Magnetohydrodynamics (MHD)

Lecturer: Professor Valery Nakariakov, *V.Nakariakov@warwick.ac.uk*

Online lecture notes: http://goo.gl/oPgmzK

MHD describes large scale, slow dynamics of plasmas. More specifically, we can apply MHD when

1. Characteristic time $\gg$ ion gyroperiod and mean free path time,
2. Characteristic scale $\gg$ ion gyroradius and mean free path length,
3. Plasma velocities are not relativistic.

In MHD, the plasma is considered as an electrically conducting fluid. Governing equations are equations of fluid dynamics and Maxwell's equations. A self-consistent set of MHD equations connects the plasma mass density ρ, the plasma velocity $\mathbf{V}$, the thermodynamic (also called gas or kinetic) pressure P and the magnetic field $\mathbf{B}$. In strict derivation of MHD, one should neglect the motion of electrons and consider only heavy ions.

The 1-st equation is mass continuity

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{V}) = 0, \tag{1}$$

and it states that matter is neither created or destroyed.

The 2-nd is the equation of motion of an element of the fluid,

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} \right] = -\nabla P + \mathbf{j} \times \mathbf{B}, \tag{2}$$

also called the Euler equation. The vector $\mathbf{j}$ is the electric current density which can be expressed through the magnetic field $\mathbf{B}$. Mind that on the lefthand side it is the total derivative, d/dt.

The 3-rd equation is the energy equation, which in the simplest *adiabatic* case has the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{P}{\rho^\gamma} \right) = 0, \tag{3}$$

where γ is the ratio of specific heats C_p/C_V, and is normally taken as 5/3.

The temperature T of the plasma can be determined from the density ρ and the thermodynamic pressure p, using the state equation (e.g. the ideal gas law). For example, in a pure hydrogen plasma, this equation is

$$P = 2\frac{k_B}{m_p}\rho T, \tag{4}$$

where m_p is the mass of a proton and k_B is Boltzmann's constant.

Now, let us derive the equation for the magnetic field using Maxwell's equations. Start with Ohm's law,

$$\mathbf{j} = \sigma \mathbf{E}', \tag{5}$$

where σ is electrical conductivity (the physical quantity inverse to the resistivity) and $\mathbf{E}'$ is the electric field experienced by the plasma (fluid) element in its *rest* frame. When the plasma is moving (with respect to the external magnetic field) at the velocity $\mathbf{V}$, applying the Lorentz transformation we obtain

$$\mathbf{E}' = \mathbf{E} + \mathbf{V} \times \mathbf{B}. \tag{6}$$

Now, Eq. (5) can be re-written as

$$\frac{1}{\sigma}\,\mathbf{j} = \mathbf{E} + \mathbf{V}\times\mathbf{B}. \tag{7}$$

In the case of perfect conductivity, $\sigma \to \infty$, we have

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B}. \tag{8}$$

Calculating the curl of the electric field $\mathbf{E}$ and using one of Maxwell's equation,

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{9}$$

we can exclude the electric field and obtain

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{V}\times\mathbf{B}), \tag{10}$$

which is the 4-th MHD equation — the "induction equation". In particular, this equation describes the phenomenon of magnetic dynamo.

To close the set of MHD equations, we have to express the current density $\mathbf{j}$ through the magnetic field $\mathbf{B}$. Consider the other Maxwell's equation,

$$\nabla\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \mu_0\mathbf{j} \tag{11}$$

From Ohm's law, we had $\mathbf{E} = -\mathbf{V}\times\mathbf{B}$. Consequently, we can estimate the electric field as $E \sim V_0 B$, where V_0 is a characteristic speed of the process. Consider the ratio of two terms in Eq. (11):

$$\nabla\times\mathbf{B} \quad \text{and} \quad \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}.$$

The first term is proportional to B/l_0, where l_0 is a characteristic scale of the process, the second to $E/c^2 t_0$, where t_0 is a characteristic time of the process, $V_0 = l_0/t_0$. When the process is not relativistic, $V_0 \ll c$, the first term is very much greater than the second, and we have

$$\mathbf{j} = \frac{1}{\mu_0}\nabla\times\mathbf{B} \tag{12}$$

In addition, the magnetic field $\mathbf{B}$ must satisfy the condition $\nabla.\mathbf{B} = 0$.

Thus, the closed set of MHD equations is

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{V}) = 0, \qquad \text{Mass Continuity Eq.,}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{P}{\rho^\gamma}\right) = 0, \qquad \text{Energy Eq.,}$$

$$\rho \frac{\mathrm{d}\,\mathbf{V}}{\mathrm{d}\,t} = -\nabla P - \frac{1}{\mu_0}\mathbf{B} \times (\nabla \times \mathbf{B}), \qquad \text{Euler's Eq.,}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}), \qquad \text{Induction Eq..}$$

The equations are <u>ideal</u>, which means that all dissipative processes (finite viscosity, electrical resistivity and thermal conductivity) were neglected.

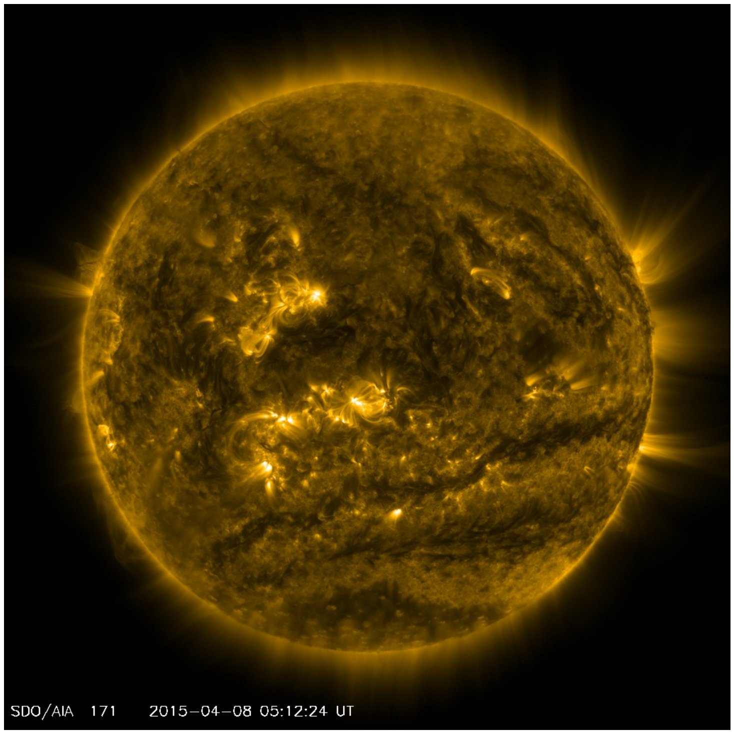
Also, the magnetic field is subject to the condition

$$\nabla \cdot \mathbf{B} = 0. \tag{13}$$

The Nobel Prize in Physics 1970 was given to Hannes Olof Gösta Alfvén "for fundamental work and discoveries in magnetohydrodynamics with fruitful applications in different parts of plasma physics".

MHD is applicable from nanometre (10^{-9} m) scales in, e.g. physics of semi-conductors, to galactic (10^{21} m) scales, e.g. galactic arms.

Example: the applicability of MHD to the solar corona:



1. Speeds are much less than the speed of light.
   (In the solar corona: $V <$ a few thousand km/s).

2. Characteristic times are much longer than the gyroperiod and the plasma period.
   In the solar corona: $f_{\rm MHD} < 1$ Hz,
   for $f_{\rm gyro} = 1.52 \times 10^3 \times B(\mathrm{G}) \approx 1.52 \times 10^4$ Hz
   and $f_{\rm plasma} = 9 \times n_e^{1/2}(\mathrm{m}^{-3}) \approx 2 \times 10^8$ Hz,
   (for $B = 10$ G and $n_e = 5 \times 10^{14}$ m^{-3}).

3. Characteristic times are much longer than the collision times. Characteristic spatial scales are larger than the mean free path length

$$\lambda \gg l_{ii}(\mathrm{m}) \approx \frac{7.2 \times 10^7 T^2(\mathrm{K})}{n(\mathrm{m}^{-3})}.$$

   For the typical conditions of the lower corona, $l_{ii} \approx 10^5 - 10^6$ m.

4. Similar estimations should be made for the spatial scales, and the conditions of applicability are well satisfied too.

MHD Equilibrium

The static equilibrium conditions are:

$$\mathbf{V} = 0, \quad \frac{\partial}{\partial t} = 0. \tag{14}$$

These conditions identically satisfy the continuity, energy and induction equations.

From Euler's equation we obtain the condition

$$-\nabla P - \frac{1}{\mu_0}\mathbf{B} \times (\nabla \times \mathbf{B}) = 0, \tag{15}$$

which is called the equation of magnetostatics. This equation should be supplemented with the condition $\nabla.\mathbf{B} = 0$.

Eq. (15) can be re-written as

$$-\nabla\left(P + \frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\mathbf{B}.\nabla)\mathbf{B} = 0. \tag{16}$$

The first term can be considered as the gradient of total pressure.
The total pressure consists of two terms, the gas (or thermodynamic) pressure P, and the magnetic pressure $B^2/2\mu_0$.
The second term is magnetic tension. The force is directed anti-parallel to the radius of the magnetic field line curvature.

Plasma-β

Compare the terms in the magnetostatic equation,

$$-\nabla P + \frac{1}{\mu_0}\mathbf{B} \times (\nabla \times \mathbf{B}) = 0. \tag{17}$$

We have that

$$\nabla P \approx \frac{P}{\lambda} \quad \text{and} \quad \frac{1}{\mu_0}\mathbf{B} \times (\nabla \times \mathbf{B}) \approx \frac{B^2}{\mu_0 \lambda}, \tag{18}$$

where λ is a characteristic scale of the problem.

The ratio of the gas pressure gradient term and the Lorentz force is known as the plasma-β,

$$\beta \equiv \frac{\text{gas pressure}}{\text{magnetic pressure}} = \frac{P}{B^2/2\mu_0}. \tag{19}$$

Plasma-β can be estimated by the formula,

$$\beta = 3.5 \times 10^{-21} n\, T\, B^{-2}, \tag{20}$$

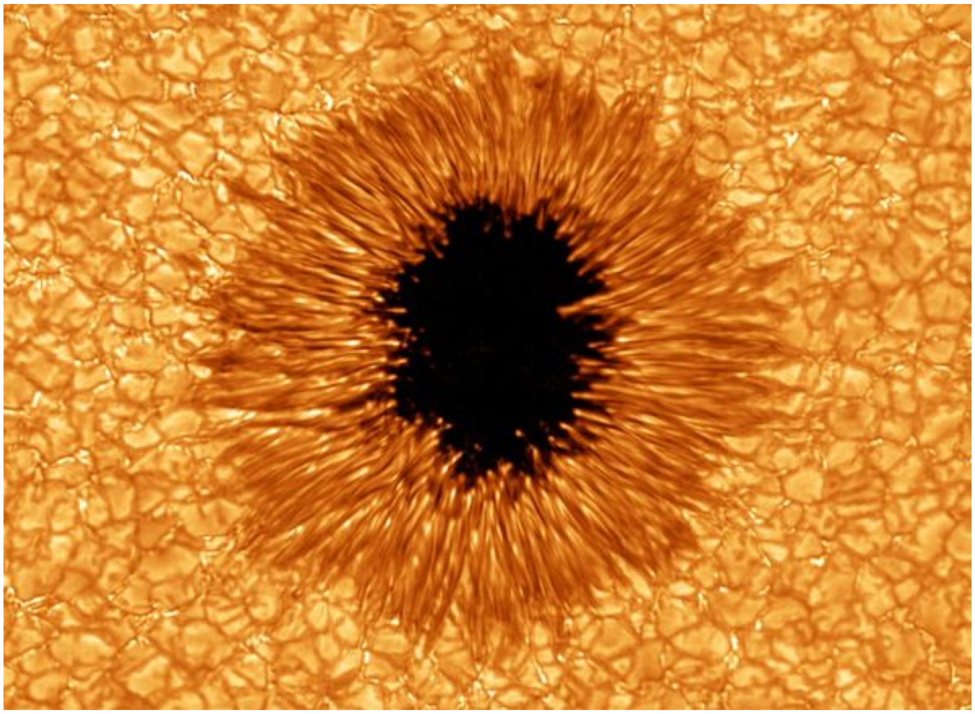
where n is in m^{-3}, T in K and B in G.

