PX358 Plasma Physics. 2006

MHD Waves

Ideal MHD connects the magnetic field **B**, plasma velocity **V**, pressure P and density ρ:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{V}) = 0, \tag{1}$$

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} \right] = -\nabla P - \frac{1}{\mu_0} \mathbf{B} \times (\nabla \times \mathbf{B}), \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}), \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{P}{\rho^\gamma} \right) = 0. \tag{4}$$

Consider an equilibrium, described by the conditions

$$\frac{\partial}{\partial t} = 0, \qquad \mathbf{V} = 0, \tag{5}$$

which gives us the *magnetostatic* equation

$$\nabla P_0 + \frac{1}{\mu_0} \mathbf{B_0} \times (\nabla \times \mathbf{B_0}) = 0. \tag{6}$$

The simplest possible solution of the magnetostatic equation is a uniform plasma:

$$P_0 = \text{const}, \qquad B_0 = \text{const}, \tag{7}$$

and the equilibrium magnetic field $\mathbf{B_0}$ is straight.

Consider *small* perturbations of the equilibrium state:

$$\left. \begin{array}{l} \mathbf{B} = \mathbf{B_0} + \mathbf{B_1}(\mathbf{r}, t) \\ \mathbf{V} = 0 + \mathbf{V_1}(\mathbf{r}, t) \\ P = P_0 + P_1(\mathbf{r}, t) \\ \rho = \rho_0 + \rho_1(\mathbf{r}, t) \end{array} \right\} \tag{8}$$

Substitute these expressions into the MHD equations (1)–(4). Neglecting terms which contain a product of two or more values with indices "1", we obtain the set of MHD equations, *linearized* near the equilibrium (7):

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \mathbf{V_1} = 0, \tag{9}$$

$$\rho_0 \frac{\partial \mathbf{V_1}}{\partial t} = -\nabla P_1 - \frac{1}{\mu_0} \mathbf{B_0} \times (\nabla \times \mathbf{B_1}), \tag{10}$$

$$\frac{\partial P_1}{\partial t} - \frac{\gamma P_0}{\rho_0} \frac{\partial \rho_1}{\partial t} = 0, \tag{11}$$

$$\frac{\partial \mathbf{B_1}}{\partial t} = \nabla \times (\mathbf{V_1} \times \mathbf{B_0}), \tag{12}$$

Let the equilibrium magnetic field $\mathbf{B_0}$ be in xz-plane,

$$\mathbf{B_0} = B_0 \sin\alpha \, \mathbf{e_x} + B_0 \cos\alpha \, \mathbf{e_z}, \tag{13}$$

where α is the angle between the magnetic field and the unit vector $\mathbf{e_z}$.

Consider plane waves, propagating along $\mathbf{e_z}$, so that all perturbed quantities are proportional to $\exp(ikz - i\omega t)$. (This gives us $\partial/\partial t = -i\omega$ and $\nabla = ik$.) Projecting equations (9)–(12) onto the axes, we have

$$-i\omega\rho_1 + ik\rho_0 V_{z1} = 0, \tag{14}$$

$$-i\omega\rho_0 V_{x1} - \frac{ikB_0\cos\alpha}{\mu_0}B_{x1} = 0, \tag{15}$$

$$-i\omega\rho_0 V_{y1} - \frac{ikB_0\cos\alpha}{\mu_0}B_{y1} = 0, \tag{16}$$

$$-i\omega\rho_0 V_{z1} + ikP_1 + \frac{ikB_0\sin\alpha}{\mu_0}B_{x1} = 0, \tag{17}$$

$$-i\omega B_{x1} + ikB_0\sin\alpha V_{z1} - ikB_0\cos\alpha\, V_{x1} = 0, \tag{18}$$

$$-i\omega B_{y1} - ikB_0\cos\alpha\, V_{y1} = 0, \tag{19}$$

$$-i\omega B_{z1} = 0, \tag{20}$$

$$-i\omega P_1 + \frac{i\omega\gamma P_0}{\rho_0}\rho_1 = 0. \tag{21}$$

The set of equations (14)–(21) splits into two partial sub-sets. The first one is formed by equations (16) and (19), describing B_{y1} and V_{y1}. The consistency condition gives us

$$\omega^2 - C_A^2\cos^2\alpha\, k^2 = 0, \tag{22}$$

where $C_A = B_0/(\mu_0\rho_0)^{1/2}$ is the Alfvén speed. This is dispersion relations for *Alfvén* waves.

The second partial set of equations is formed by equations (14), (15), (17), (18) and (21) and describes variables V_{x1}, V_{z1}, B_{x1}, P_1 and ρ_1. The consistency condition gives us

$$(\omega^2 - C_A^2\cos^2\alpha\ k^2)(\omega^2 - C_s^2k^2) - C_A^2\sin^2\alpha\,\omega^2k^2 = 0, \tag{23}$$

where $C_s = (\gamma p_0/\rho_0)^{1/2}$. The equation is bi-quadratic with respect to ω and consequently has two pairs of roots. They correspond to *fast* and *slow* *magnetoacoustic* waves.

Polar plots for phase speeds (ω/k) and group speeds ($\mathrm{d}\omega/\mathrm{d}k$) for $\beta < 1$: