A simple avalanche model as an analogue for magnetospheric activity

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Abstract. The power law dependence of the power spectrum of auroral indices, and in-situ magnetic field observations in the earth's geotail, may be evidence that the coupled solar wind-magnetospheric system exhibits scale free self organised criticality and can to some extent be described by avalanche models. In contrast, the intensity of, and time interval between, substorms both have well defined probability distributions with characteristic scales. We present results from a simple cellular automaton that models avalanches in a one dimensional "sandpile"; here we examine the simplest case of constant inflow. This model generates a probability distribution of energy discharges due to internal reorganization that is a power law implying SOC, whereas systemwide discharges (flow of "sand" out of the system) form a distinct group which do not exhibit SOC. The energy dissipated in a systemwide discharge follows a probability distribution with a well defined mean, as does the time interval between one systemwide discharge and the next. Internal and external avalanches can therefore in principle be identified with distinct processes in the dynamic geotail. If so, the avalanche model places restrictions on the class of physical process that may be invoked to explain the observed geomagnetic dynamics.

Introduction

The characterisation of global energy transport in the solar wind-magnetosphere-ionosphere system is of fundamental importance to space physics. The solar wind provides a source of kinetic energy and magnetic flux; this is transferred to, stored in, and ultimately released from the magnetosphere by a range of mechanisms, in which magnetotail dynamics and substorms play a central role. This tripartite overall behavior (inflow; storage; reconfiguration/detectable energy release events) is replicated elsewhere in certain laboratory and astrophysical plasma systems, to which "sandpile" modelling and cellular automata have recently been applied (a review in the plasma confinement context is given in [Dendy and Helander, 1997]). The concept of self organised criticality (SOC) [Bak et al., 1987] that is exemplified in the sandpile studies is an important new paradigm in contemporary physics [Bak, 1997]. Briefly, a confinement system ("sandpile") is in a state of SOC when the statistics of the

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energy release events ("avalanches") reveal no characteristic length or time scales. In contrast to the scale invariant phenomena that can arise close to critical points (for example at phase transitions) in many systems [Huang, 1987], self organised criticality is robust and arises spontaneously without fine tuning of system parameters.

An important avenue of substorm research has been the study of the state space of the coupled solar wind-magnetosphere-ionosphere system (see the review [Klimas et al., 1996]). The auroral electrojet index (AE) provides a diagnostic of energy release from the magnetosphere into the ionosphere. It has long been realised (e.g.[Baker et al, 1981; Bargatze et al, 1985]) that magnetospheric indices such as AE must in some sense be sampling this state space. Although AE is a non-stationary time series, it is self affine [Takalo et al., 1993], with power spectrum that is of broken power law form with a break at roughly 1/(5 hours) [Tsurutani et al., 1990]. Recent analysis of a "burst size" distribution for AE has revealed a power law over three orders of magnitude indicating the absence of any characteristic scales in the magnetospheric system [Consolini, 1997]. However, Consolini [1997] also argued that a region in the power spectrum of AE is an indirect measurement of large scale outflow from the system (storms and substorms) and that power law (1/f) behaviour in this region, and by implication, in the large scale outflow, provides the fingerprint of SOC in the magnetosphere. Here, we shall instead argue that self similar behaviour in AE is indicative of SOC in the internal dynamics of the magnetosphere (an important caveat is that AE will also exhibit the response of the ionospheric conductivity to energy inflow into the ionosphere).

