MHD Equilibrium

The static equilibrium conditions are:

$$\mathbf{V} = 0, \quad \frac{\partial}{\partial t} = 0. \tag{1}$$

These conditions identically satisfy the continuity, energy and induction equations.

From Euler's equation we obtain the condition

$$-\nabla P - \frac{1}{\mu_0}\mathbf{B} \times (\nabla \times \mathbf{B}) = 0, \tag{2}$$

which is called the equation of magnetostatics. This equation should be supplemented with the condition $\nabla.\mathbf{B} = 0$.

Eq. (2) can be re-written as

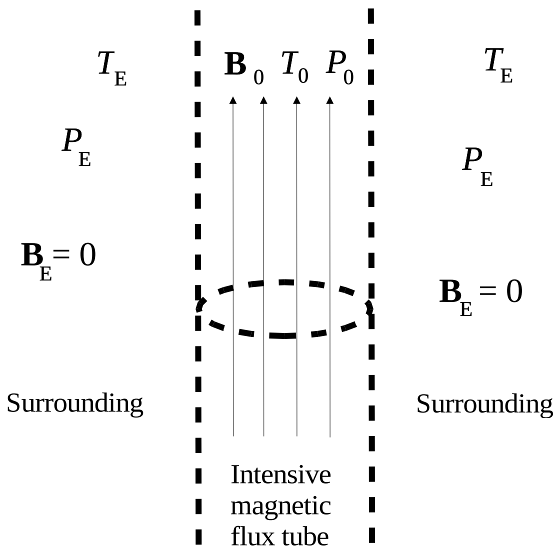
$$-\nabla\left(P + \frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\mathbf{B}.\nabla)\mathbf{B} = 0. \tag{3}$$

The first term can be considered as the gradient of total pressure. The total pressure consists of two terms, the kinetic (or thermodynamic) pressure P, and the magnetic pressure $B^2/2\mu_0$. The second term is magnetic tension.

Example: Sunspots.

Sunspots appear as dark spots on the surface of the Sun. They typically last for several days, although very large ones may live for several weeks. Sunspots are magnetic regions on the Sun with magnetic field strengths thousands of times stronger than the Earth's magnetic field.

Consider a sunspot as a vertical magnetic flux tube. The magnetic field $\mathbf{B}_0$ is vertical. The kinetic pressure is P_0 and P_E inside and outside, respectively. The plasma temperature is T_0 inside the sunspot and T_E outside.



Sunspots are long-durational objects with no fast flows of plasma. So, it is naturally to describe their structure in terms of magnetostatics. As the magnetic field is not bent, the last term in Eq. (2), responsible for the magnetic tension, is zero. The equilibrium condition becomes

$$\nabla \left(p + \frac{B^2}{2\mu_0} \right) = 0, \tag{4}$$

This means that the total pressure must be equal inside and outside the sunspot,

$$P_E = P_0 + \frac{B_0^2}{2\mu_0}. \tag{5}$$

Let us assume that the density of the plasmas inside and outside the sunspot are equal, $\rho_O = \rho_E$. Now, we divide Eq. (5) by ρ_0,

$$\frac{P_E}{\rho_E} = \frac{P_0}{\rho_0} + \frac{B_0^2}{2\mu_0 \rho_0}. \tag{6}$$

Using the state equations,

$$P_E = 2\frac{k_B}{m_i}\rho_E T_E, \qquad P_0 = 2\frac{k_B}{m_i}\rho_0 T_0, \tag{7}$$

we obtain from Eq. (6)

$$\frac{2k_B}{m_i}T_E = \frac{2k_B}{m_i}T_0 + \frac{B_0^2}{2\mu_0\rho_0}. \tag{8}$$

This gives us

$$\frac{T_0}{T_E} = 1 - \frac{B_0^2}{2\mu_0}\frac{m_i}{2k_B\rho_E T_E} = 1 - \frac{B_0^2}{2\mu_0 P_E} \tag{9}$$

Thus, in a sunspot, $T_E > T_0$. Indeed, temperatures in the dark centers of sunspots drop to about 3700 K, compared to 5700 K for the surrounding photosphere. This is why sunspots are seen to be darker than the surrounding.