Unitarity Triangle Angles Explained: a Predictive New Quark Mass Matrix Texture

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arXiv: 2501.18508 with Bill Scott, Rutherford Appleton Laboratory

Outline of Talk

- Mysteries of the quark mass and mixing spectra
- Weak Interaction flavour structure
- SM origins of masses and mixings
- Historical efforts to explain

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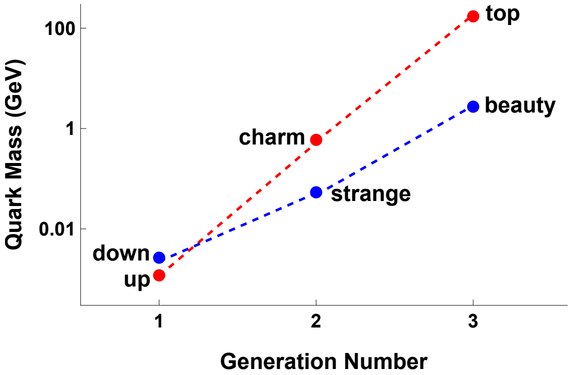
- Mysteries of the quark mass and mixing spectra
- Weak Interaction flavour structure
- SM origins of masses and mixings
- Historical efforts to explain
- Mysteries of the Unitarity Triangle
- The new mass matrix texture
- Confronting the data
- Symmetries of the texture
- Discussion and conclusions

Mystery of Quark Mass Spectra

- Quark masses show marked hierarchical structure:



- Is quasi-"geometric":

$$\frac{m_c}{m_t} \simeq 0.0035$$
$$\frac{m_u}{m_c} \simeq 0.0020$$

$$\frac{m_s}{m_b} \simeq 0.020$$
$$\frac{m_d}{m_s} \simeq 0.050$$

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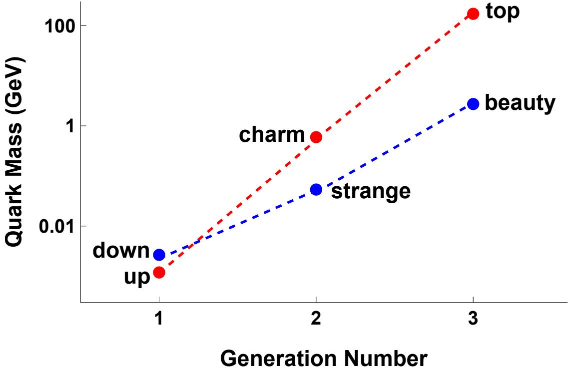

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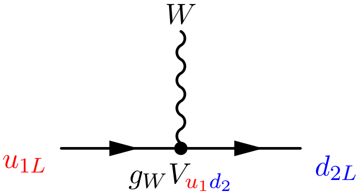
- Masses not predicted in the SM
- Hierarchy certainly not explained within SM

Mystery of Quark Mixing Spectrum

- CKM quark mixing matrix:

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$



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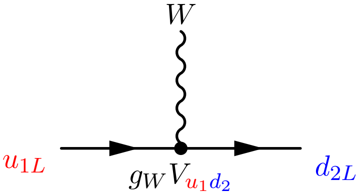

$$\sim \begin{pmatrix} 1 & \lambda & A\lambda^3(\overline{\rho} - i\overline{\eta}) \\ -\lambda & 1 & A\lambda^2 \\ A\lambda^3(1 - \overline{\rho} - i\overline{\eta}) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4),$$

where $\lambda \equiv |V_{us}| \simeq 0.22$

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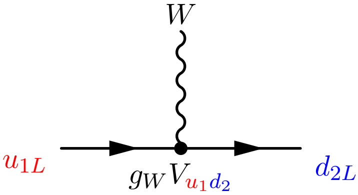

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where $\lambda \equiv |V_{us}| \simeq 0.22$

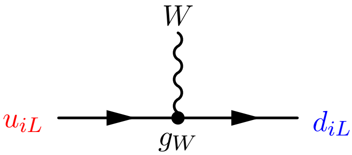
A, $\overline{\rho}$ and $\overline{\eta} \lesssim \mathcal{O}(1)$

