PX358 Plasma Physics. 2006

A Concept of Adiabatic Invariants

Some physical quantities change very slowly (or 'adiabatically') compared with some typical periodicities of the particle motion. These physical quantities are called *adiabatic invariants*. They are not absolute constants of motion, like total energy or total moment.

In particular, in weakly non-uniform magnetic fields, the magnetic moment

$$\mu = W_\perp / B,$$

where $W_\perp$ is the perpendicular kinetic energy, varies slowly with respect to the gyroperiod. So, in this case, the magnetic moment is an adiabatic invariant and can be considered as a constant of motion.

Let us demonstrate

that the magnetic moment of a charged particle does not change when the particle moves into stronger or weaker magnetic fields.

Consider the case $\nabla B \,||\, \mathbf{B} \parallel \mathbf{e_z}$,
there are no other fields (gravitational, electric, etc.).

The particle total energy is

$$W = W_\perp + W_{||}, \tag{1}$$

where $W_{||}$ is the kinetic energy of the longitudinal component of the particle motion and $W_\perp$, the perpendicular component.

Since W is a constant in the absence of non-magnetic fields,

$$\frac{\mathrm{d}W}{\mathrm{d}\,t} = \frac{\mathrm{d}W_\perp}{\mathrm{d}\,t} + \frac{\mathrm{d}W_{||}}{\mathrm{d}\,t} = 0 \tag{2}$$

Consider $W_\perp$:

From the definition of the magnetic moment, $W_\perp = \mu B$.

$$\Rightarrow \qquad \frac{\mathrm{d}W_\perp}{\mathrm{d}\,t} = \mu \frac{\mathrm{d}B}{\mathrm{d}\,t} + B\frac{\mathrm{d}\mu}{\mathrm{d}\,t}. \tag{3}$$

Here

$$\frac{\mathrm{d}B}{\mathrm{d}\,t} = V_{||}\frac{\mathrm{d}B}{\mathrm{d}\,z}$$

is the variation of the magnetic field as seen by the particle along its guiding centre trajectory.

Consider $W_{||}$:

From the equation of motion,

$$m\frac{\mathrm{d}V_{||}}{\mathrm{d}\,t} = -\mu\frac{\mathrm{d}B}{\mathrm{d}\,z} \tag{4}$$

(see the case of $\nabla B \,||\, \mathbf{B}$ discussed before).

Multiplying the LHS of Eq. (4) with $V_{||}$ and the RHS with $\mathrm{d}\,z/\mathrm{d}\,t$ $(= V_{||})$, we obtain

$$\frac{\mathrm{d}W_{||}}{\mathrm{d}\,t} = -\mu \frac{\mathrm{d}B}{\mathrm{d}\,t} \tag{5}$$

Now, returning to Eq. (2) and using Eqs. (3) and (5), we obtain

$$\frac{\mathrm{d}W_{\perp}}{\mathrm{d}\,t} + \frac{\mathrm{d}W_{||}}{\mathrm{d}\,t} = \mu \frac{\mathrm{d}B}{\mathrm{d}\,t} + B \frac{\mathrm{d}\mu}{\mathrm{d}\,t} - \mu \frac{\mathrm{d}B}{\mathrm{d}\,t} = 0. \tag{6}$$

Consequently,

$$\frac{\mathrm{d}\mu}{\mathrm{d}\,t} = 0$$

and the magnetic moment is conserved (or, in other words, is an invariant of the motion).

The magnetic moment μ is <u>not</u> affected by <u>small</u> changes in the gyrofrequency and the Larmor radius which occur when the magnetic field changes <u>weakly</u> along the particle path.