A distinct approach is the contemporary search for low dimensional chaos in AE, which motivated studies of its fractal dimension (see the reviews [Klimas et al., 1996; Sharma, 1995]). However, as pointed out in the space plasma physics context by Chang [1992] low dimensionality is also a property of a system near criticality. Critical behaviour (such as fluctuations on all length scales and scale invariance) had previously been known in systems at phase transitions and hence over relatively small regions of parameter space. The "sandpile" cellular automaton constructed by Bak et al., [1987] exhibited criticality as a natural consequence of its time evolution, without parameters having to be adjusted "by hand", and was thus dubbed self-organised criticality (SOC). Such a non-equilibrium system dissipates energy by means of many avalanches of all sizes, but returns to its out of equilibrium critical stationary state rather than relaxing to a non critical one. Hence an alternative framework for explanation of the low dimensionality, the existence of bursty bulk flows and the power law magnetic field spectra in the magnetotail, has been given by postulating the magnetotail to be an open, dissipative dynamical system at a (self organised) critical state[Chang, 1998].
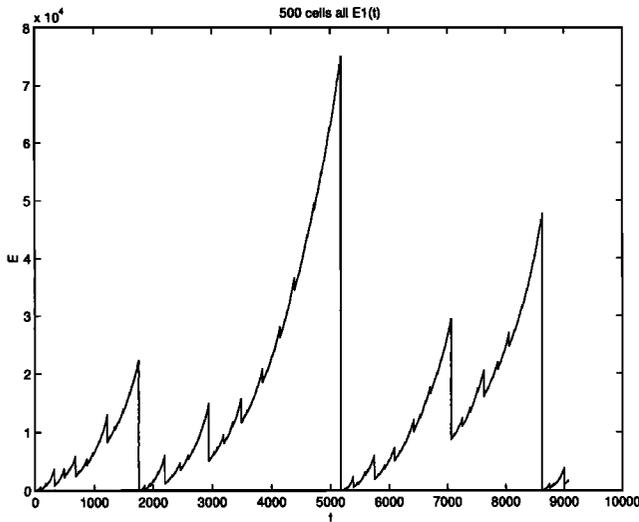
Figure 1. Energy in the sandpile as a function of time (normalization given in the text). Zero energy corresponds to the energy of the sandpile at the angle of repose, hence drops in energy to values > 0 are internal avalanches and to values $= 0$ are systemwide avalanches.

Figure 2. A loglog plot of the probability distributions of the energy dissipated per avalanche for two sets of data: the first 50000 avalanches (O) and the first 10000 systemwide avalanches (*) to occur in a sandpile with 500 cells. Where there is no overlap, the "0" group are internal avalanches only and their probability distribution follows a power law.

Other indicators such as substorm chorus events seen at the ground and particle injection events in the near earth magnetosphere indicate the outflow of energy from the system due to large scale systemwide reorganizations. These are typically not self similar, and not completely deterministic, however the intensity and time intervals between one substorm and the next have a probability distribution with a well defined mean ([*Pritchard et al,* 1996], [*Smith et al.,* 1996]; also [*Borovsky et al.,* 1993] and references therein). We shall see that a model that internally exhibits SOC can simultaneously generate non-SOC global instability (systemwide events), without recourse to any other mechanism.
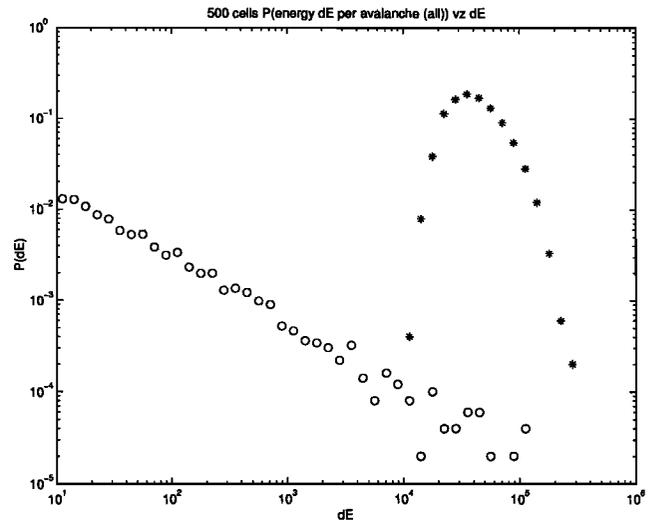
The Sandpile Model

The sandpile cellular automaton used here is described in more detail in [*Dendy and Helander,* 1998]. The sandpile is represented by a 1D grid of N equally spaced cells unit 1 apart, each with sand at height h_j and local gradient $z_j = h_j - h_{j+1}$. The sandpile is assumed to have an "angle of repose" below which it is always stable, the heights h_j and the gradients z_j are those which are in excess of the sandpile at the angle of repose. Each cell is assigned a critical gradient z; if the local gradient exceeds this then the sand is redistributed (an avalanche occurs).

The selection rule for the critical gradients on the N nodes is $F(z) = R$ where R is a uniformly distributed random number in the range [0, 1] and

$$F(z) = 1 - exp\left(\frac{-z^{1+y}}{1+y}\right) \qquad (1)$$

where y is an adjustable parameter. The probability of a given node being assigned a critical gradient in the range $[z, z + dz]$ is then given by:

$$P(z)dz = \frac{dF}{dz}dz \qquad (2)$$

so that

$$\int_{z=0}^{\infty} P(z)dz = \int_{z=0}^{\infty} dF = 1 \qquad (3)$$

as required, and the probability distribution of the critical gradients is

$$P(z) = \frac{dF}{dz} = z^y exp\left(\frac{-z^{1+y}}{1+y}\right) \qquad (4)$$

Sand is then added at cell 1 at a constant rate $g \ll 1$, we normalize length and time to this loading rate so that unit volume sand is added in unit time, (ie after $1/g$ iterations). Once the critical gradient is exceeded at cell 1, the sand is redistributed. The redistribution rule (see [*Dendy and Helander,* 1998]) is conservative and instantaneous; sand will propagate to cell 2 and if the local critical gradient there is exceeded, to cell 3 and so on. Within this avalanche, the sandpile is "flattened" back to the angle of repose ($z_j = 0$) at which the sandpile is stable. The critical gradients within the avalanche are then regenerated by rule (1). The avalanche may be entirely an internal rearrangement of sand or may continue until it spreads across all N cells of the pile (a systemwide discharge), in which case the entire sandpile is emptied and returns to the angle of repose. The energy dissipated by avalanche events (both internal and systemwide) is given by the difference in the potential energy in the entire sandpile before and after the avalanche:

$$dE = \sum_{j=1}^{N} h_j^2 \mid_{before} - \sum_{j=1}^{N} h_j^2 \mid_{after} \qquad (5)$$

This model is an attempt to incorporate, with minimum complexity, several generic properties which are expected to characterize the magnetospheric system dynamics. The random selection of critical gradients represents the possibility that more than one competing process may be responsible for instability, and that locally each of these processes

has a different threshold for instability. The system has a (constant) finite extent. The evolution of a single avalanche takes place on a timescale that is assumed to be negligible compared with the timescale for loading (inflow and storage of sand/energy). Internal avalanches conserve matter but dissipate energy, i.e. sand can only be lost via systemwide discharges when sand exits the system. The system exhibits both internal and systemwide avalanches under constant loading rate. We will now numerically explore the phenomenology of this simple model.

Results

Numerical experiments were performed on sandpiles with system lengths ranging over an order of magnitude (50-1000 cells) and in each case the sandpile model was iterated until 10^4 systemwide discharges occurred. The results shown here are for a 500 cell system, with $g = 0.01, y = 0.3$. The energy in the sandpile in addition to that associated with sand at the (stable) angle of repose is shown for a short interval of this run after $t = 0$ in Figure 1. The energy starts at zero (the sandpile begins at the angle of repose) and grows on average quadratically as sand is added at a constant rate (since $E \sim h^2$). As the sandpile evolves, small internal drops in energy can initially be seen; these are internal avalanches in which no sand leaves the system. Eventually, a systemwide avalanche occurs (at $t \simeq 1700$) and the energy shown drops to zero as all excess sand leaves the system and the sandpile returns to the angle of repose. This cycle of events repeats, but in what appears to be a non deterministic fashion.

We can make some statements about the probability distributions of the time intervals dt between successive avalanches, and the change in energy to the system dE associated with them. The normalized (binned) distribution for energy change (as defined by (5)) due to all avalanches (both internal and systemwide) is shown on a loglog plot in Figure 2. On this plot the internal avalanches all lie approximately on a straight line of exponent ~ -0.75 and occupy a range in dE of three orders of magnitude, whilst the systemwide avalanches form a distinct group. This power law dependence indicates that the internal avalanches have no intrinsic scale and is taken as evidence that the system exhibits SOC. If the sandpile model were embedded in a larger scale system we would therefore expect to see break points in this distribution at the "edges" (at $dE \sim 10^1, 10^4$). This power law form of $P(dE)$ is found to be independent of the choice of selection rule for the critical gradients (1). In particular, it persists if we simply choose a single value for the critical gradient throughout the sandpile; in this system, the output of the sandpile is fractal and is completely deterministic. The appearance of a power law signature of SOC in data therefore does not distinguish the nature of the detailed mechanism(s) required for instability.