- Elements not predicted by the SM
- Strong hierarchy certainly not explained within SM
- But masses and mixings both arise in the Yukawa/Mass matrices

The Weak Interaction

- In the gauge theory
- 3 generations of quarks:

$$\begin{pmatrix} u_1 \\ d_1 \end{pmatrix}_L \quad \begin{pmatrix} u_2 \\ d_2 \end{pmatrix}_L \quad \begin{pmatrix} u_3 \\ d_3 \end{pmatrix}_L$$



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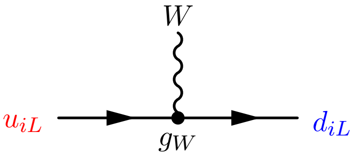
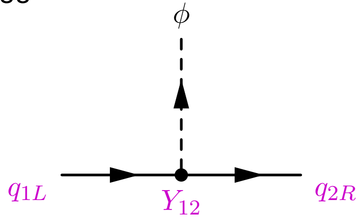


- Write $\underline{u}_w = (u_1, u_2, u_3)^T$ and $\underline{d}_w = (d_1, d_2, d_3)^T$
- $W^{\pm}$ couplings initially Flavour-diagonal:

$$\mathcal{L}_W \sim g_W \, \bar{\underline{u}}_{wL} \cdot \underline{d}_{wL} \, W^+ + H.C.$$

Masses and Mixings from SM

- Fermion Masses and Mixings have common origin in (Yukawa) couplings of the Higgs to fermions



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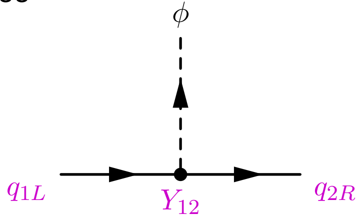
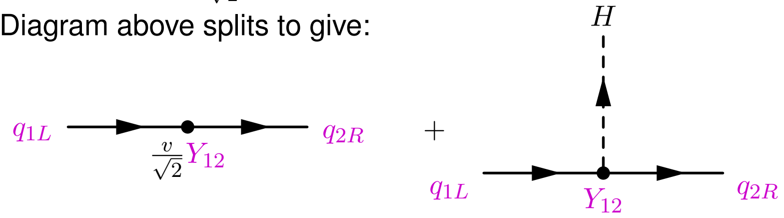


- After SSB, $\phi \to \frac{v}{\sqrt{2}} + H$
- Diagram above splits to give:



Taking all Permutations

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- Recall $\underline{u}_w = (u_1, u_2, u_3)^T$ and $\underline{d}_w = (d_1, d_2, d_3)^T$
- After SSB, Lagrangian for the quark masses is (dropping L/R labels):
$$\mathcal{L}_{Mass} \sim \frac{v}{\sqrt{2}}\,\bar{\underline{u}}_w \cdot Y_u \cdot \underline{u}_w + \frac{v}{\sqrt{2}}\,\bar{\underline{d}}_w \cdot Y_d \cdot \underline{d}_w$$
- We identify $\frac{v}{\sqrt{2}} Y_u \equiv M_u$ and $\frac{v}{\sqrt{2}} Y_d \equiv M_d$

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- We identify $\frac{v}{\sqrt{2}} Y_u \equiv M_u$ and $\frac{v}{\sqrt{2}} Y_d \equiv M_d$
- M_u and M_d are clearly not diagonal
- Can choose basis where they are Hermitian without observable consequences.

Physical Particles?

- Identified as the eigenstates of the mass matrices (MMs)
- So, we diagonalise to find them:
  $(u, c, t)^T \equiv \underline{u} = U_u \cdot \underline{u}_w$ and $(d, s, b)^T \equiv \underline{d} = U_d \cdot \underline{d}_w$
- Then (chiral labels dropped):

$$\mathcal{L}_{M+W} = \underline{\bar{u}} \cdot D_u \cdot \underline{u} + \underline{\bar{d}} \cdot D_d \cdot \underline{d} + g_W \, \underline{\bar{u}} \cdot U_u \cdot {U_d}^{\dagger} \cdot \underline{d} \, W^+ + \ldots$$

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- where
  $D_u \equiv U_u \cdot M_u \cdot {U_u}^\dagger = \mathrm{diag}(m_u, m_c, m_t)$

  $D_d \equiv U_d \cdot M_d \cdot {U_d}^\dagger = \mathrm{diag}(m_d, m_s, m_b)$

  and $V_{CKM} \equiv U_u \cdot {U_d}^\dagger$ is unitary.
- Thus masses and mixings have common origin in the MMs

Historical Context

- So, can mass ratios and mixing values be related to each other?
- One way is with "texture zeroes" - pioneering idea by Harald Fritzsch (1976-78)
- E.g. $M_d \equiv M_F(m_b, a_d, b_d)$

$$= m_b \begin{pmatrix} 0 & a_d & 0 \\ a_d^* & 0 & b_d \\ 0 & b_d^* & 1 \end{pmatrix} : \text{diagonalise} \Rightarrow \begin{array}{c} |a_d| = \sqrt{\dfrac{m_d}{m_b}\dfrac{m_s}{m_b}} \sim 0.0044 \\ |b_d| = \sqrt{\dfrac{m_s}{m_b}} \sim 0.14 \end{array}$$

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- Diagonalised by:

$$U_d{}^\dagger \sim \begin{pmatrix} 1 & s_1^d & s_1^d s_2^d \\ -s_1^d & 1 & s_2^d \\ 0 & -s_2^d & 1 \end{pmatrix} \quad \text{with} \quad \begin{array}{c} s_1^d \equiv \sin\theta_{12}^d \simeq \sqrt{\frac{m_d}{m_s}} \sim 0.224 \\ s_2^d \equiv \sin\theta_{23}^d \simeq \sqrt{\frac{m_s}{m_b}} \sim 0.14 \end{array}$$

- Already somewhat encouraging.

Fritzsch Texture Prediction for V_{us}

- But s_2^d, s_3^d are too big, AND need to treat M_u similarly to M_d.
- Do by writing $M_u = M_F(m_t, a_u, b_u)$ [NB. 8 params for 10 obs ✓]
- Since

$$V_{CKM} = U_u {U_d}^\dagger (\dagger \Rightarrow \text{ inverse for unitary matrix}),$$

is like a rotation and a rotation back.

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$$\tilde{\delta} \equiv arg(a_u) - arg(a_d)$$

- Together give:

$$\begin{aligned} V_{us} \sim \lambda &= |s_1^d - s_1^u e^{i\tilde{\delta}}| \\ &= |\sqrt{\frac{m_d}{m_s}} - \sqrt{\frac{m_u}{m_c}} e^{i\tilde{\delta}}| \end{aligned}$$

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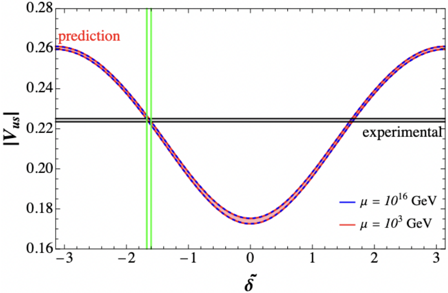
- Good fit ✓



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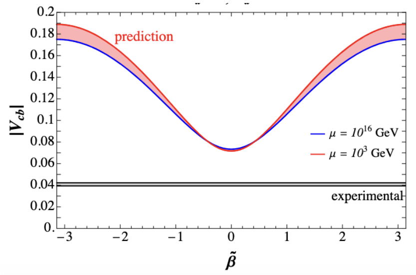

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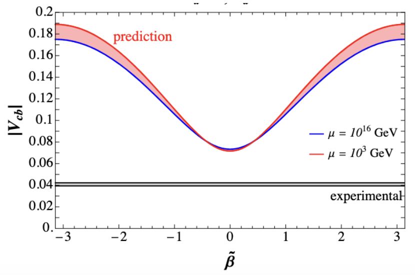

