Recurrence Relations

CIS008-2 Logic and Foundations of Mathematics

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Outline

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1. Recursive Algorithms
2. Recurrence Relations

Recursive Algorithms

- An algorithm is a specific set of instructions for carrying out a procedure or solving a problem, usually with the requirement that the procedure terminate at some point.
- The process of applying an algorithm to an input to obtain an output is called a computation.
- A recursive process is one in which objects are defined in terms of other objects of the same type. Using some sort of recurrence relation, the entire class of objects can then be built up from a few initial values and a small number of rules.

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Recurrence Relations

Definition

A **recurrence relation** for a sequence $a_0, a_1, \ldots$ is an equation that relates a_n to certain of its predecessors $a_0, a_1, \ldots, a_{n-1}$.

Initial conditions for the sequence $a_0, a_1, \ldots$ are explicitly given values for a finite number of terms of the sequence.

Example - Fibonacci sequence

The Fibonacci sequence is defined by the recurrence relation

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 3$$

and initial conditions

$$f_1 = 1, \quad f_2 = 1$$

Example - Towers of Hanoi

The *towers of Hanoi* is a puzzle invented by E. Lucas in 1883. Given a stack of n disks arranged from largest on the bottom to smallest on top placed on a rod, together with two empty rods, the towers of Hanoi puzzle asks for the minimum number of moves required to move the stack from one rod to another, where moves are allowed only if they place smaller disks on top of larger disks.

Example - Towers of Hanoi

We let c_n denote the number of moves our solution takes to solve the n-disk puzzle. Specifically, we would solve the $(n-1)$-disk problem twice and then explicitly move the last (largest) disk once. Therefore

$$c_n = 2c_{n-1} + 1, \quad n > 1$$

where the inital condition is $c_c = 1$

Example - Towers of Hanoi

We can find an explicit formula for c_n:

$$
\begin{aligned}
c_n &= 2c_{n-1} + 1 \\
< 2- > &= 2(2c_{n-2} + 1) + 1 \\
&= 2^2 c_{n-2} + 2 + 1 \\
&= 2^2 (2c_{n-3} + 1) + 2 + 1 \\
&= 2^3 c_{n-3} + 2^2 + 2 + 1 \\
&\vdots \\
&= 2^{n-1} c_n + n^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\
&= 2^n - 1
\end{aligned}
$$

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