_3}$$

Tensor notation

- The tension force

$$(\mathbf{B}\cdot\nabla)\mathbf{B} = B_\alpha \frac{\partial B_\beta}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}[B_\alpha B_\beta] - B_\beta \frac{\partial B_\alpha}{\partial x_\alpha}$$

- Can be simplified using the divergence free magnetic field

$$\frac{\partial B_\alpha}{\partial x_\alpha} = \nabla \cdot \mathbf{B} = 0$$

$$\frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} = \frac{\partial}{\partial x_\alpha}\left[\frac{B_\alpha B_\beta}{\mu_0}\right] \qquad S_{\alpha\beta} = \frac{1}{\mu_0} B_\alpha B_\beta$$

Divergence of a tensor

Tensor notation

- Now write also the pressure force as a tensor

$$\frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} = \frac{\partial}{\partial x_\alpha}\left[\frac{B_\alpha B_\beta}{\mu_0}\right]$$

$$\nabla p = \frac{\partial p}{\partial x_\beta} = \delta_{\alpha\beta}\frac{\partial p}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}\left[p\delta_{\alpha\beta}\right]$$

$\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $\delta_{\alpha\beta} = 1$ for $\alpha = \beta$.

- The force can then be written as

$$\frac{\partial}{\partial x_\alpha}\left[\left(\frac{B^2}{2\mu_0}+p\right)\delta_{\alpha\beta} - \frac{B_\alpha B_\beta}{\mu_0}\right] = \frac{\partial S_{\alpha\beta}}{\partial x_\alpha}$$

Virial theorem

- Assume an equilibrium exists, then the force must be zero

$$\frac{\partial}{\partial x_\alpha}\left[\left(\frac{B^2}{2\mu_0}+p\right)\delta_{\alpha\beta}-\frac{B_\alpha B_\beta}{\mu_0}\right]=\frac{\partial S_{\alpha\beta}}{\partial x_\alpha}=0$$

- Build a scalar quantity, to investigate the implication

$$\int d^3\mathbf{V}\, x_\beta \frac{\partial S_{\alpha\beta}}{\partial x_\alpha}=0.$$

$$x_1\frac{\partial S_{\alpha 1}}{\partial x_\alpha}=\frac{\partial}{\partial x_\alpha}[x_1 S_{\alpha 1}]-\frac{\partial x_1}{\partial x_\alpha}S_{\alpha 1}=\frac{\partial}{\partial x_\alpha}[x_1 S_{\alpha 1}]-S_{11}$$

$$x_\beta\frac{\partial S_{\alpha\beta}}{\partial x_\alpha}=\frac{\partial}{\partial x_\alpha}[x_\beta S_{\alpha\beta}]-\sum_{\alpha=1}^{3}S_{\alpha\alpha}$$

Virial theorem

- Using the relation

$$x_\beta \frac{\partial S_{\alpha\beta}}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}[x_\beta S_{\alpha\beta}] - \sum_{\alpha=1}^{3} S_{\alpha\alpha}$$

- One can rewrite the integral using Gauss's theorem

$$\int d^3\mathbf{V}\, x_\beta \frac{\partial S_{\alpha\beta}}{\partial x_\alpha} = \int \mathrm{d}^2\mathbf{S}\, x_\beta S_{\alpha\beta} n_\alpha + \sum_\alpha \int d^3\mathbf{V}\, S_{\alpha\alpha} = 0$$

- Consider the plasma in a finite volume and extent the integration limits to infinity

$$S_{\alpha\beta} = \left(p + \frac{B^2}{2\mu_0}\right)\delta_{\alpha\beta} - \frac{1}{\mu_0} B_\alpha B_\beta$$

Zero outside the plasma   Scales as the magnetic field squared

Virial theorem

$$\int d^3\mathbf{V}\, x_\beta \frac{\partial S_{\alpha\beta}}{\partial x_\alpha} = \int \mathrm{d}^2\mathbf{S}\, x_\beta S_{\alpha\beta} n_\alpha + \sum_\alpha \int d^3\mathbf{V}\, S_{\alpha\alpha} = 0$$

- The magnetic field in the far field limit scales as

$$B \propto 1/r^3 \longrightarrow \mathrm{d}^2 S x B^2 \propto 1/r^3$$

- So for a sufficient large radius

$$\int \mathrm{d}^3\mathbf{V}\, x_\beta \frac{\partial S_{\alpha\beta}}{\partial x_\alpha} = -\sum_\alpha \int \mathrm{d}^3\mathbf{V}\, S_{\alpha\alpha} = 0.$$

Virial theorem

- For a sufficient large volume

$$\int \mathrm{d}^3\mathbf{V}\, x_\beta \frac{\partial S_{\alpha\beta}}{\partial x_\alpha} = -\sum_\alpha \int \mathrm{d}^3\mathbf{V}\, S_{\alpha\alpha} = 0.$$

- We then have

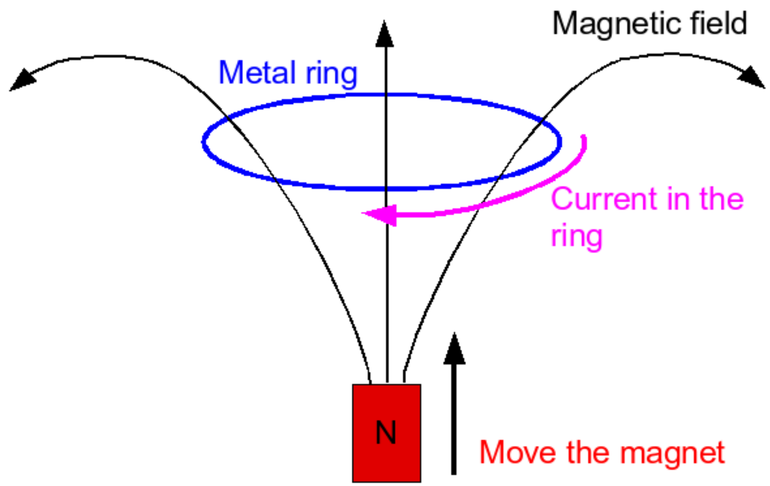
$$\sum_\alpha S_{\alpha\alpha} = 3\left[\frac{B^2}{2\mu_0} + p\right] - \frac{B_1^2 + B_2^2 + B_3^2}{\mu_0} = \frac{B^2}{2\mu_0} + 3p.$$

$$-\int d^3\mathbf{V}\left[3p + \frac{B^2}{8\pi}\right] < 0$$

- Contradiction: NO equilibrium can exist

Flux conservation

- When trying to change the magnetic flux through a metal ring an electric field is generated (Faraday) which drives a current such that it tries to conserve the flux
- The current eventually decays due to the resistivity
- A perfect conductor, however, would conserve the magnetic flux



Flux conservation

- A plasma is like a metal (electrons are free)
- A hot plasma has a small resistivity
- As a first approximation it is perfectly conducting
- Flux is then conserved but the fluid can be moving
- Flux is transported along with the fluid