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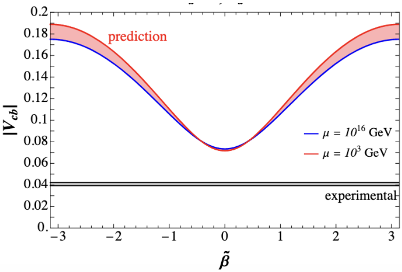

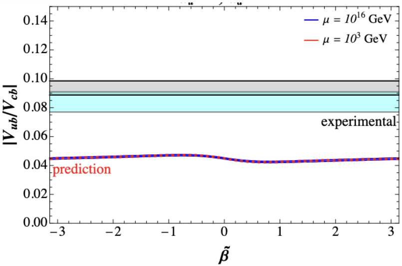

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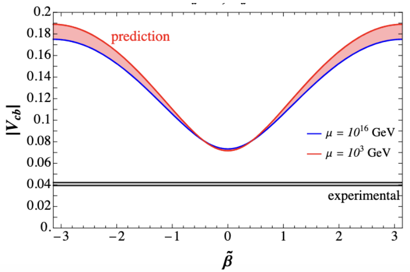

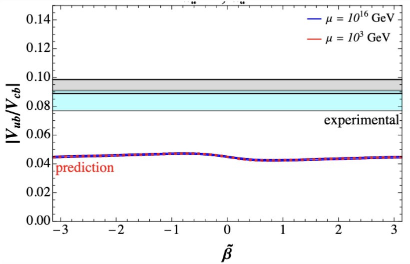

- B. Belfatto and Z. Berezhiani, arXiv: 2305.00069, for figs. Also modern approach to revive Fritzsch using non-Hermitian MMs (but 10 pars ✗)

The Unitarity Triangle

- $$V_{CKM} \equiv U_u U_d{}^{\dagger} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \text{ is unitary.}$$
- ie. complex dot-product of every pair of columns (or rows) is zero. E.g.
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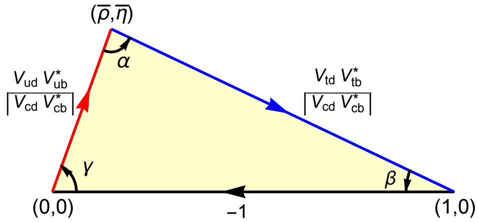
- ie. complex dot-product of every pair of columns (or rows) is zero. E.g.
$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$$
- $\Rightarrow$ triangle in complex plane (normalise by $1/|V_{cd}V_{cb}^*|$):

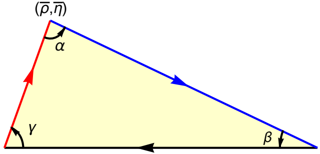


- Base length unity
- 2 parameters:
  2 angles or
  top vertex = $\overline{\rho} + i\overline{\eta}$
- Area = $\frac{1}{2}\overline{\eta}$
- All CP-violating observables $\propto$ Area.

Mysteries of the Unitarity Triangle

- Sides/Angles of UT are arbitrary in SM.
- But all three angles are, experimentally, close to “special” values:

$$\alpha = (91.6 \pm 1.4)^\circ$$
$$\beta = (22.6 \pm 0.4)^\circ$$
$$\gamma = (65.7 \pm 1.3)^\circ$$



i.e.

$$(\alpha, \beta, \gamma) \simeq (\tfrac{\pi}{2}, \tfrac{\pi}{8}, \tfrac{3\pi}{8}) \equiv (\alpha_0, \beta_0, \gamma_0).$$

- Seems striking!

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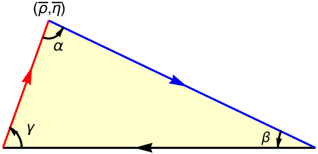

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- Seems striking!
- Coincidence or a smoking gun?
- Decided to take it as a clue to what lies behind.
- So, try to build it into a MM texture.

Our Texture

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$$M_q^{HS} \equiv n_q \begin{pmatrix} c_q \lambda_q^4 & b\, \boldsymbol{\lambda_q^3} & 0 \\ b\, \boldsymbol{\lambda_q^{*3}} & b\lambda_q^2 & A_0 \boldsymbol{\lambda_q^2} \\ 0 & A_0 \boldsymbol{\lambda_q^{*2}} & 1 \end{pmatrix}, \quad \begin{matrix} q = u, d, \\ \boldsymbol{\lambda_q} \text{ complex} \end{matrix}$$

- Describes 10 observables with 7 real parameters:
  - 2 normalisations $n_q \simeq m_t, m_b$ for $q = u, d$.
  - 4 real coefficients (A_0, b, c_u, c_d) of magnitude $\lesssim \mathcal{O}(1)$.
  - Overall normalisations of the $\boldsymbol{\lambda_q}$ – free, but close to λ:

$$\lambda_0 \equiv |\boldsymbol{\lambda_d} + \boldsymbol{\lambda_u}| = \lambda + \mathcal{O}(\lambda^3).$$

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  - Overall normalisations of the $\boldsymbol{\lambda_q}$ – free, but close to λ:

$$\lambda_0 \equiv |\boldsymbol{\lambda_d} + \boldsymbol{\lambda_u}| = \lambda + \mathcal{O}(\lambda^3).$$

- Absolute phases of $\boldsymbol{\lambda_q}$ unobservable.
- Complex ratio is *fixed constant:*

$$\frac{\boldsymbol{\lambda_u}}{\boldsymbol{\lambda_d}} \equiv -i\,\tan\tfrac{\pi}{8}.$$

  - Relative phase, $-i$, is sole source of CP violation
  - Magnitude $|\frac{\boldsymbol{\lambda_u}}{\boldsymbol{\lambda_d}}| \simeq 0.41$ controls relative strength of "u" and "d" mass hierarchies

Leading-order Solution (Quark Masses)

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- Diagonalise $\rightarrow$ masses:

$$D_q = U_q M_q^{HS} U_q^\dagger = m_3^q \begin{pmatrix} (c_q - b)\lambda_q^4 & 0 & 0 \\ 0 & b\lambda_q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = u, d,$$

- Good for hierarchy ($\lambda_u, \lambda_d << 1$) ✓

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- Good for hierarchy ($\lambda_u, \lambda_d << 1$) ✓
- 3 free parameters (at LO): b, c_u, c_d (to fit 4 mass ratios)
- $\Rightarrow$ one constraint/prediction (LO):

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Leading-order Solution (Quark Masses)

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- Diagonalise $\rightarrow$ masses:

$$D_q = U_q M_q^{HS} U_q^\dagger = m_3^q \begin{pmatrix} (c_q - b)\lambda_q^4 & 0 & 0 \\ 0 & b\lambda_q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = u, d,$$

- Good for hierarchy ($\lambda_u, \lambda_d << 1$) ✓
- 3 free parameters (at LO): b, c_u, c_d (to fit 4 mass ratios)
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- Fits any m_u, m_d ✓ (no prediction here).

Leading-order Solution (Quark Mixing)

- Diagonalised by 2×2 (complex) rotations in 23 and 12 spaces.
- Small entries induced in the 13 elements of U_q:

$$U_q \simeq \begin{pmatrix} 1 & \pm\boldsymbol{\lambda_q} & A_0\,\boldsymbol{\lambda_q^3} \\ \mp\boldsymbol{\lambda_q^*} & 1 & -A_0\,\boldsymbol{\lambda_q^2} \\ 0 & A_0\,\boldsymbol{\lambda_q^{*2}} & 1 \end{pmatrix}, \quad q = u, d.$$

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- Combine U_u and U_d:

$$\Rightarrow V_{CKM} = U_u {U_d}^\dagger \simeq \begin{pmatrix} 1 & \lambda_0 & A_0\lambda_0^2\,\boldsymbol{\lambda_u} \\ -\lambda_0 & 1 & A_0\lambda_0^2 \\ A_0\lambda_0^2\,\boldsymbol{\lambda_d^*} & -A_0\lambda_0^2 & 1 \end{pmatrix}$$