For example, in the solar corona, $T = 10^6$ K, $n = 10^{14}$ m^{-3}, $B = 10$ G, and $\beta = 3.5 \times 10^{-3}$.

In photospheric magnetic flux tubes, $T = 6 \times 10^3$ K, $n = 10^{23}$ m^{-3}, $B = 1000$ G, and $\beta = 2$.

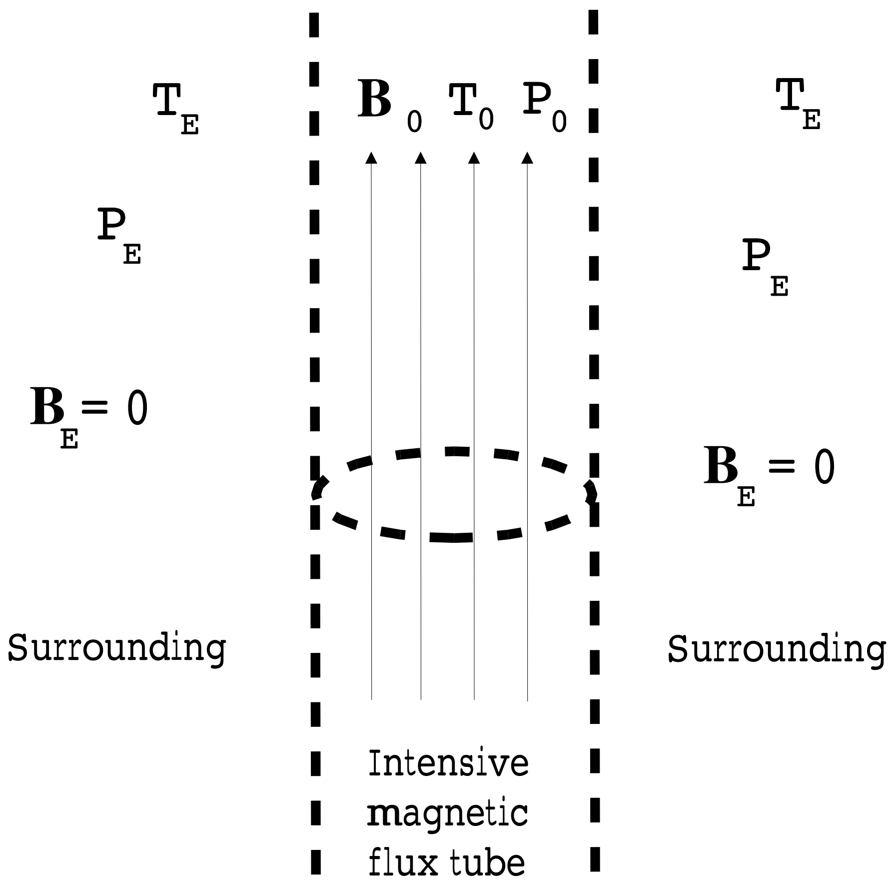
In the solar wind near the Earth's orbit, $T = 2 \times 10^5$ K, $n = 10^7$ m^{-3}, $B = 6 \times 10^{-5}$ G, and $\beta = 2$.

Example: Sunspots.



Sunspots appear as dark spots on the surface of the Sun. They typically last for several days, although very large ones may live for several weeks. Sunspots are magnetic regions on the Sun with magnetic field strengths thousands of times stronger than the Earth's magnetic field.

Consider a sunspot as a vertical magnetic flux tube. The magnetic field $\mathbf{B}_0$ is vertical. The kinetic pressure is P_0 and P_E inside and outside, respectively. The plasma temperature is T_0 inside the sunspot and T_E outside.



Sunspots are long-durational objects with no fast flows of plasma. So, it is naturally to describe their structure in terms of magnetostatics. As the magnetic field is not bent, the last term in Eq. (15), responsible for the magnetic tension, is zero. The equilibrium condition becomes

$$\nabla \left(P + \frac{B^2}{2\mu_0} \right) = 0, \tag{21}$$

This means that the total pressure must be equal inside and outside the sunspot,

$$P_E = P_0 + \frac{B_0^2}{2\mu_0}. \tag{22}$$

Let us assume that the density of the plasmas inside and outside the sunspot are equal, $\rho_0 = \rho_E$. Now, we divide Eq. (22) by ρ_0,

$$\frac{P_E}{\rho_E} = \frac{P_0}{\rho_0} + \frac{B_0^2}{2\mu_0\rho_0}. \tag{23}$$

Using the state equations,

$$P_E = 2\frac{k_B}{m_i}\rho_E T_E, \qquad P_0 = 2\frac{k_B}{m_i}\rho_0 T_0, \tag{24}$$

we obtain from Eq. (23)

$$\frac{2k_B}{m_i}T_E = \frac{2k_B}{m_i}T_0 + \frac{B_0^2}{2\mu_0\rho_0}. \tag{25}$$

This gives us

$$\frac{T_0}{T_E} = 1 - \frac{B_0^2}{2\mu_0}\frac{m_i}{2k_B\rho_E T_E} = 1 - \frac{B_0^2}{2\mu_0 P_E} \tag{26}$$

Thus, in a sunspot, $T_E > T_0$. Indeed, temperatures in the dark centers of sunspots drop to about 3700 K, compared to 5700 K for the surrounding photosphere. This is why sunspots are seen to be darker than the surrounding.

MHD Waves

Ideal MHD connects the magnetic field $\mathbf{B}$, plasma velocity $\mathbf{V}$, pressure P and density ρ:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{V}) = 0, \tag{27}$$

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} \right] = -\nabla P - \frac{1}{\mu_0} \mathbf{B} \times (\nabla \times \mathbf{B}), \tag{28}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}), \tag{29}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{P}{\rho^\gamma} \right) = 0. \tag{30}$$

Consider an equilibrium, described by the conditions

$$\frac{\partial}{\partial t} = 0, \qquad \mathbf{V} = 0, \tag{31}$$

which gives us the *magnetostatic* equation

$$\nabla P_0 + \frac{1}{\mu_0} \mathbf{B_0} \times (\nabla \times \mathbf{B_0}) = 0. \tag{32}$$

The simplest possible solution of the magnetostatic equation is a uniform plasma:

$$P_0 = \text{const}, \qquad B_0 = \text{const}, \tag{33}$$

and the equilibrium magnetic field $\mathbf{B_0}$ is straight.

Consider *small* perturbations of the equilibrium state:

$$\left. \begin{array}{l} \mathbf{B} = \mathbf{B_0} + \mathbf{B_1}(\mathbf{r}, t) \\ \mathbf{V} = 0 + \mathbf{V_1}(\mathbf{r}, t) \\ P = P_0 + P_1(\mathbf{r}, t) \\ \rho = \rho_0 + \rho_1(\mathbf{r}, t) \end{array} \right\} \tag{34}$$

Substitute these expressions into the MHD equations (27)–(30). Neglecting terms which contain a product of two or more values with indices "1", we obtain the set of MHD equations, *linearized* near the equilibrium (33):

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \mathbf{V_1} = 0, \tag{35}$$

$$\rho_0 \frac{\partial \mathbf{V_1}}{\partial t} = -\nabla P_1 - \frac{1}{\mu_0} \mathbf{B_0} \times (\nabla \times \mathbf{B_1}), \tag{36}$$

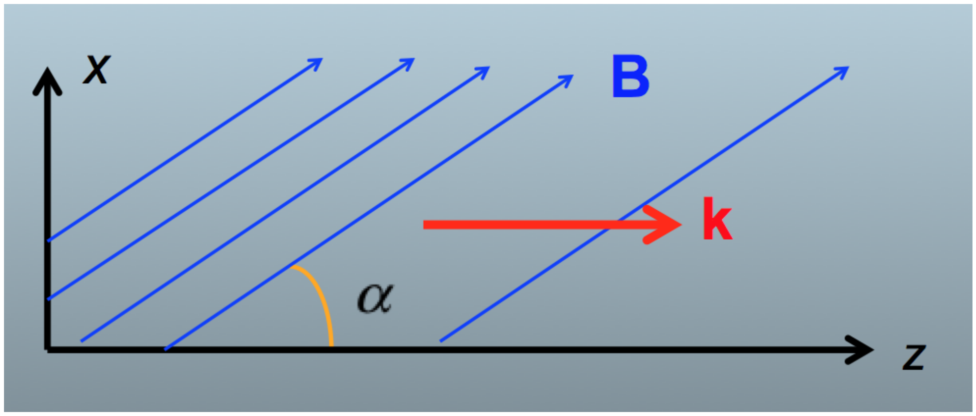
$$\frac{\partial P_1}{\partial t} - \frac{\gamma P_0}{\rho_0} \frac{\partial \rho_1}{\partial t} = 0, \tag{37}$$

$$\frac{\partial \mathbf{B_1}}{\partial t} = \nabla \times (\mathbf{V_1} \times \mathbf{B_0}), \tag{38}$$

Let the equilibrium magnetic field $\mathbf{B_0}$ be in xz-plane,

$$\mathbf{B_0} = B_0 \sin\alpha \, \mathbf{e_x} + B_0 \cos\alpha \, \mathbf{e_z}, \tag{39}$$

where α is the angle between the magnetic field and the unit vector $\mathbf{e_z}$:



Consider plane waves, propagating along $\mathbf{e_z}$, so that all perturbed quantities are proportional to $\exp(ikz - i\omega t)$. (This gives us $\partial/\partial t = -i\omega$ and $\nabla = ik$.) Projecting equations (35)–(38) onto the axes, we have

$$-i\omega\rho_1 + ik\rho_0 V_{z1} = 0, \tag{40}$$

$$-i\omega\rho_0 V_{x1} - \frac{ikB_0\cos\alpha}{\mu_0} B_{x1} = 0, \tag{41}$$

$$-i\omega\rho_0 V_{y1} - \frac{ikB_0\cos\alpha}{\mu_0} B_{y1} = 0, \tag{42}$$

$$-i\omega\rho_0 V_{z1} + ikP_1 + \frac{ikB_0\sin\alpha}{\mu_0} B_{x1} = 0, \tag{43}$$

$$-i\omega B_{x1} + ikB_0\sin\alpha V_{z1} - ikB_0\cos\alpha\, V_{x1} = 0, \tag{44}$$

$$-i\omega B_{y1} - ikB_0\cos\alpha\, V_{y1} = 0, \tag{45}$$

$$-i\omega B_{z1} = 0, \tag{46}$$

$$-i\omega P_1 + \frac{i\omega\gamma P_0}{\rho_0}\rho_1 = 0. \tag{47}$$

The set of equations (40)–(47) splits into two partial sub-sets. The first one is formed by equations (42) and (45), describing B_{y1} and V_{y1}. The consistency condition gives us

$$\omega^2 - C_A^2\cos^2\alpha\, k^2 = 0, \tag{48}$$

where $C_A \equiv B_0/(\mu_0\rho_0)^{1/2}$ is the Alfvén speed. This is dispersion relations for *Alfvén waves*.

Main properties of Alfvén waves:

- they are transverse, $\mathbf{V} \perp \mathbf{k}$;
- Alfvén waves can be linearly polarised, elliptically polarised, or circularly polarised;
- they are essentially incompressive: they do not modify the density of the plasma, $\nabla.\mathbf{V} = 0$;
- their group speed is always parallel to the magnetic field, $\mathbf{V}_{\text{group}} \parallel \mathbf{B_0}$; while the phase speed can be oblique to the field, $\mathbf{V}_{\text{phase}}$ may be $\nparallel \mathbf{B_0}$; $\mathbf{V}_{\text{phase}} \nparallel \mathbf{V}_{\text{group}}$;
- the absolute value of the group speed equals the Alfvén speed, C_A.

The second partial set of equations is formed by equations (40), (41), (43), (44) and (47) and describes variables V_{x1}, V_{z1}, B_{x1}, P_1 and ρ_1. The consistency condition gives us

$$(\omega^2 - C_A^2 \cos^2\alpha\ k^2)(\omega^2 - C_s^2 k^2) - C_A^2 \sin^2\alpha\,\omega^2 k^2 = 0, \tag{49}$$

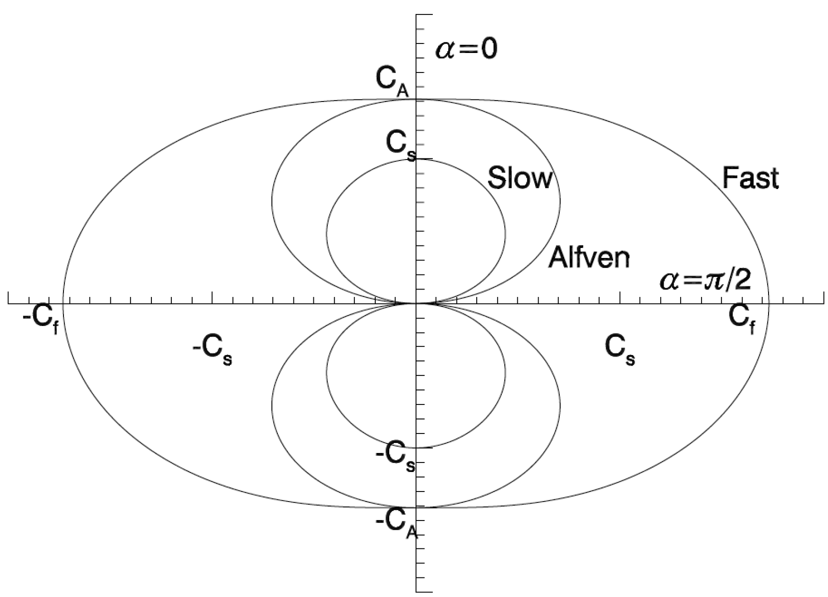
where $C_s \equiv (\gamma p_0/\rho_0)^{1/2}$ is the sound (or acoustic) speed.

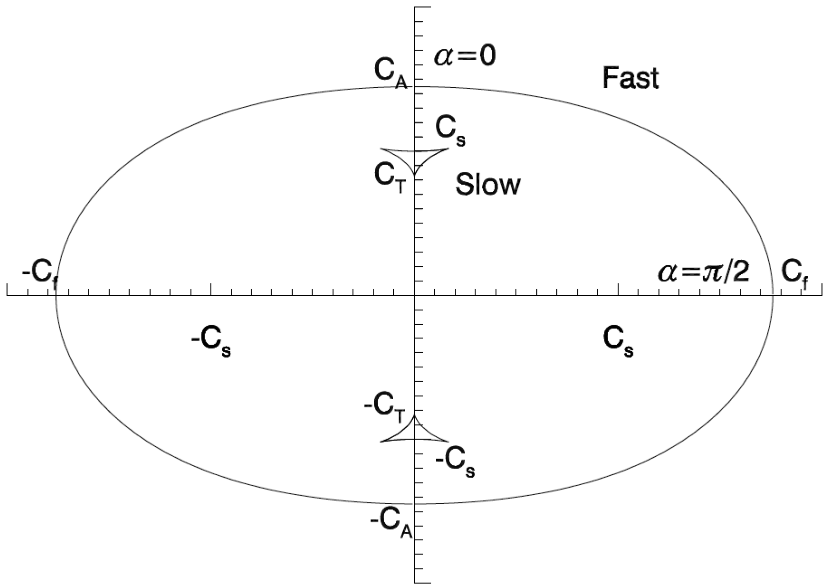
Equation (49) is bi-quadratic with respect to ω and consequently has two pairs of roots. They correspond to *fast* and *slow* *magnetoacoustic waves*.

Main properties of magnetoacoustic waves:

- they are longitudinal, $\mathbf{V}\|\mathbf{k}$;
- they are essentially compressive: they always perturb the density of the plasma;
- the fast wave can propagate in the direction perpendicular to the field at the speeds V_{phase} and V_{group} equal to the *fast* speed, $V_F \equiv \left(C_A^2 + C_s^2\right)^{1/2}$;
- the fast wave cannot propagate along the field — if $\mathbf{k}\|\mathbf{B_0}$ the fast wave becomes incompressive and degenerates to the Alfvén wave;
- in the $\beta < 1$ case, the slow wave propagates along the field at the speed C_s and degenerates to the usual acoustic wave; the slow wave cannot propagate across the field;
- in the slow wave the density and the absolute value of the magnetic field are perturbed in anti-phase, while in the fast wave in phase.

Polar plots for phase speeds (ω/k) and group speeds ($\mathrm{d}\omega/\mathrm{d}k$) for $\beta < 1$:





$\alpha=0$
C_A
Fast
C_s
C_T
Slow
$-C_f$
$\alpha=\pi/2$
C_f
$-C_s$
C_s
$-C_T$
$-C_s$
$-C_A$

Non-Ideal MHD Equations

We can account for non-ideal (e.g. dissipative) effects. In this case the set of MHD equations become

$$\rho\frac{\partial \mathbf{V}}{\partial t} + \rho(\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla P - \frac{1}{\mu}\mathbf{B}\times\nabla\times\mathbf{B} + \mathcal{F}, \tag{50}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{V}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}, \tag{51}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) = 0, \tag{52}$$

$$\frac{\rho^\gamma}{\gamma-1}\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = -\mathcal{L}. \tag{53}$$

The parameter η is the magnetic diffusivity, connected with the electrical conductivity σ,

$$\eta = 1/(\mu\sigma). \tag{54}$$

The term $\mathcal{F}$ is an external force acting on a unit of volume of the plasma. For example, if we take into account the gravity and the viscosity,

$$\mathcal{F} = -\rho\,\mathbf{g} + \nu\rho\left[\nabla^2\mathbf{V} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{V})\right], \tag{55}$$

where $\mathbf{g}$ is the gravity acceleration and ν is the coefficient of kinematic viscosity (assumed uniform).

Incompressible limit. Consider the situation when $\rho = \rho_0 = \text{const}$. Then, from the continuity Eq.,

$$0 = \frac{\partial \rho_0}{\partial t} = -\nabla\cdot(\rho_0\mathbf{V}) = 0, \tag{56}$$

$$\rho_0\nabla\cdot(\mathbf{V}) = 0, \qquad \nabla\cdot(\mathbf{V}) = 0 \tag{57}$$

Motions which satisfy this condition (e.g. Alfvén waves) are incompressible.

The ratio of specific heats γ is usually about 5/3. In some cases, isothermal ($T = \mathrm{const}$) processes can be considered, with $\gamma = 1$.

The righthand side of equation (53) contains the energy loss/gain function $\mathcal{L}$, discussed later. When $\mathcal{L} = 0$, the equation reduces to the adiabatic equation.

In addition, the electric current density $\mathbf{j}$, the electric field $\mathbf{E}$ and the temperature T can be determined from the equations:

$$\mathbf{j} = \nabla \times \mathbf{B}/\mu, \tag{58}$$

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \mathbf{j}/\sigma, \tag{59}$$

$$P = \frac{k_B}{m}\rho T, \tag{60}$$

where k_B is the Boltzmann constant and m is the mean particle mass.

In equation (60), the expression $k_B/m = \tilde{R}/\tilde{\mu}$, where $\tilde{R}$ and $\tilde{\mu} = m/m_p$ are the gas constant and the mean atomic weight (the average mass per particle in units of the proton mass), is often used.

If there are only protons and electrons, $n_e = n_p$,

$$m = \frac{n_e m_e + m_p n_p}{n_e + n_p} \approx \frac{m_p n_p}{n_e + n_p} = 0.5 m_p, \tag{61}$$

c.f. Eq. (4).

For example, in the solar corona the presence of He (alpha-particles) and other elements (in addition to H) makes $m/m_p = \tilde{\mu} \approx 0.6$.

The total number of particles per unit volume

$$n = \begin{array}{l} n_p + n_e \approx 2n_e \\ n_p + n_e + n_{\mathrm{other}} \approx 1.9 n_e \end{array} . \tag{62}$$

Consequently,

$$\rho = n_p m_p + n_e m_e + n_{\mathrm{other}} m_{\mathrm{other}} \approx n_e m_p \tag{63}$$

The adiabatic equation ((53) with $\mathcal{L} = 0$) can also be taken in several different forms, e.g.

$$\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\gamma p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0 \tag{64}$$

(Exercise: Derive it from equation (53))

Non-adiabatic effects in the energy equations

The right hand side of equation (53) is the energy loss/gain function,

$$\mathcal{L} = \nabla \cdot \mathbf{q} + L_{\mathrm{r}} - j^2/\sigma - H, \tag{65}$$

where

q is the heat flux due to the thermal conduction, $\mathbf{q} = -\kappa \nabla T$, with κ being the thermal conductivity.

L_{r} is the radiation function, in the optically thick plasma of the solar interior it is $L_{\mathrm{r}} = -\kappa_r \nabla^2 T$, with κ_r being the coefficient of radiative conductivity;

j^2/σ is the ohmic dissipation; and

H represents the sum of all the other heating sources.

In rarified and magnetised plasmas,

$$\mathbf{q} = -\hat{\kappa}\, \nabla T, \tag{66}$$

where $\hat{\kappa}$ is the thermal conduction tensor. In this case

$$\nabla \cdot \mathbf{q} = \nabla_{||} \cdot (\kappa_{||} \nabla_{||} T) + \nabla_{\perp} \cdot (\kappa_{\perp} \nabla_{\perp} T). \tag{67}$$

Thermal conduction <u>along</u> the field is primarily by electrons,

$$\kappa_{||} = 10^{-11} T^{5/2} \mathrm{W m^{-1} K^{-1}}. \tag{68}$$

Conduction perpendicular to the field is mainly by protons, and

$$\frac{\kappa_{\perp}}{\kappa_{||}} = 2 \times 10^{-31} \frac{n^2}{T^3 B^2}, \tag{69}$$

where the field is in teslas.

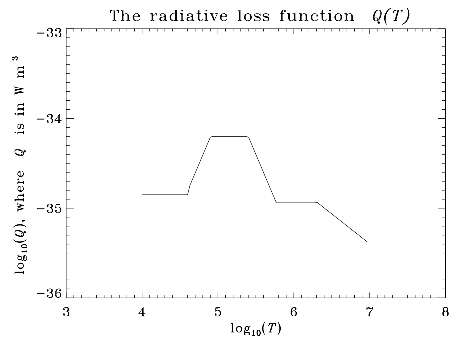
In strongly magnetised plasmas, thermal conduction across the magnetic field is dramatically depressed.

Estimate the ratio for the coronal parameters, $n = 10^{15}$ m^{-3}, $T = 10^6$ K and $B = 10$ G $= 10^{-3}$ T.

In the *optically thin* part of the solar atmosphere (where $T \geq 2 \times 10^4$ K)
(what are the parts of the atmosphere where this condition is fulfilled?)
the radiation function takes the form

$$L_{\mathrm{r}} = n_{\mathrm{e}} n_{\mathrm{H}} Q(T) \approx n_{\mathrm{e}}^2 Q(T), \tag{70}$$

where n_{e} is the electron concentration and n_{H} is the hydrogen concentration, ($n_{\mathrm{e}} \approx n_{\mathrm{H}}$) and $Q(T)$ is a function of temperature T. Often, the function can be approximated as χT^{α}, where χ and α are constant:



When the plasma pressure P remains constant (isobaric processes), a convenient alternative form of equation (53) is

$$\rho c_{\mathrm{p}} \frac{\mathrm{d}T}{\mathrm{d}t} = -\mathcal{L}, \tag{71}$$

where c_{p} is specific heat at constant pressure,

$$c_{\mathrm{p}} = \frac{\gamma}{\gamma - 1} \frac{k_{\mathrm{B}}}{m}.$$

4. Energetics

In MHD, three different types of energy are considered:

- internal energy $\leftrightarrow$ entropy,
- EM energy $\leftrightarrow$ Poynting flux,
- mechanical energy $\leftrightarrow$ kinetic energy

$$\begin{array}{c}\text{increase}\\ \text{in entropy}\end{array} = \text{heat flux} - \text{radiation} + \text{heat sources}$$

$$\begin{array}{c}\text{inflow of EM}\\ \text{energy } \mathbf{E}\times\mathbf{H}\end{array} = \begin{array}{c}\text{electrical}\\ \text{energy } \mathbf{E}\cdot\mathbf{j}\end{array} + \begin{array}{c}\text{a rise in magnetic}\\ \text{energy } B^2/2\mu\end{array}$$

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{V^2}{2}\right) = -\mathbf{V}\cdot\nabla P + \mathbf{V}\cdot\mathbf{j}\times\mathbf{B} + \mathbf{V}\cdot\mathcal{F}$$

Consequences of the Induction Equation

Consider the induction equation with the diffusive term

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \tag{72}$$

Compare the right hand side terms.

