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Robustness of collective behaviour in strongly driven avalanche models: magnetospheric implications

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Abstract. The hypothesis of self-organised criticality (SOC) predicts that certain open dissipative systems evolve to a critical state where all energy release statistics display power law distributions for event occurrence, size and duration. This has motivated "sandpile" simulations of magnetospheric energy confinement and release events ("avalanches"), previous examples of which have taken the limit where energy inflow ("fuelling") is slow relative to dissipation, and either uniform or random. However the magnetospheric system has both slow and fast periods mixed together in observations, and naturally modulated fuelling. We have developed an avalanche model with variable, modulated fuelling rate. The power law form for the distribution of energy release events is the least ambiguous current indicator of SOC; we show that this is preserved for the large avalanches in such a system under both constant and varying loading and so such systems are remarkably efficient at eliminating small scale information about their fuelling.

Introduction

Most investigations of the magnetospheric substorm problem have focused on single substorm episodes or event groups. An alternative is to use long-term statistics of magnetospheric energy release to infer constraints on the underlying physical processes. This has usually required the definition of substorm "events" via a thresholding process. There is, however, growing interest in relating the characteristics of global energy transport in the solar wind-magnetosphere-ionosphere system to "sandpile models" [*Consolini*, 1997; *Dendy and Helander*, 1998; *Chapman et al.*, 1998; *Uritsky and Pudovkin*, 1998], which dissipate energy by means of avalanches. This follows the suggestion [*Chang*, 1992; *Chang*, 1998a; *Chang*, 1998b] that the magnetosphere is in a self organised critical (SOC) state. In sandpile models, the distribution of avalanche magnitude and frequency may display scale free, inverse power law statistics that are characteristic of SOC [*Bak et al.*, 1987; *Lu*, 1995]. Observational motivation is provided by power law features of magnetospheric index data, notably AE which is an indicator of energy dissipated by the magnetosphere into the ionosphere. A significant result [*Chapman et al.*, 1998] is that a class of sandpile models [*Dendy and Helander*, 1998], see also [*Pinho and Andrade*, 1998], yield systemwide avalanches where the statistics have a well defined mean (intrinsic scale) whereas the internal avalanche statistics are scale free; analogous to the differing characteristics of internal and external magnetospheric energy release events.

A limitation of standard sandpile models is that their fuelling, which would correspond in principle to magnetospheric loading, is assumed to be slow. This raises a central question. Given that in reality magnetospheric loading can be strong, and fluctuate substantially, to what extent does standard sandpile phenomenology persist? Does the sandpile (or magnetosphere) act as a nonlinear filter, with the observed avalanches retaining magnitude and frequency information contained in the fuelling (or loading) process?

Tsurutani et al. [1990] described a broken power law AE spectrum; this has been claimed as evidence of SOC [*Consolini*, 1997; *Uritsky and Pudovkin*, 1998], but a power law spectrum for AE cannot be uniquely interpreted as the output of an SOC system because other types of physical system can also produce scale free spectra. Furthermore AE is a compound index in that it mixes driven and unloading effects, a fact reflected both by its power spectrum [*Freeman et al.*, 1998] and structure function [*Takalo and Timonen*, 1998]. One may also define individual energy release events, and examine the statistics of their distribution; the signature of an SOC system would be a "power law" probability density for this quantity. Consolini [*personal communication*, 1998] used AE data taken over a 10 year period to construct the probability density $D(s)$ of a burst measure s; the density obtained was fitted by an exponentially cutoff power law $D(s) \sim (s^{-\alpha})e^{-s/\beta}$ with $\alpha = 1.23 \pm 0.01$ and $\beta = (1.17 \pm 0.01) \times 10^6$, extending the result obtained for the year 1978 by *Consolini*, [1997]. A generic feature of this type of statistical experimental evidence is that long runs of data are required, and in the magnetospheric system it is unavoidable that both the instantaneous value and the recent mean of the loading rate will have strong variation. Another class of evidence is derived from in situ data. Here, long runs of data are not available, but a given region in the magnetosphere can be observed repeatedly under

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different conditions. Of particular interest as evidence for avalanche-type events are bursty bulk flows [*Angelopoulos et al.*, 1996] in the geomagnetic tail.

