Magnetohydrodynamics (MHD).

MHD describes large scale, slow dynamics of plasmas. More specifically, we can apply MHD when

1. Characteristic time $\gg$ ion gyroperiod and mean free path time,
2. Characteristic scale $\gg$ ion Larmor radius and mean free path length,
3. Plasma velocities are not relativistic.

In MHD, the plasma is considered as an electrically conducting fluid. Governing equations are equations of fluid dynamics and Maxwell's equations. A self-consistent set of MHD equations connects the plasma mass density ρ, the plasma velocity $\mathbf{V}$, the thermodynamic (also called kinetic) pressure P and the magnetic field $\mathbf{B}$. In strict derivation of MHD, one should neglect the motion of electrons and consider only heavy ions.

The 1-st equation is mass continuity

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{V}) = 0, \tag{1}$$

and it states that matter is neither created or destroyed.

The 2-nd is the equation of motion of an element of the fluid,

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} \right] = -\nabla P + \mathbf{j} \times \mathbf{B}, \tag{2}$$

also called the Euler equation. The vector $\mathbf{j}$ is the electric current density which can be expressed through the magnetic field $\mathbf{B}$.

The 3-rd equation is the energy equation, which in the simplest *adiabatic* case has the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{p}{\rho^\gamma} \right) = 0, \tag{3}$$

where γ is the ratio of specific heats C_p/C_V, and is normally taken as 5/3.

The temperature T of the plasma can be determined from the density ρ and the thermodynamic pressure p, using the state equation. For example, in a pure hydrogen plasma, this equation is

$$P = 2\frac{k_B}{m_i}\rho T, \tag{4}$$

where m_i is the mass of a proton and k_B is Boltzmann's constant.

Now, let us derive the equation for the magnetic field using Maxwell's equations. Start with Ohm's law,

$$\mathbf{j} = \sigma \mathbf{E}', \tag{5}$$

where σ is electric conductivity (the physical quantity inversed to the resistivity) and $\mathbf{E}'$ is the electric field experienced by the plasma (fluid) element in its *rest* frame. When the plasma is moving (with respect to the external magnetic field) at the velocity $\mathbf{V}$, applying the Lorentz transformation we obtain

$$\mathbf{E}' = \mathbf{E} + \mathbf{V} \times \mathbf{B}. \tag{6}$$

Now, Eq. (5) can be re-written as

$$\frac{1}{\sigma}\,\mathbf{j} = \mathbf{E} + \mathbf{V} \times \mathbf{B}. \tag{7}$$

In the case of perfect conductivity, $\sigma \to \infty$, we have

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B}. \tag{8}$$

Calculating the curl of the electric field $\mathbf{E}$ and using one of Maxwell's equation,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{9}$$

we can exclude the electric field and obtain the 4-th MHD equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}), \tag{10}$$

To close the set of MHD equations, we have to express the current density $\mathbf{j}$ through the magnetic field $\mathbf{B}$. Consider the other Maxwell's equation,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{11}$$

From Ohm's law, we had $\mathbf{E} = -\mathbf{V}\times\mathbf{B}$. Consequently, we can estimate the electric field as $E \sim V_0 B$, where V_0 is a characteristic speed of the process. Consider the ratio of two terms in Eq. (11):

$$\nabla \times \mathbf{B} \quad \text{and} \quad \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

The first term is proportional to B/l_0, where l_0 is a characteristic scale of the process, the second to $E/c^2 t_0$, where t_0 is a characteristic time of the process, $V_0 = l_0/t_0$. When the process is not relativistic, $V_0 \ll c$, the first term is very much greater than the second, and we have

$$\mathbf{j} = \frac{1}{\mu_0}\nabla \times \mathbf{B} \tag{12}$$

In addition, the magnetic field $\mathbf{B}$ must satisfy the condition $\nabla.\mathbf{B} = 0$.

Thus, the closed set of MHD equations is

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{V}) &= 0, && \text{Mass Continuity Eq.,}\\
\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{P}{\rho^\gamma}\right) &= 0, && \text{Energy Eq.,}\\
\rho \frac{\mathrm{d}\,\mathbf{V}}{\mathrm{d}\,t} &= -\nabla P - \frac{1}{\mu_0}\mathbf{B} \times (\nabla \times \mathbf{B}), && \text{Euler's Eq.,}\\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{V} \times \mathbf{B}), && \text{Induction Eq..}
\end{aligned}$$

The equations are ideal, which means that all dissipative processes (finite viscosity, electric conductivity and thermal conductivity) were neglected.