Let the plasma have the typical speed V_0 and the length scale l_0, then

$$\text{“convective term”:} \quad \nabla \times (\mathbf{V} \times \mathbf{B}) \approx \frac{V_0 B}{l_0}, \tag{73}$$

$$\text{“diffusive term”:} \quad \eta \nabla^2 \mathbf{B} = \frac{\eta B}{l_0^2}. \tag{74}$$

Their ratio is

$$\frac{V_0 B}{l_0} \frac{l_0^2}{\eta B} = \frac{l_0 V_0}{\eta} = \mathcal{R}_m. \tag{75}$$

This dimensionless parameter $\mathcal{R}_m$ is called the magnetic Reynolds number.

1. Diffusive Limit

If $\mathcal{R}_m \ll 1$, the convective term can be neglected with respect to the diffusive term, and the induction equation becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}. \tag{76}$$

This is the *diffusion equation.* It implies that field variations on a length scale l_0 are destroyed over a diffusion time scale,

$$\tau_{\rm d} = l_0^2/\eta. \tag{77}$$

The smaller the length-scale, the faster the magnetic field diffuses away.

In a fully-ionised plasma,

$$\tau_{\rm d} \approx 10^{-9} l_0^2 T^{3/2}, \tag{78}$$

where the length scale is in m and the temperature in in K.

E.g., in the solar corona, $T = 10^6$ K, and the typical length scale is 1 Mm$= 10^6$ m, thus

$$\tau_{\rm d} \approx 10^{-9} 10^{12} 10^9 = 10^{12} \text{ s} = 30,000 \text{ years (!!)} \tag{79}$$

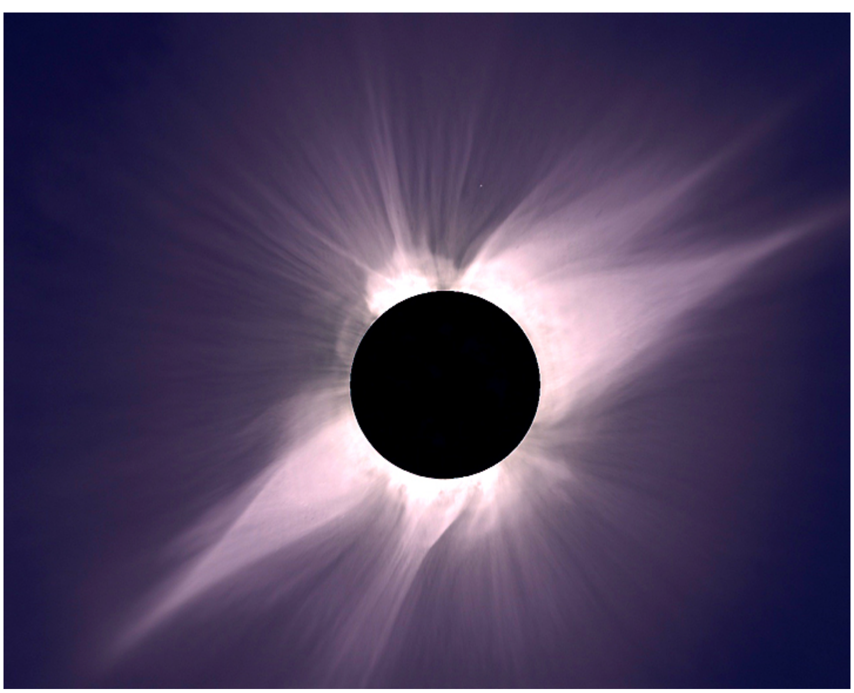
Since solar flares represent a release of magnetic energy over a time-scale of 100 or 1000 s, it seems that a length-scale as small as

$$\tau_{\rm d} = 10^2 \approx 10^{-9} l_0^2 10^9 \quad \rightarrow \quad l_0 \approx 10 - 100 \text{ m} \tag{80}$$

Example: Consider the diffusion of a unidirectional magnetic field $\mathbf{B} = B(x,t)\mathbf{e_y}$ with the initial step-function profile:

$$B(x,0) = \begin{cases} +B_0, x > 0 \\ -B_0, x < 0 \end{cases} \tag{81}$$

("a current sheet", e.g. in helmet streamers).



(Image of the corona taken during a solar eclipse. Several helmet streamers are well seen.)

Suppose the field remains unidirectional. Then Eq. (76) becomes

$$\frac{\partial B}{\partial t} = \eta \frac{\partial^2 B}{\partial x^2}. \tag{82}$$

The PDE should be supplemented by initial conditions (81) and the boundary conditions, e.g.

$$B(\pm\infty, t) = \pm B_0. \tag{83}$$

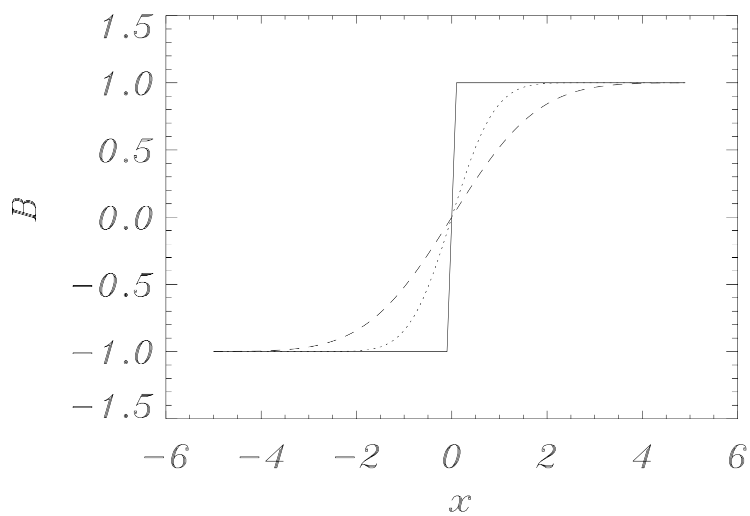
The solution which satisfies the boundary conditions is

$$B(x,t) = B_0 \,\mathrm{erf}(\xi), \tag{84}$$

where $\xi = x/(4\eta t)^{1/2}$, and

$$\mathrm{erf}(\xi) = \frac{2}{\pi^{1/2}} \int_0^\xi \exp(-u^2)\,\mathrm{d}u, \tag{85}$$

is the error function.



(Here, there are three curves shown, corresponding to $t = 0$, $t = 1$ and $t = 2$. Mark the curves in Figure with the appropriate value of t.)

The gradient of the magnetic field causes the current:

$$\mathbf{j} = \frac{1}{\mu}\nabla \times \mathbf{B} \quad \Rightarrow \quad j_z = \frac{1}{\mu}\frac{\mathrm{d}B}{\mathrm{d}x} \tag{86}$$

What happens with the other components of the current density? Draw the structure of the current in Figure for different times.

The width of the current sheet behaves like $l = 4(\eta t)^{1/2}$. Notice that the field density at large distances remains constant in time. The field lines in the sheet are not moving outwards, since those at large distances are unaffected. Rather, the field in the sheet is diffusing away, and so it is being annihilated. (The magnetic energy is being converted into heat by ohmic dissipation).

2. Perfectly Conducting Limit

When $\mathcal{R}_m \gg 1$, the induction equation reduces to

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) \tag{87}$$

In this large magnetic Reynolds number limit, the *frozen-flux (Alfvén's) theorem* holds:

In a perfectly conducting plasma, magnetic field lines behave as if they move with the plasma.

In other words:

The total amount of magnetic flux passing through any closed circuit moving with the local fluid velocity is constant in time

Proof: We show the above statement by proving that the time rate of change of the magnetic flux through such a circuit is zero.

Consider a closed curve C bounding a surface $\mathbf{S}$ which is moving with the plasma.

The magnetic flux of the field $\mathbf{B}$ through the elementary area A is

$$\mathbf{B} \cdot \mathbf{n}\, \mathrm{d}A,$$

where $\mathrm{d}A$ is a differential of area enclosed within the circuit and $\mathbf{n}$ is the unit vector normal to A.

The flux through a circuit may change if

either the field strength at a point enclosed by the circuit changes

or the motion of the boundary results in change in the amount of the field enclosed.

The first type of change is given by

$$\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, \mathrm{d}A. \tag{88}$$

The total change is given by integrating over the entire surface S.

The second type of change can be visualised by imagining a piece $\mathbf{dl}$ of boundary moving at velocity $\mathbf{V}$ past a magnetic field $\mathbf{B}$. The change in the amount of flux enclosed within the area bounded by the curve due to the motion of $\mathbf{dl}$ is

$$\mathbf{B} \cdot (\mathbf{V} \times \mathbf{dl}). \tag{89}$$

Using the vector identity

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c},$$

we can rewrite Eq. (89) as

$$\mathbf{B} \cdot (\mathbf{V} \times \mathbf{dl}) = (\mathbf{B} \times \mathbf{V}) \cdot \mathbf{dl}) = -(\mathbf{V} \times \mathbf{B}) \cdot \mathbf{dl}. \tag{90}$$

The total change in flux through the circuit due to motion of the contour C, is obtained by integrating the above expression around C.

Combining both the effects, the total change of the flux through a circuit:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathbf{n}\, \mathrm{d}A = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \mathrm{d}A - \oint_C (\mathbf{V} \times \mathbf{B}) \cdot \mathbf{dl}. \tag{91}$$

We can then use the Stokes theorem,

$$\int_S (\nabla \times \mathbf{Q}) \cdot \mathbf{n}\, \mathrm{d}A = \oint_C \mathbf{Q} \cdot \mathbf{dl},$$

and get

$$\oint_C (\mathbf{V} \times \mathbf{B}) \cdot \mathbf{dl} = \int_S [\nabla \times (\mathbf{V} \times \mathbf{B})] \cdot \mathbf{n} \, \mathrm{d}A. \tag{92}$$

Consequently, Eq. (91) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathbf{n} \, \mathrm{d}A = \int_S \left[\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{V} \times \mathbf{B}) \right] \cdot \mathbf{n} \, \mathrm{d}A, \tag{93}$$

and, together with the induction equation, it gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathbf{n} \, \mathrm{d}A = 0. \tag{94}$$

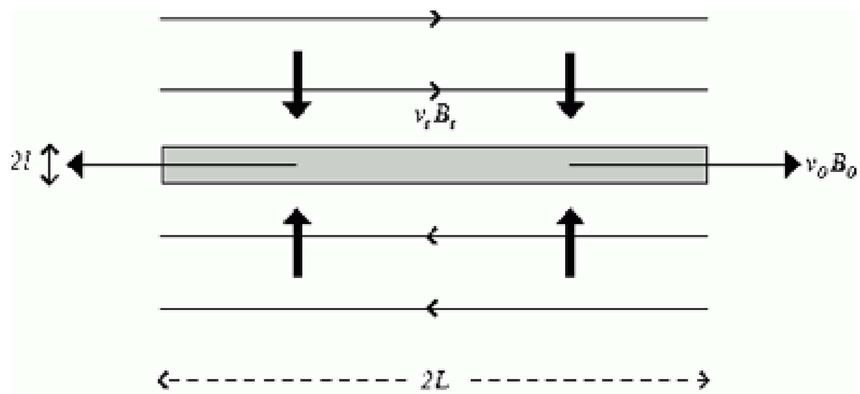
Thus, the magnetic flux passing through the circuit C is constant.

Consequently, the magnetic field lines are <u>frozen into the plasma</u>: plasma can move freely <u>along</u> field lines, but, in motion <u>perpendicular</u> to them, either the field lines are dragged with the plasma or the field lines push the plasma.

Alfvén's Theorem prohibits reconnection of magnetic field lines.

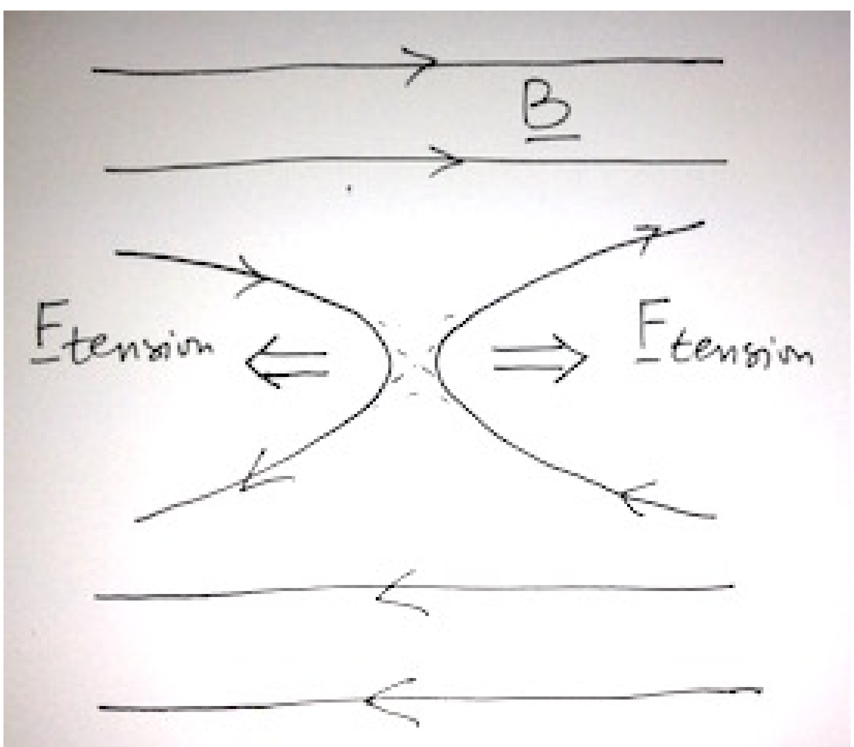
Magnetic reconnection

Consider a current sheet. In its very vicinity the resistivity can be taken as finite.



The plasma diffuses into the current layer at some relatively small inflow velocity V_{i}. (More specifically: there is the total pressure balance across the current sheet; in the vicinity of the current sheet there are large gradients of the field and hence diffusion; the total pressure outside the current sheet is getting higher, resulting into a pressure gradient forces, moving field lines toward the current sheet from the top and bottom).

In the current sheet the oppositely directed magnetic field lines get reconnected, resulting into the magnetic tension forces in the horizontal direction. These forces drive the frozen-in plasma — the **sling shot effect**.



The plasma is accelerated along the layer (in the sketch in the horizontal direction), and eventually expelled from its ends at some relatively large velocity V_{o}, which can be shown to be about the Alfvén speed C_{A}. The (incompressible) mass conservation condition gives us $LV_{\mathrm{i}} = lV_{\mathrm{o}}$, hence $V_{\mathrm{i}} \ll V_{\mathrm{o}}$.

It is the Sweet–Parker stationary reconnection.

Energy conversion in magnetic reconnection:

- The input energy is the energy stored in the magnetic field.
- Change of $\mathbf{B}$ because of reconnection generates steep gradients of $\mathbf{B}$, hence increase in $\nabla \times \mathbf{B}$. It leads to the increase in the current density $\mathbf{j}$.
- As the diffusivity is not negligible in the reconnection region (in the vicinity of the current sheet), the current is subject to Ohmic dissipation, hence increase in internal energy of the plasma.
- Also, the slingshot effect generates bulk flows of plasma, hence increase in its kinetic energy.
- The electric field $\mathbf{E} = -\mathbf{V} \times \mathbf{B}$ accelerates plasma particles: non-thermal high energy particles.

There is however a problem: for typical parameters of the corona, the characteristic time of energy release by magnetic reconnection is about a few tens of days. This is too long to explain dynamical phenomena (e.g. flares and CME) in the solar atmosphere. The problem of "fast reconnection" is one of the key problems of modern solar and space plasma physics. Possible solutions: anomalous resistivity, non-MHD processes...

Hydrostatic Pressure Balance

The magnetohydrostatic equilibrium condition is

$$0 = -\nabla P + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g}, \tag{95}$$

coupled with

$$\nabla \cdot \mathbf{B} = 0, \tag{96}$$

$$\mu \mathbf{j} = \nabla \times \mathbf{B}, \tag{97}$$

$$P = \frac{\rho R T}{\tilde{\mu}}, \tag{98}$$

and T satisfies an energy equation.

Before investigating any specific phenomena we need to consider the basic pressure balance when the magnetic field does not exert any force.

Consider the simple case of a uniform vertical magnetic field. For simplicity we assume that the temperature is known.

Thus,

$$\mathbf{B} = B_0 \hat{\mathbf{z}}, \qquad \mathbf{g} = -g \hat{\mathbf{z}}.$$

Hence, $\mathbf{j} = 0$ and there is no Lorentz force.

In addition, the pressure is $P = P(z)$ and (95) becomes

$$\frac{\mathrm{d}P}{\mathrm{d}z} = -\rho(z) g = -\frac{g\tilde{\mu}}{RT(z)} P(z) = -\frac{P(z)}{\Lambda(z)}, \tag{99}$$

where

$$\Lambda(z) = \frac{RT(z)}{\tilde{\mu} g}, \tag{100}$$

is the pressure scale height.

Eq. (99) is a separable, first order ordinary differential equation so that

$$\frac{\mathrm{d}P}{P} = -\frac{1}{\Lambda(z)} \mathrm{d}z,$$

and integrating gives

$$\log P = -n(z) + \log P(0),$$

where

$$n(z) = \int_0^z \frac{1}{\Lambda(u)} du,$$

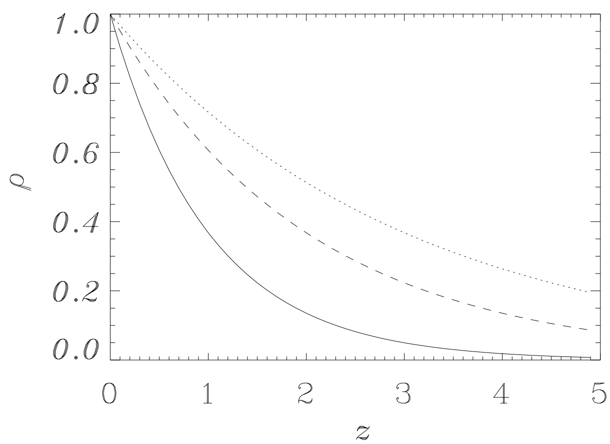
is the 'integrated number' of scale heights between the arbitrary level at which the pressure is $P(0)$ and the height z. Therefore,

$$p(z) = p(0) \exp\left[-n(z)\right]. \tag{101}$$

If the atmosphere is isothermal so that both T and Λ are constant, then (101) gives

$$P(z) = P(0) \exp\left(-z/\Lambda\right), \qquad \rho(z) = \rho(0) \exp\left[-z/\Lambda\right], \tag{102}$$

so that the pressure decreases exponentially on a typical length scale given by the pressure scale height Λ:



(Here, three curves are shown, corresponding to $\Lambda = 1$, $\Lambda = 2$ and $\Lambda = 3$. Mark the curves in the figure with the appropriate value of Λ.)

Consider typical values of the pressure scale height. Taking the solar gravitational constant as $g = 274\ \mathrm{ms}^{-2}$ and $R = 8.3 \times 10^3\ \mathrm{J\ K}^{-1}\ \mathrm{mol}^{-1}$ then Λ takes the following values:

1. In the photosphere $T = 6,000$ K and $\tilde{\mu} = 1.3$ so that

$$\Lambda = \frac{\mathrm{R}T}{\tilde{\mu} g} = \frac{8.3 \times 10^3 \times 6 \times 10^3}{1.3 \times 274} = 140 \text{ km}.$$

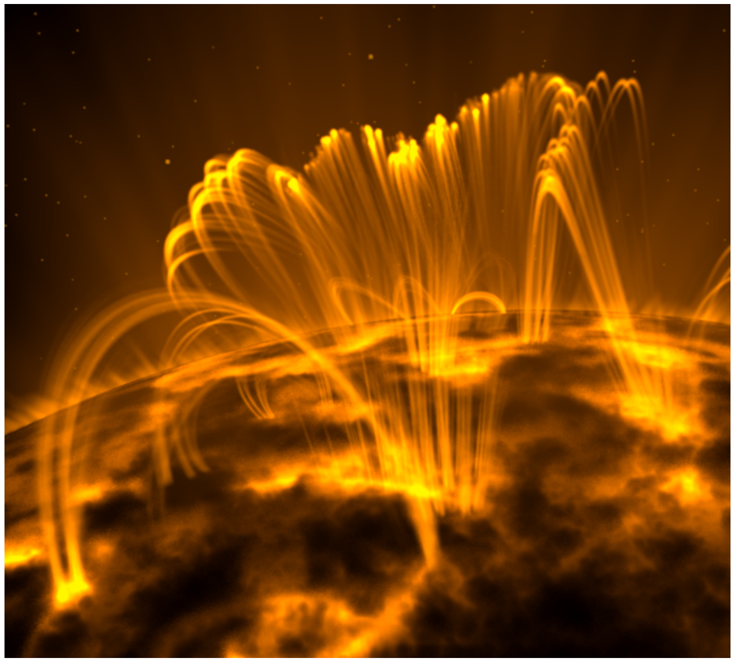
2. In the corona $T > 10^6$ K and $\tilde{\mu} = 0.6$ giving

$$\Lambda = \frac{\mathrm{R}T}{\tilde{\mu} g} = \frac{8.3 \times 10^3 T}{0.5 \times 274} \approx 50.5T \text{ m}.$$

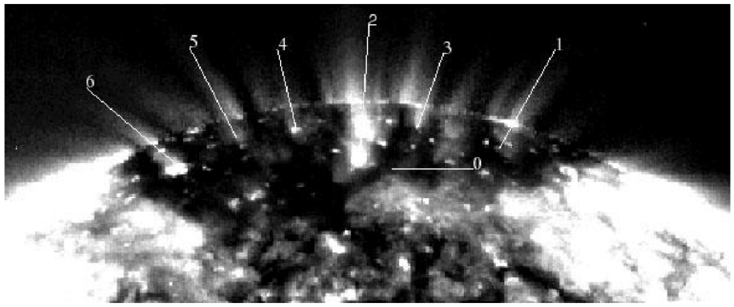
   Thus, the scale height can be estimated as

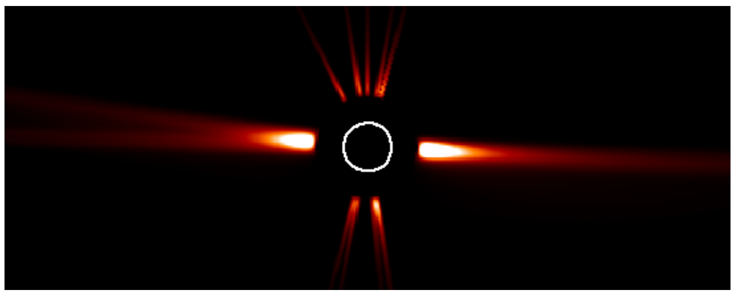
$$\Lambda/\mathrm{Mm} \approx 50T/\mathrm{MK}.$$

   E.g., the scale height of the corona observed by TRACE-171Å (the temperature is 1 MK) is 50 Mm. *(This figure is comparable to the size of a loop).*

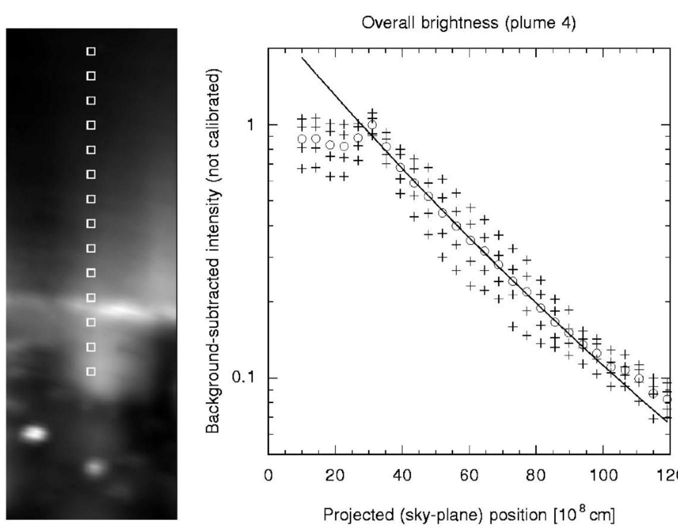


Example: Density stratification in a polar plume.





Polar plumes are cool, dense, linear, magnetically open structures that arise from predominantly unipolar magnetic footpoints in the solar polar coronal holes.



The solid line shows the hydrostatic solution for $T = 10^6$ K.