This returns us to the central issue of robustness. If power law statistics still emerge from sandpile models that have substantially varying loading, they remain useful as a characteristic signature to test for in energy release event time series. The appearance of power law (scale free) statistics in such data would then support the hypothesis that the underlying physics remained governed by critical gradients leading to local redistribution generating avalanches on all scales. A limiting case of the sandpile algorithm used in this paper [*Dendy and Helander,* 1998] has recently been shown to display self-similar properties [*Helander et al.*, 1999]. In general, the conditions necessary for SOC to occur constrain the inflow rate (see [*Lu,* 1995] for some proposed criteria); to emphasise this *Jensen*, [1998] has proposed a category of slowly-driven, interaction-dominated, thresholded (SDIDT) systems. The slowly driven condition is of particular interest in the magnetospheric context because most SOC simulations have been conducted in the limit where the rate of inflow is "slow" relative to dissipation. In this paper we pose the question: how robust are magnetospherically relevant sandpile models to changes in the inflow ?

Scale free behaviour under slow, finite loading

In a typical sandpile algorithm, redistribution of sand at cell j is triggered if the local gradient z_j exceeds a critical gradient z_c. Redistribution may in turn cause the neighbouring cell to become unstable, triggering further redistribution and, by iteration, an avalanche. Avalanches can be classified as internal, meaning rearrangements of sand within the pile, or systemwide where the entire sandpile returns to its angle of repose and excess sand actually leaves the system. In the model considered in [*Chapman et al.,* 1998] the values for the critical gradients at each cell were drawn from a probability distribution. The sand added to trigger an avalanche is spread by the avalanche over j cells, at each of which the average height of sand has increased. Let $h_{j,0}$ and $h_{j,1}$ denote the heights at cell j immediately before and after the avalanche respectively, and $< g >$ the mean (ensemble average) fuelling rate. The amount of sand then transferred to the second cell can be written

$$\alpha_1(h_{1,0} - h_{2,0} + < g >) \tag{1}$$

where $\alpha_1 < 1$ by construction. Similarly, an amount

$$\alpha_2\{h_{2,0} - h_{3,0} + \alpha_1(h_{1,0} - h_{2,0} + < g >)\} \tag{2}$$

is transferred to the third cell, where $\alpha_2 < 1$, and so on. For the final transfer to the jth cell, it follows that $< g >$ is multiplied by $\Pi_{i=1}^{j-1}\alpha_i$, whereas the nearest-neighbour height difference $h_{j-1,0} - h_{j,0}$ is multiplied by α_{j-1}, the next nearest neighbour is multiplied by $\alpha_{j-2}\alpha_{j-1}$, and so on. Since $\alpha_1\alpha_2...\alpha_k \leq \alpha_1\alpha_2...\alpha_{k-1} \leq \alpha_1\alpha_2...\alpha_{k-2}$ and similarly for higher powers in α_i, it follows that the relative importance of $< g >$ in determining the size of the avalanche diminishes as the scale j of the avalanche increases. We consider only the class of sandpile models which have scale free power law avalanche statistics in the limit of small but non-zero $< g >$ ("quiet time"). To give such statistics $\Pi_{i=1}^{j-1}\alpha_i < g >$ must decline sufficiently rapidly with j for small oscillations in g (with $< g >$ small) to be effectively damped by the sandpile.

Sandpile with strong driving

Let us consider the effects of rapid fuelling on a physical system in which energy transport and dissipation are caused only by avalanches. If the system displays SOC over some dynamic range, the event statistics within that range will be of power law form. There are three characteristic timescales implicit in any sandpile algorithm for such a system: the relaxation time τ_r over which an avalanche takes place; the average time required, following an avalanche, for instability to recur at the first cell (we assume central fuelling for simplicity), τ_u; and the time for one iteration of the algorithm, Δt. The amount of sand added at cell 1 per timestep is again g, which may be constant, modulated over time, or drawn from some random distribution; its ensemble averaged value is $< g >$. Following the physically motivated arguments of *Dendy and Helander*[1998] and *Chapman et al.*[1998] we also assume that the critical gradient z_{crit} for any grid cell and time step is drawn from some probability distribution $P(z_c)$ with mean $< z_c >$. In the limit where $P(z_c)$ is a delta function, $< z_c >$ becomes the "standard" uniform sandpile-wide z_{crit}. It is clear that $\tau_u \sim< z_c > / < g >$ and hence, for instantaneously relaxing sandpile models ($\tau_r \ll \Delta t$) we identify slow and fast loading regimes:

$$\text{slow} : \tau_r \ll \Delta t \ll \tau_u \tag{3}$$

$$\text{fast} : \tau_r \ll \tau_u \sim \Delta t \tag{4}$$

respectively. In the latter case, instability is likely to be triggered at each timestep. We reiterate, that in the literature, the ordering of time scales given by (3) has often been cited as a condition for SOC rather than one of several regimes in which aspects of SOC behaviour may be observed.

Normalising Δt to unity, that is one timestep per in unit time, the fast loading condition becomes $< g >\sim< z_c >$. For a given loading rate, there will be an effective minimum avalanche length needed to dissipate the energy associated with the sand added in each Δt. In the slow limit this corresponds to less than one cell, in the fast limit to many. If the mean fuelling rate $< g >$ is increased towards and beyond $< z_c >$, the smallest scale avalanches will be increasingly eliminated. This will be reflected in the lower bound of any range of "power law" event statistics for energy release in the system.

Sandpile with variable driving

We now examine how far information concerning the modulation or fluctuation of g is transmitted in avalanches, and whether its consequences may become visible in the event statistics ("the sandpile as filter"). Insofar as fluctuations in g take the sandpile intermittently into the fast driving regime, the comments in the preceding section apply. In any event, there will exist a maximum avalanche scale (number of cells n_f) above which the effect of fluctuations in g becomes vanishingly small in the avalanche statistics. We then have two possible cases. The first is that n_f is sufficiently large for there to be a detectable signature in the internal statistics (disruption of the power law), but is still much smaller than the system size. The second has n_f of order the system size, in which case the event statistics of both internal and external (or systemwide) events reveal a signature reflecting both the mean level and the fluctuation spectrum of g. In the latter case we expect a sandpile model to yield good correlation between fluctuations in the inflow and in the external (systemwide) timeseries in the extremely strongly driven limit.

Results and discussion

As a concrete example of the preceeding arguments we modified the algorithm used by *Dendy and Helander* [1998] and *Chapman et al.* [1998] to incorporate variable fuelling. We take a one dimensional, unit spaced grid of N cells. Sand height is given by h_j, so the local gradient becomes $z_j = h_j - h_{j+1}$. The pile is assumed stable at the angle of repose. Both h_j, z_j are then measured from this stable state. Each jth cell has a critical gradient z_{cj}. In common with all other avalanche models the model of *Dendy and Helander* [1998] has a fuelling rule, an instability rule and a redistribution rule, which together form a repeating cycle. Fuelling occurs by adding sand at the first cell. We normalise L and T to the average loading rate g so that unit volume sand is added in unit time, (i.e. after $1/g$ iterations). The conservative redistribution rules [*Dendy and Helander*, 1998; *Chapman et al.*, 1998] are such that "avalanches", i.e. redistribution events, occur on all scales up to the size of the pile.

Each avalanche releases energy (but not necessarily sand) from the sandpile

$$\Delta E = \sum_{j=1}^{N} h_j^2 \,|_{before} - \sum_{j=1}^{N} h_j^2 \,|_{after} \qquad (5)$$

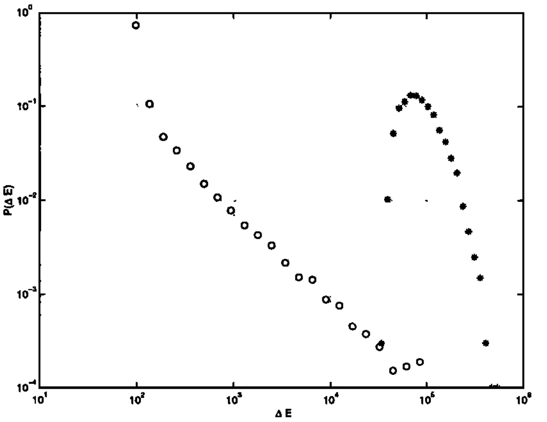

Figure 1. Normalised histograms of probability of energy release of size ΔE for all (o) and systemwide only (*) avalanches for a 5000 cell sandpile with constant fuelling $g = 10$.

The extent to which the loading rate g is "fast" or "slow" can be measured by comparing g to the mean value of z_c from (3) and (4). We have previously studied [*Chapman et al.*, 1998] avalanche statistics under slow ($g \ll \; < z_c >$) constant loading rate, for a 500 cell system, with $g = 0.01$. Here we present results for a larger grid of 5000 cells, with $P(z_c)$ a "top hat" distribution covering $[0.5, 1.5]$ (see also [*Chapman et al.*, 1999]). The normalised histograms of probability of energy release events of size ΔE in the plots shown here are comprised of over 150,000 internal avalanches and 20,000 systemwide avalanches. In Figure 1 we show results for constant $g = 10$. The systemwide event statistics remain unaffected by the fast loading, illustrating the fact that large avalanches are "well insulated" against the fuelling process; however the internal avalanche distribution displays a dropout in the number of the smallest events. In this constant loading rate case, there is a sharp cutoff at $\Delta E \sim 100$ and events below this size are of low probability. Above this, the emergent power law form of the energy release distribution is preserved up to ΔE of 10^5. This behaviour may be compared with Figure 2 in which the input fuelling rate has been varied. The mean $< g >$ is still 10 but each value of g is drawn from a uniform random distribution, in the range $[5, 15]$. We see that the effect is to "soften" the peak at ΔE of 100, and to leave some avalanches down to

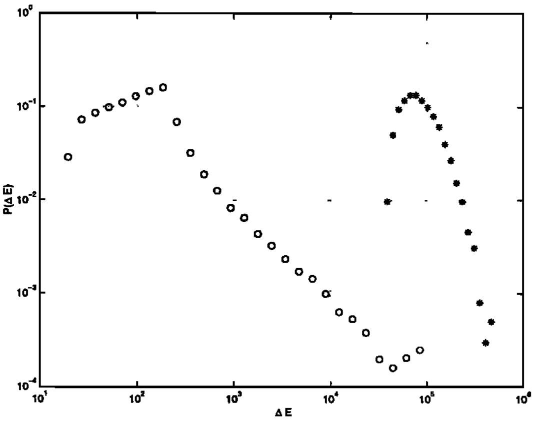

Figure 2. Normalised histograms of probability of energy release of size ΔE for all (o) and systemwide only (*) avalanches for a 5000 cell sandpile with randomly fluctuating fuelling in the range $[5, 15]$ and $< g >= 10$.

$\Delta E = 10$. We interpret this as being due to the range of possible fuelling rates, which means that the smallest avalanches are no longer automatically ruled out in a given iteration (i.e. for some iterations g is relatively small).

Conclusions

Over long periods, the magnetospheric driver (that is, the solar wind) is characterised by both strong variability about the mean, and a large dynamic range of mean energy input. Any statistical analysis of energy release events will therefore require long uninterrupted runs of data. In order to use such data to test the hypothesis that the magnetosphere may be in a state of SOC, it is necessary to examine how resilient the main observational signatures of SOC (notably scale invariant power law statistics) will be under circumstances of varying fuelling rate. Our model shows that the crucial power law form of the burst size distribution of large events is surprisingly robust under fast loading. Accordingly, we expect inverse power law avalanche distributions to be a persistent feature in long runs of data that include "fast" inflow conditions, if the underlying system is governed by SOC. The relative stability of this distribution suggests that a long run of data including periods with high driven activity will give a power law, in an SOC magnetosphere. Furthermore restricting data to active times would give a power law with a tail-off at small events.

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