The probability distribution for the time intervals between internal avalanches is straightforward. Since for any avalanche to occur, instability must first occur at cell 1, and the critical gradient at cell 1 has probability distribution (4), the time between all avalanches must also have probability distribution (4). This distribution in data is therefore a strong indicator of the range of critical gradients, that is, the threshold for instability in the detailed mechanism(s) invoked for instability.
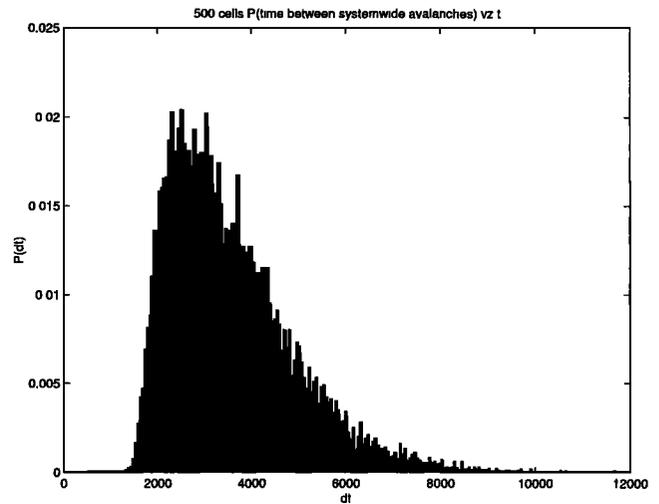
Figure 3. The probability distribution of the time interval between one systemwide avalanche and the next for the data shown in the previous figures. Normalization given in the text.

The behaviour of the systemwide discharges is shown in Figure 3. Both the energy released in systemwide discharges (see Figure 2), and the time intervals between each systemwide discharge and the next, follow probability distributions with well defined means; they have a characteristic scale and hence do not show evidence of SOC. These probability distributions arise as a consequence of the randomized critical gradients in the sandpile; a sandpile with a single critical gradient produces systemwide discharges that are precisely periodic in time and have the same energy. The probability distributions of the random sandpile scale with the system size and the rate of inflow of energy (sand) to the system. Hence, an output distribution that is not in SOC, and exhibits some characteristic scale, does not indicate a system's preference for a single internal threshold for instability per se. The characteristic scale may simply reflect the rate of inflow of energy and the scale size of the system.

Finally, the results presented here are restricted to the case of constant rate of inflow of energy/"sand" g. The detailed behaviour of the model under variable inflow rate is complex and will be the subject of a future paper. However, since time is normalized to the inflow rate we can infer that fluctuations (increase/decrease) in inflow rate will be reflected in fluctuations (decrease/increase) in the mean time interval between all avalanches. In this sense the model will naturally incorporate "driven" behaviour.

Summary

The dynamic global magnetosphere exhibits two properties. Internal to the global system there is evidence of SOC (absence of an intrinsic scale) both in magnetospheric activity as monitored by AE and in magnetic field spectra taken in- situ in the geotail. The output from the magnetosphere as seen by substorm chorus events, injection events and so forth on the other hand has a well defined probability distribution with a mean time interval between events such as substorms. Here, we have presented a simple avalanche (sandpile) model which reproduces this behaviour. The model requires a box of fixed size, random critical gradients and

conservative internal redistribution rule. Results were presented for constant rate of inflow of energy to the system. The implications of the model are:

- The appearance of a power law signature of SOC in data does not in itself distinguish the nature of the detailed mechanism(s) required for instability.

- The probability distribution of time intervals between internal avalanches in data is a strong indicator of the range of critical gradients, that is, the range of thresholds for instability in the detailed mechanism(s) invoked for instability.

- An output distribution that is not deterministic but is not in SOC, and exhibits some characteristic scale, does not, per se, indicate a system's preference for a single internal threshold for instability. The characteristic scale may simply reflect the rate of inflow of energy and the scale size of the system.

It should be stressed that the avalanche model discussed here does not solve the equations describing the underlying plasma physics, and as such can only provide an analogue for the system evolution. It is a major challenge to synthesise the avalanche model with plasma physics, and identifying that this route is likely to lead to solution is a critical first step. This has been the aim of this paper. Even without this synthesis, the sandpile model places constraints on instability mechanisms invoked to explain the evolution of the dynamic magnetosphere. The observations in turn place constraints on the avalanche model. An example may be the following: since the grid size is constant, increasing the rate of inflow g will decrease the mean time between systemwide discharges (substorms). Evidence to the contrary [*Freeman and Farrugia* , 1998] would imply that the system size is also a function of the inflow rate.

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