3. In the Earth's atmosphere $T = 300$ K, $g = 9.81$ ms^{-1}, $\tilde{\mu} = 29$ in air and so

$$\Lambda = \frac{8.3 \times 300}{29 \times 9.81} = 8.7 \text{ km}.$$

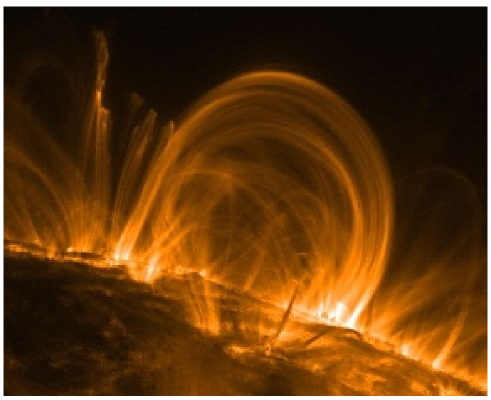
Note that the height of Mount Everest is about 8.8 km. Thus, the air pressure at the summit of Everest is about $1/e = 0.37$ that of the air pressure at sea level.

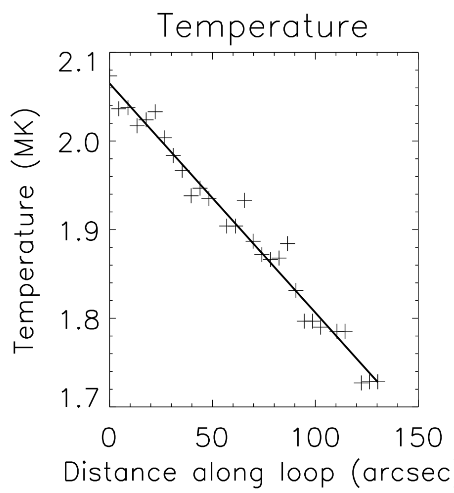
Thermal equilibrium

In more realistic models, the hydrostatic equilibrium should be supplemented with the thermal equilibrium between thermal conduction, radiation and heating (see Eq. (65)):

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\kappa_0 T^{5/2}\frac{\mathrm{d}T}{\mathrm{d}s}\right) = \chi n_e^2 T^{-1/2} - H, \tag{103}$$

where s is the coordinate along the magnetic field.





In particular, for short coronal loops, with the major radius shorter than the scale height of the stratification, ($R_\mathrm{L} < \Lambda$) the loop pressure $p(s)$ can be taken to be constant,

$$P(s) = P_0 \tag{104}$$

and, consequently, from the state equation, the density is

$$n_\mathrm{e}(s) = P_0/2k_\mathrm{B}T(s). \tag{105}$$

Assuming that all three terms in Eq. (103) are of the same order, we get, comparing the terms on RHS of Eq. (103),

$$H \approx \frac{P_0^2 \chi T^{-5/2}}{4k_\mathrm{B}^2}. \tag{106}$$

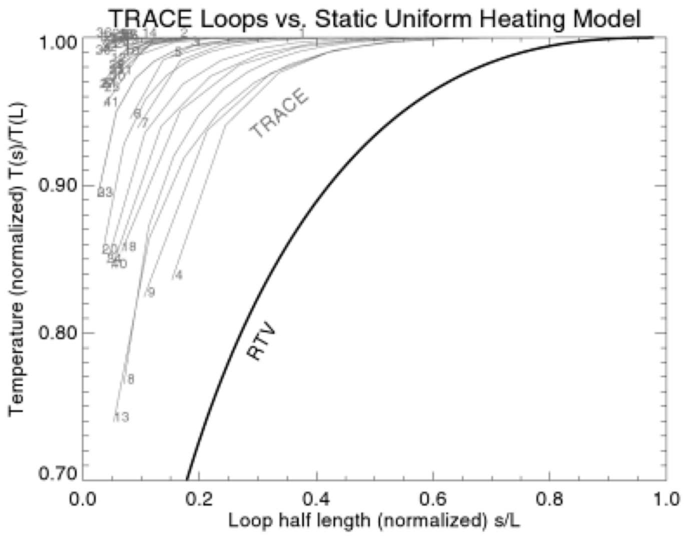
and the LHS and RHS terms,

$$\frac{\kappa_0 T^{7/2}}{R_\mathrm{L}^2} \approx \frac{P_0^2 \chi T^{-5/2}}{4k_\mathrm{B}^2}, \tag{107}$$

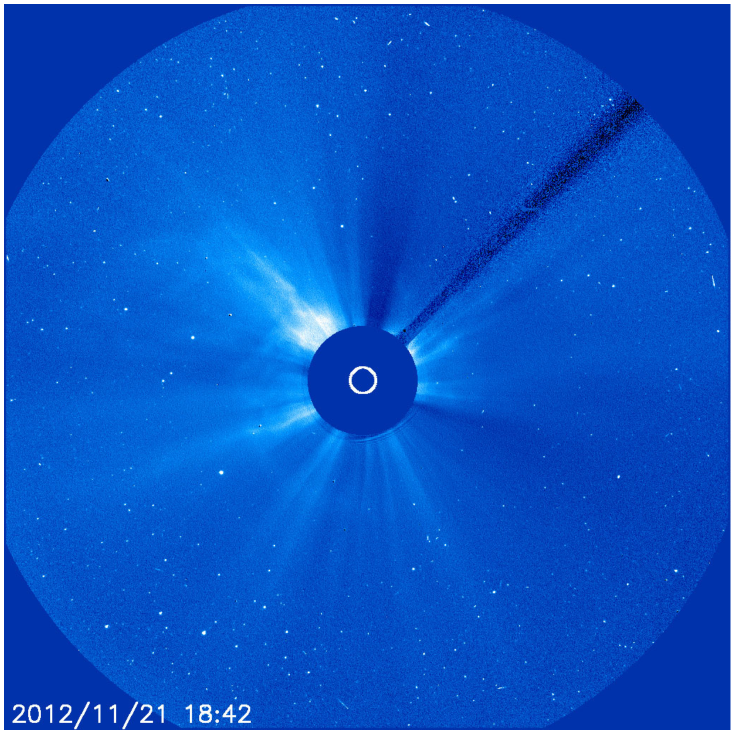
the scaling law:

$$T \propto (P_0 R_\mathrm{L})^{1/3}. \tag{108}$$

This scaling law is called *RTV* (after Rosner, Tucker & Vaiana) was, as the zero-order approximation, confirmed observationally in the soft X-ray and EUV band:


TRACE Loops vs. Static Uniform Heating Model
Temperature (normalized) T(s)/T(L)
Loop half length (normalized) s/L
TRACE
RTV
1.00
0.90
0.80
0.70
0.0
0.2
0.4
0.6
0.8
1.0

Hydrostatic equilibrium at larger heights



(Image of the solar corona taken with a coronograph).

Determination of the density stratification on larger scale, e.g. in coronal holes, requires taking into account the effects of the spherical geometry and the change of the gravitational acceleration with height,

$$g(r) = \frac{GM_\odot}{r^2} \tag{109}$$

where r is the radial coordinate.

The magnetic field is assumed to be strictly radial,

$$B = \frac{B_0 R_\odot^2}{r^2}. \tag{110}$$

In the following we consider spherically symmetric isothermal ($T = \text{const}$) atmosphere.

Again, there is no the Lorenz force. The magnetostatic equation is similar to (99),

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\rho(r) g(r), \tag{111}$$

which, with the use of the state equation,

$$P = \frac{\rho \mathrm{R} T}{\tilde{\mu}}$$

can be rewritten as

$$\frac{\mathrm{d}\rho(r)}{\mathrm{d}r} = -\frac{R_\odot^2}{r^2}\frac{1}{\Lambda}\rho(r). \tag{112}$$

Here, the scale height Λ was defined by substituting the value of the gravitational acceleration $g(R_\odot) = GM_\odot/R_\odot^2$ at the solar surface into Eq. (100).

ODE (112) is separable,

$$\int \frac{\mathrm{d}\rho}{\rho} = -\int \frac{R_\odot^2}{\Lambda r^2}\,\mathrm{d}r \tag{113}$$

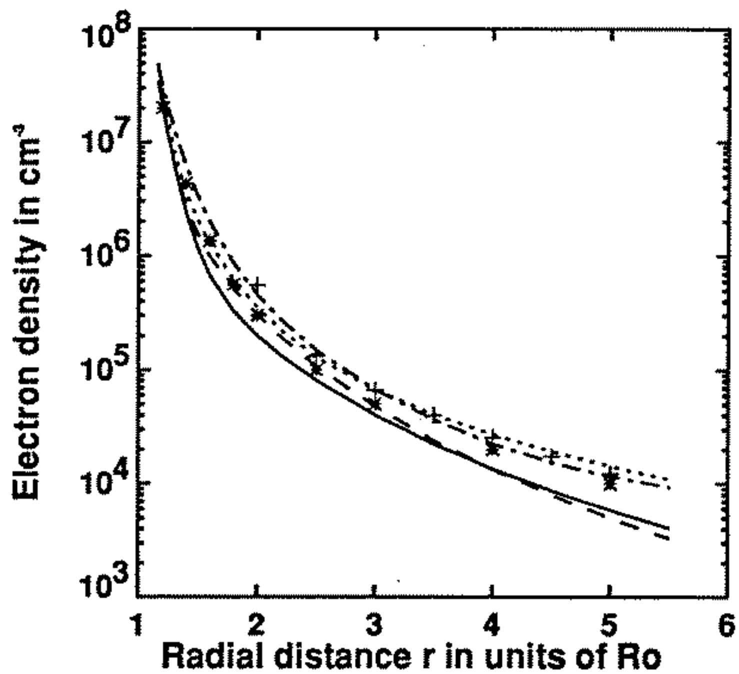
with the solution

$$\rho(r) = C \exp\left(\frac{R_\odot^2}{\Lambda r}\right). \tag{114}$$

Determining the constant C from the condition $\rho(R_\odot) = \rho_0$, we obtain

$$\rho(r) = \rho_0 \exp\left[\frac{R_\odot}{\Lambda}\left(\frac{R_\odot}{r} - 1\right)\right]. \tag{115}$$

This solution coincides well with the observationally determined empirical dependence. E.g., the profile of plasma concentration in polar coronal holes determined with SPARTAN 201-01 (from Fisher & Guhathakurta 1995):



An empirical model was constructed by Esser et al. (1999),

$$n_\mathrm{e} = \frac{2.494 \times 10^6}{r^{3.76}} + \frac{1.034 \times 10^7}{r^{9.64}} + \frac{3.711 \times 10^8}{r^{16.86}}, \tag{116}$$

which corresponds to the theoretical dependence reasonably well.

Notice that Eq. (115) gives infinite density at $r \to \infty$, so can be applied at the heights below 5–6 $R_\odot$ only. At larger distances from the Sun steady flows of plasma (the solar wind) must be accounted for.

Potential and Force-Free Fields
Magnetic Field Lines

If the magnetic field $\mathbf{B} = (B_x, B_y, B_z)$ is known as a function of position, then the magnetic lines of force, called the magnetic field lines, are defined by

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} = \frac{ds}{B}. \tag{117}$$

The solution to (117), a system of ordinary differential equations, defines a curve in three dimensional space that is the field line. In parametric form, in terms of the parameter s, the field lines satisfy

$$\frac{dx}{ds} = \frac{B_x}{B}, \qquad \frac{dy}{ds} = \frac{B_y}{B}, \qquad \frac{dz}{ds} = \frac{B_z}{B}, \tag{118}$$

where the parameter s is the distance along the field line.

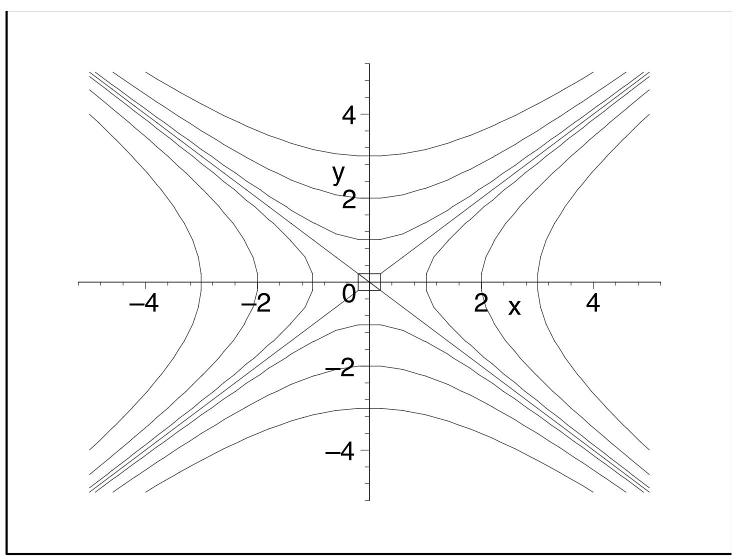
Example: Consider the field given by $\mathbf{B} = B_0(y/a, x/a, 0)$, where B_0 and a are constants, calculate the equations of the field lines. Using (117), the field lines are given by

$$\frac{dx}{(y/a)} = \frac{dy}{(x/a)}, \qquad \Rightarrow xdx = ydy,$$

and so

$$x^2 - y^2 = \pm c^2 = \text{constant}.$$

Therefore the field lines are hyperbolae.



This is a neutral point or an X-point.

Potential Fields

If $\beta \ll 1$, we may also neglect the gas pressure (with respect to the magnetic pressure!) magnetostatic equation (95) reduces to the low β plasma approximation

$$\mathbf{j} \times \mathbf{B} = 0 \tag{119}$$

and the magnetic field is called force-free.

A simple solution to (119) is given by assuming that the current density $\mathbf{j}$ is identically zero so that the magnetic field is potential. Thus, the field must satisfy the conditions

$$\mathbf{j} = \frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{0}, \tag{120}$$

and

$$\nabla \cdot \mathbf{B} = 0, \tag{121}$$

The most general solution to (120) is

$$\mathbf{B} = \nabla\phi, \tag{122}$$

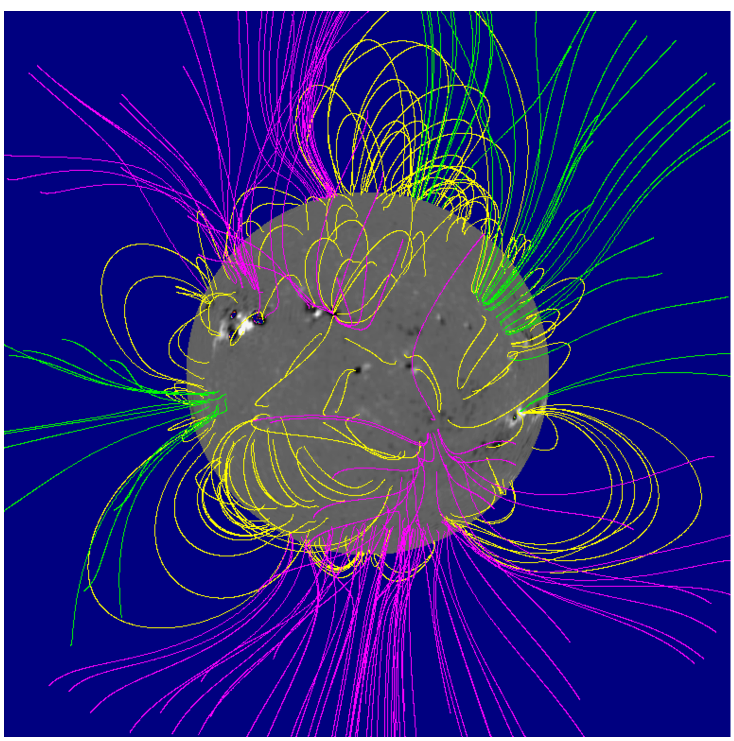
where ϕ is the scalar magnetic potential.

Substituting (122) into condition (121), we get

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0, \tag{123}$$

which is Laplace's equation. It is commonly used for the determination of the basic geometry of the magnetic field. Eq. (123).

Example: Potential extrapolation of the coronal field Solving it we obtain the magnetic field geometry e.g. in the corona. Measurements of the magnetic field at the photosphere provide us with a boundary condition for



However, potential fields do not have electric currents that are necessary for plasma heating and impulsive energy releases, e.g. flares and coronal mass ejections (CME).

Force-Free Fields

Again, we assume that $\lambda \ll \Lambda$ and $\beta \ll 1$ and we again have the force-free field equation (119). If the magnetic field is <u>not</u> potential ($|\mathbf{j}| \neq 0$) then the general solution is that the current must be parallel to the magnetic field. Thus,

$$\mu_0 \mathbf{j} = \alpha \mathbf{B}, \qquad \Rightarrow \nabla \times \mathbf{B} = \alpha \mathbf{B}, \tag{124}$$

for some scalar function α which may be a function of position and time.

Property of α: The scalar function $\alpha(\mathbf{r})$ is not completely arbitrary since $\mathbf{B}$ must satisfy the conditions:

- $\nabla \cdot \mathbf{B} = 0$ and
- the vector identity $\nabla \cdot (\nabla \times \mathbf{B}) = 0$.

So using (124) we obtain

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{B}) &= \nabla \cdot (\alpha \mathbf{B}) \\ &= \alpha \nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla \alpha. \end{aligned}$$

Hence,

$$\mathbf{B} \cdot \nabla \alpha = 0, \tag{125}$$

so that α is constant along each field line, although it may vary from field line to field line. If $\alpha = 0$, then the magnetic field reduces to the potential case already considered.

If α is constant everywhere then

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \qquad \Rightarrow \nabla \times (\nabla \times \mathbf{B}) = \nabla \times (\alpha \mathbf{B}) = \alpha \nabla \times \mathbf{B} = \alpha^2 \mathbf{B}.$$

However, $\nabla \times \nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$ and so

$$-\nabla^2 \mathbf{B} = \alpha^2 \mathbf{B}. \tag{126}$$

This is a Helmholtz equation.

If α is a function of position, i.e. $\alpha(\mathbf{r})$, then we have

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{B}) &= \nabla \times (\alpha \mathbf{B}) = \alpha \nabla \times \mathbf{B} + \nabla \alpha \times \mathbf{B} \\ &= \alpha^2 \mathbf{B} + \nabla \alpha \times \mathbf{B} \end{aligned}$$

Hence, we get two coupled equations for $\mathbf{B}$ and α, namely

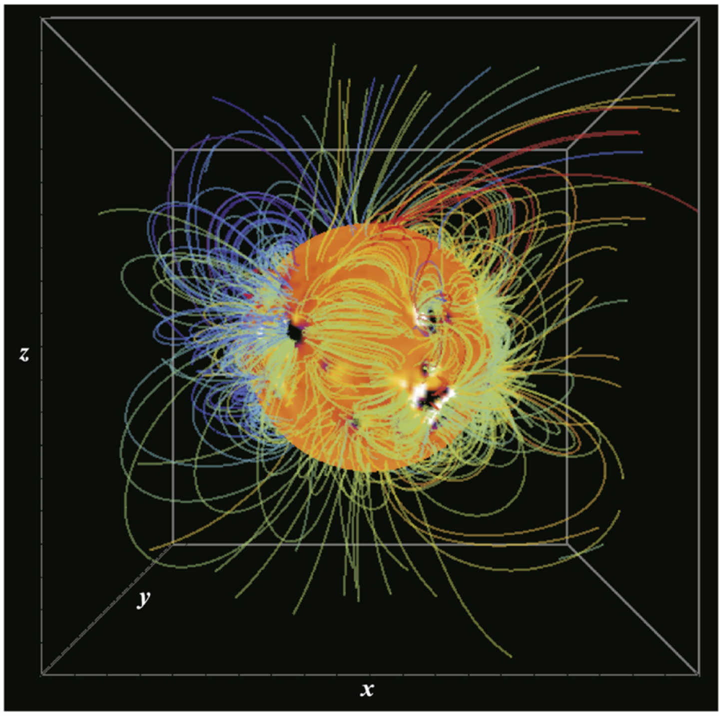
$$\nabla^2 \mathbf{B} + \alpha^2 \mathbf{B} = \mathbf{B} \times \nabla \alpha, \tag{127}$$

and

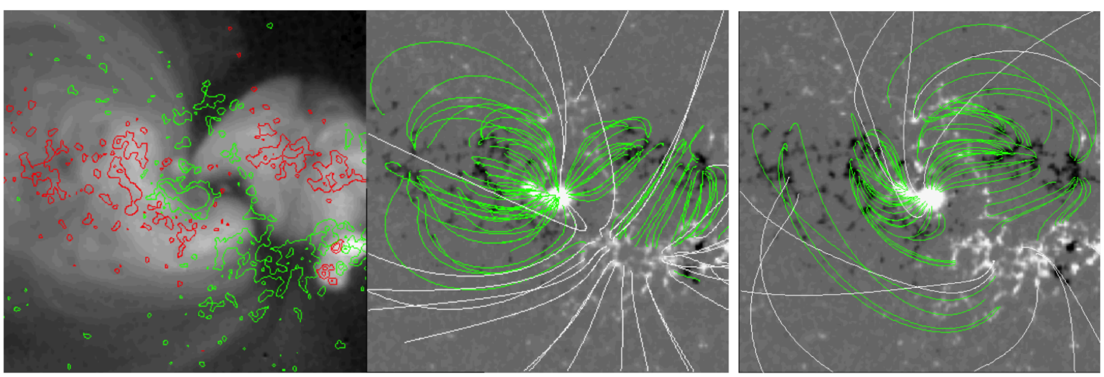
$$\mathbf{B} \cdot \nabla \alpha = 0. \tag{128}$$

They are usually solved numerically.

Example: Nonlinear Force-Free (NLFF) extrapolation of photospheric sources:



Example: Comparison of potential and NLFF extrapolations



Parker's solar wind model

The corona cannot remain in static equilibrium but is continually expanding. The continual expansion is called the solar wind.

Assume that the expanding plasma of the solar wind is isothermal and steady.

The governing equations can be obtained from the MHD equations setting $\partial/\partial t = 0$:

$$\nabla \cdot (\rho \mathbf{V}) = 0, \tag{129}$$

$$\rho(\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla P + \rho \mathbf{g}, \tag{130}$$

$$P = \rho R T, \tag{131}$$

and

$$T = T_0. \tag{132}$$

Also, we restrict our attention to the spherically symmetric solution. The velocity $\mathbf{V}$ is taken as purely radial, $\mathbf{V} = v\mathbf{e_r}$ and the gravitational acceleration $\mathbf{g} = g\mathbf{e_r}$ obeys the inverse square law,

$$g = -\frac{GM_\odot}{r^2}. \tag{133}$$

The temperature and, consequently, the sound speed

$$C_\mathrm{s}^2 = p/\rho, \tag{134}$$

are constant.

We are interested, for simplicity, in the dependence on the r coordinate only. Thus, the expressions for the differential operators in the spherical coordinates are

$$\nabla a = \frac{\mathrm{d}a}{\mathrm{d}r}, \qquad \nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 A_r\right).$$

In the spherical geometry, the governing equations describing and the radially symmetric values are

$$\rho v \frac{\mathrm{d}\,v}{\mathrm{d}\,r} = -\frac{\mathrm{d}\,P}{\mathrm{d}\,r} - \frac{GM_\odot \rho}{r^2}, \tag{135}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^2 \rho v) = 0 \qquad \Rightarrow \qquad r^2 \rho v = \mathrm{const.} \tag{136}$$

Substituting (134) into (135) we exclude the pressure from the equations,

$$\rho v \frac{\mathrm{d}\,v}{\mathrm{d}\,r} = -C_\mathrm{s}^2 \frac{\mathrm{d}\,\rho}{\mathrm{d}\,r} - \frac{GM_\odot \rho}{r^2}, \tag{137}$$

or

$$v \frac{\mathrm{d}\,v}{\mathrm{d}\,r} = -C_\mathrm{s}^2 \frac{1}{\rho}\frac{\mathrm{d}\,\rho}{\mathrm{d}\,r} - \frac{GM_\odot}{r^2}. \tag{138}$$

To exclude ρ, we use (136),

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^2 \rho v) = \rho \frac{\mathrm{d}}{\mathrm{d}r}(r^2 v) + r^2 v \frac{\mathrm{d}\rho}{\mathrm{d}r} = 0, \tag{139}$$

and obtain

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{1}{r^2 v}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 v). \tag{140}$$

Now, Eq. (138) becomes

$$v\frac{\mathrm{d}\,v}{\mathrm{d}\,r} = \frac{C_s^2}{r^2 v}\frac{\mathrm{d}}{\mathrm{d}\,r}(r^2 v) - \frac{GM_\odot}{r^2}. \tag{141}$$

Rewriting this equation, we obtain

$$\left(v - \frac{C_\mathrm{s}^2}{v}\right)\frac{\mathrm{d}\,v}{\mathrm{d}\,r} = \frac{2C_\mathrm{s}^2}{r} - \frac{GM_\odot}{r^2}, \tag{142}$$

and, then

$$\left(v - \frac{C_\mathrm{s}^2}{v}\right)\frac{\mathrm{d}\,v}{\mathrm{d}\,r} = 2\frac{C_\mathrm{s}^2}{r^2}\left(r - r_\mathrm{c}\right), \tag{143}$$

where $r_\mathrm{c} = GM_\odot/(2C_\mathrm{s}^2)$ is the critical radius showing the position where the wind speed reaches the sound speed, $v = C_\mathrm{s}$.

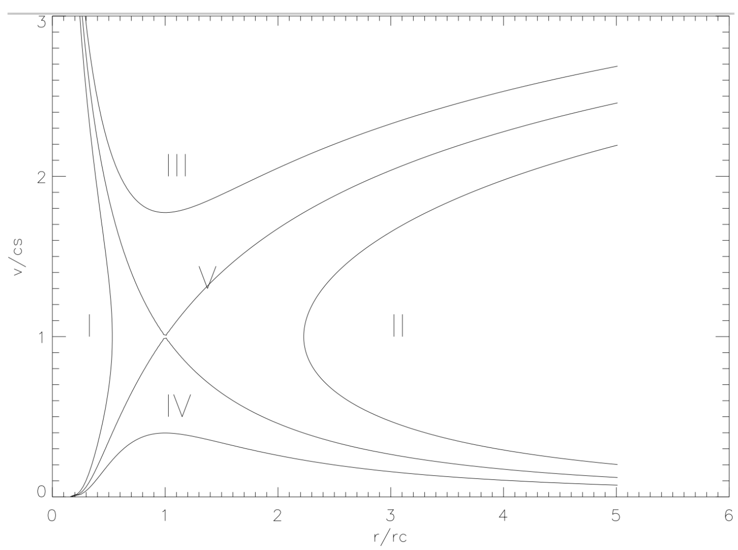
This is a separable ODE, which can readily be integrated,

$$\int \left(v - \frac{C_\mathrm{s}^2}{v}\right)\mathrm{d}v = \int 2\frac{C_\mathrm{s}^2}{r^2}\left(r - r_\mathrm{c}\right)\mathrm{d}r, \tag{144}$$

giving the solution

$$\left(\frac{v}{C_\mathrm{s}}\right)^2 - \log\left(\frac{v}{C_\mathrm{s}}\right)^2 = 4\log\left(\frac{r}{r_\mathrm{c}}\right) + 4\frac{r_\mathrm{c}}{r} + C. \tag{145}$$

The constant of integration C can be determined from boundary conditions, and it determines the specific solution. Several types of solution are present in the figure:



Types I and II are double valued (two values of the velocity at the same distance), and are non-physical.

Types III has supersonic speeds at the Sun which are not observed.

Types IV seem also be physically possible. (The "solar breeze" solutions).

The unique solution of type V passes through the critical point ($r = r_c, v = C_s$) and is given by $C = -3$. It can be obtained from the general solution (145) by putting the coordinates of the critical point. This is the "solar wind" solution (Parker, 1958). It was discovered by Soviet Luna-2, Luna-3 and Venera-1 probes in 1959.

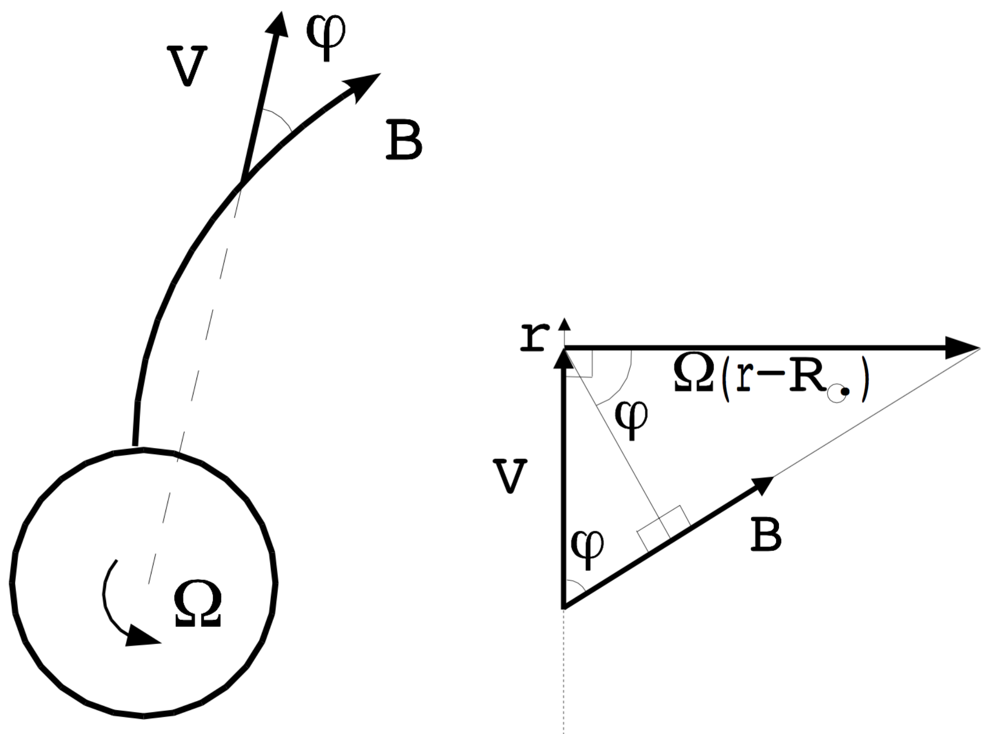
Let us estimate the critical radius r_c. For a typical coronal sound speed of about 10^5 m/s, and the critical radius is

$$r_\mathrm{c} = \frac{GM_\odot}{2C_\mathrm{s}^2} \approx 6\times 10^9\ \mathrm{m} \approx 9 - 10R_\odot. \tag{146}$$

At the Earth's orbit, the solar wind speed can be obtained by substituting $r = 214R_\odot$ to Eq. (145), which gives $v = 310$ km/s.

For the radial flow, the rotation of the Sun makes the solar magnetic field twist up into a spiral.

Suppose the magnetic field is inclined at an angle ϕ to the radial solar wind velocity:



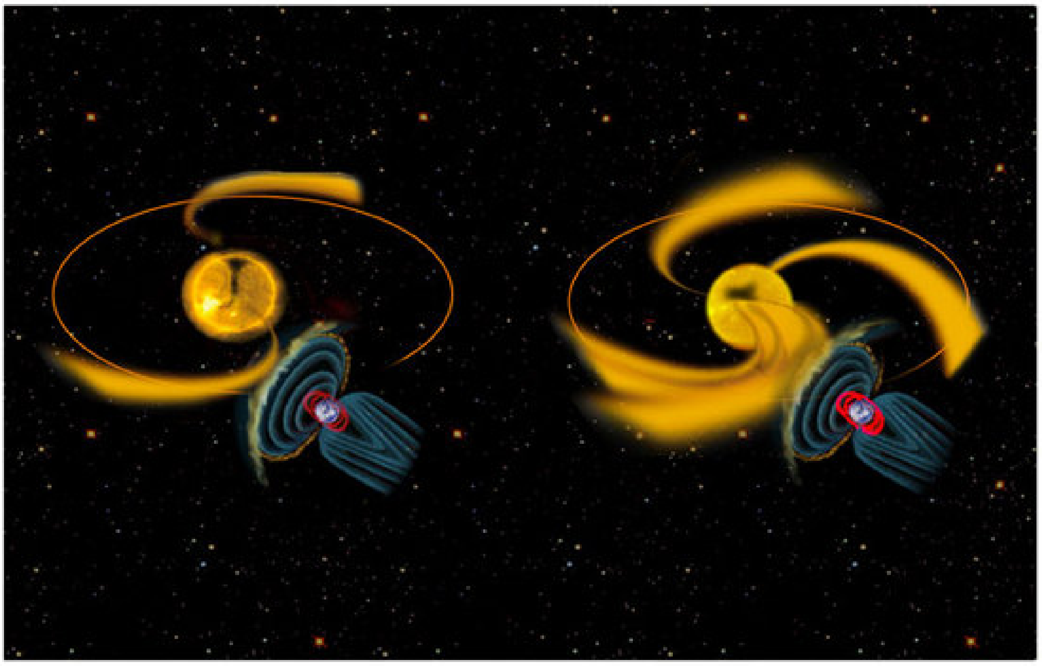
The component of the vector **V** perpendicular to the vector **B**, $v\ \sin\phi$, must equal the speed of the field line in that direction, because the field is frozen in the plasma. But, the field is dragged by the solar rotation with the angular frequency Ω. The normal component of the speed of the field line is $\Omega(r - R_\odot)$. Consequently,

$$v \sin\phi = \Omega(r - R_\odot)\cos\phi, \tag{147}$$

which gives us

$$\tan\phi = \frac{\Omega(r - R_\odot)}{v} \tag{148}$$

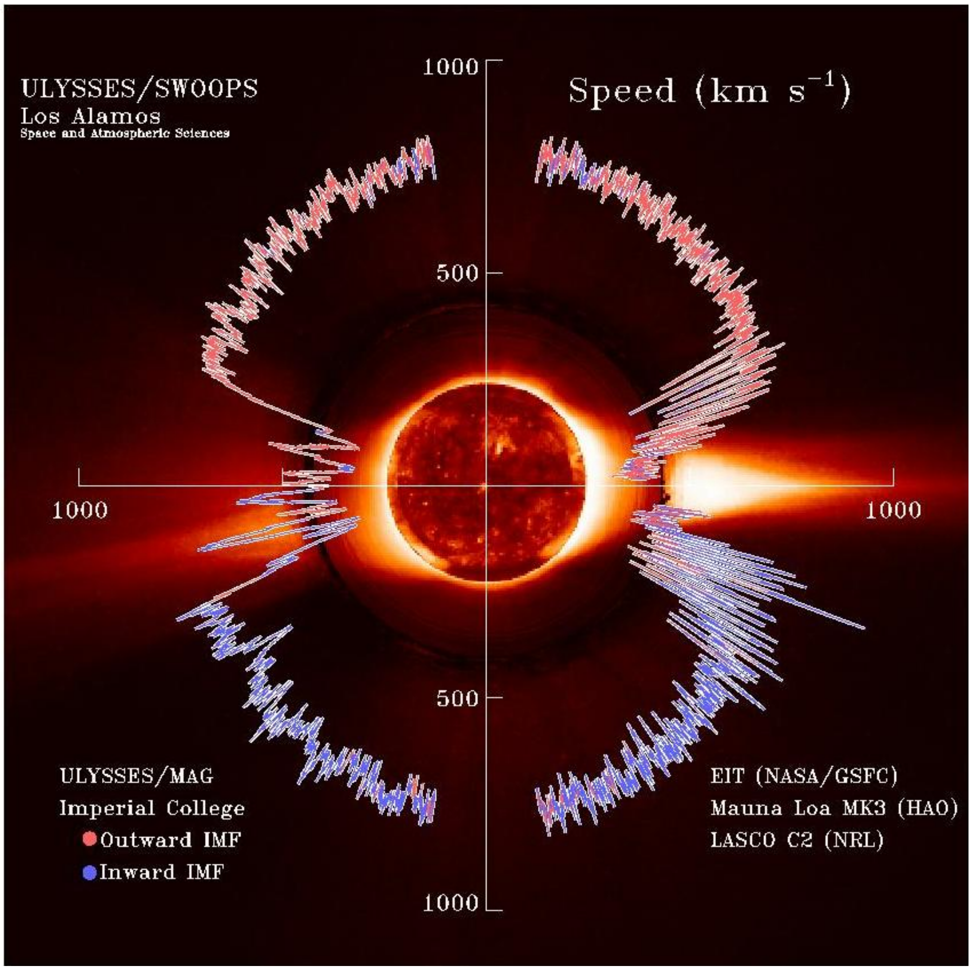
Taking $v \approx 310$ km/s and calculating the frequency of the equatorial rotation period, which is about 26 days, $\Omega = 2\pi/(26 \times 24 \times 60 \times 60) \approx 2.8 \times 10^{-6}$ rad/s, we obtain that near the Earth's orbit, $r \approx 214R_\odot$, the angle is about 45°.



In-situ observations have established that there are actually two component in the solar wind,

- relatively low-speed streams ($v < 350$ km/s) - the "slow solar wind" and
- high-speed streams (v up to 800 km/s) - the "fast wind".

The slow wind is denser and carries greater flux of particles. The presence of the fast wind has been observed at higher solar latitudes.



Realtime monitoring of the solar wind near the Earth's orbit:
http://www.swpc.noaa.gov/products/ace-real-time-solar-wind