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- C.f. Wolfenstein form:

$$V_{CKM} = \begin{pmatrix} 1 & \lambda & A\lambda^3(\bar\rho - i\bar\eta) \\ -\lambda & 1 & A\lambda^2 \\ A\lambda^3(1-\bar\rho - i\bar\eta) & -A\lambda^2 & 1 \end{pmatrix} \Rightarrow \begin{cases} \lambda \simeq \lambda_0 \checkmark \\ A \simeq A_0 \checkmark \\ (\bar\rho + i\bar\eta) \simeq \dfrac{\boldsymbol{\lambda_u^*}}{\lambda_0} \end{cases}$$

The UT Angles

- Have deduced that:

$$\boldsymbol{\lambda}_u^* \simeq \lambda(\overline{\rho} + i\overline{\eta}) = \frac{V_{ub}^*}{A\lambda^2}$$

$$\boldsymbol{\lambda}_d^* \simeq \lambda(1 - \overline{\rho} - i\overline{\eta}) = \frac{V_{td}}{A\lambda^2}$$

$$\Rightarrow \gamma \simeq \arg \boldsymbol{\lambda}_u^*$$

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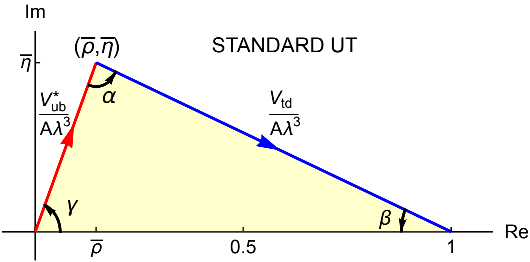

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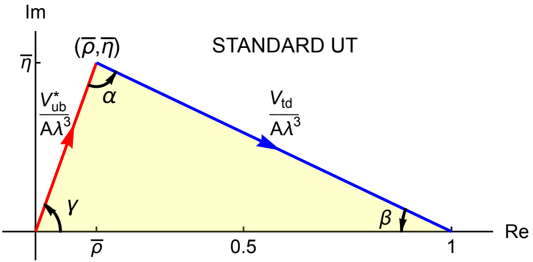

- Recall, HS texture asserts $\frac{\boldsymbol{\lambda}_u}{\boldsymbol{\lambda}_d} = -i \tan \frac{\pi}{8}$.
  - $\Rightarrow \alpha \simeq \frac{\pi}{2}$ ✓
  - $\Rightarrow \tan \beta = |\frac{\boldsymbol{\lambda}_u}{\boldsymbol{\lambda}_d}|$ (see Figure).
  - $\Rightarrow \beta \simeq \frac{\pi}{8}$ ✓

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- Disaster?
- Not necessarily!
- Because these quantities “run” with renormalisation scale
- $\sim$ 14% from weak to GUT scales: $A(\uparrow), m_c/m_t(\uparrow)$ and $m_s/m_b(\downarrow)$.
- λ, α, β, m_u/m_c and m_d/m_s are $\sim$ invariant.
- $\Rightarrow$ vary μ

Precision Fit to the Data

- Fit $\chi^2/\text{d.o.f} \simeq 1.01/2$

Observable	Input Renormalised to $\mu = 10^4$ TeV	Fitted Value at $\mu = 10^4$ TeV
$\lvert m_u/m_c \rvert\, (\times 10^3)$	2.00 ± 0.05	2.00
$\lvert m_d/m_s \rvert\, (\times 10^2)$	4.97 ± 0.06	4.97
$m_c/m_t\, (\times 10^3)$	3.46 ± 0.03	3.46
$m_s/m_b\, (\times 10^2)$	1.968 ± 0.008	1.968
λ	0.2250 ± 0.0007	0.2250
A	0.88 ± 0.02	0.88
$\overline{\rho}$	0.159 ± 0.009	0.152
$\overline{\eta}$	0.352 ± 0.007	0.348
UT Angles	–	Prediction from Fit
$\alpha\, (^\circ)$	91.6 ± 1.4	91.3
$\beta\, (^\circ)$	22.6 ± 0.4	22.3
$\gamma\, (^\circ)$	65.7 ± 1.3	66.4

Some Details of the Fit

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- Best fit renormalisation scale: $\mu \sim (0.3 \to 3) \times 10^4$ TeV.
- Fitted values of the free parameters:
  - $\lambda_0 = 0.22646$
  - $A_0 = 0.854$
  - $b = 0.462$
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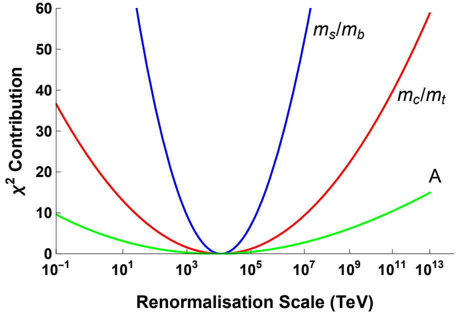
- Three curves minimise at common scale $\sim 10^4$ TeV
- Fitted values of observables in table represent predictions



The Leading Order UT (LO-UT)

- Define useful complex constants:

$$\boldsymbol{z_0} \equiv \boldsymbol{\lambda_u^*}/\lambda_0 = i s_0\, e^{-i\beta_0} = \rho_0 + i\eta_0,$$
$$\boldsymbol{\bar{z}_0} \equiv \boldsymbol{\lambda_d^*}/\lambda_0 = c_0\, e^{-i\beta_0} = 1 - \boldsymbol{z_0},$$

where

$$s_0 \equiv \sin\beta_0; \quad c_0 \equiv \cos\beta_0; \quad \eta_0 = s_0 c_0 = \frac{1}{2\sqrt{2}} \quad \text{and} \quad \rho_0 = s_0^2.$$

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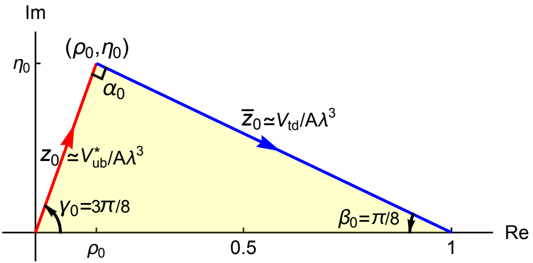
- Use to construct LO-UT

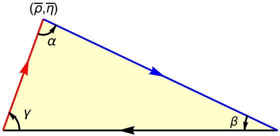


Symmetries of the M_q^{HS}

- Properties of the *paired system* (M_u, M_d), rather than of either in isolation
- Could be viewed as consequence of forms, or, preferably, as ab initio symmetries which constrain (M_u, M_d) forms
- Elucidated below

CP Transformation and Rephasing

- CP:
- Under CP, all complex numbers in the MMs are complex-conjugated.



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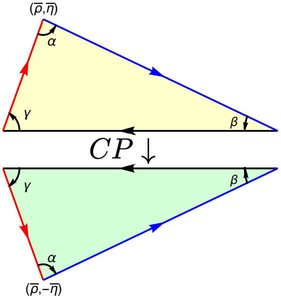
- Observable effect is to flip orientation of UT in complex plane ($\overline{\eta} \to -\overline{\eta}$)
- Unless $\overline{\eta} = 0$ (CP is conserved)



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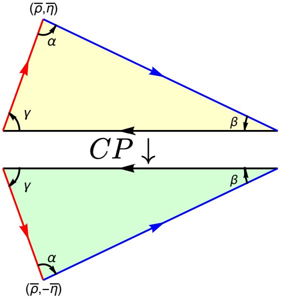
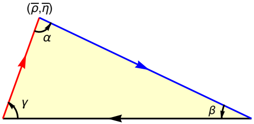


- Rephasing:
- Simultaneous *phase changes* of M_u and M_d unobservable



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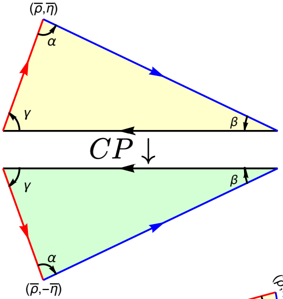
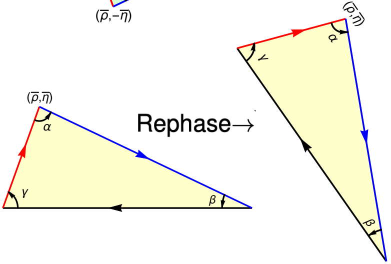
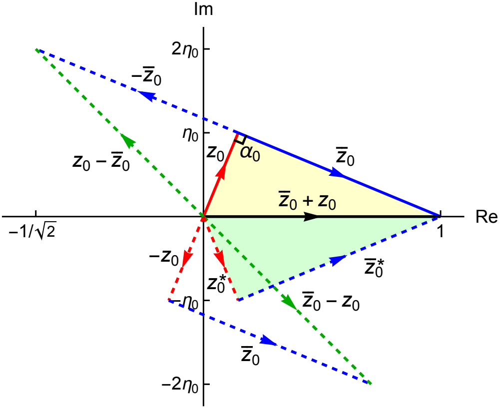


- Rephasing:
- Simultaneous *phase changes* of M_u and M_d unobservable
- UT simply rotates in complex plane
- (Physical) shape and size invariant



Symmetry for $\alpha_0 = \frac{\pi}{2}$

- In HS texture, simple sign change of z_0 (or of $\overline{z}_0$, but not both), flips orientation of the UT (see fig)
- Is only observable effect
- But iff $\alpha = \pm\frac{\pi}{2}$



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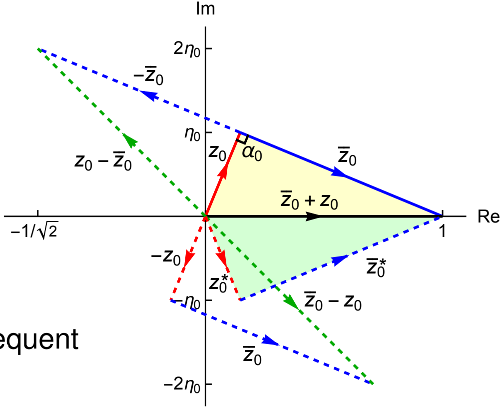
- It is equivalent to a CP transformation
- Can be reversed by a subsequent actual CP transformation
- Symmetry is good to all orders

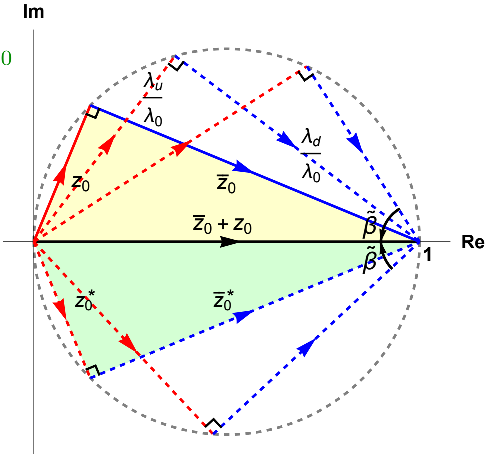


Symmetry for $\beta_0 = \frac{\pi}{8}$

- First consider $\beta_0 = \tilde\beta \neq \frac{\pi}{8}$ (fig$\rightarrow$) keeping $\alpha = \frac{\pi}{2}$ and $\boldsymbol{\lambda_u} + \boldsymbol{\lambda_d} = \lambda_0$
- Clearly now

$$\left|\frac{\boldsymbol{\lambda_u}}{\boldsymbol{\lambda_d}}\right| = \tan\tilde\beta,$$

and $-\frac{\pi}{2} < \tilde\beta < \frac{\pi}{2}$



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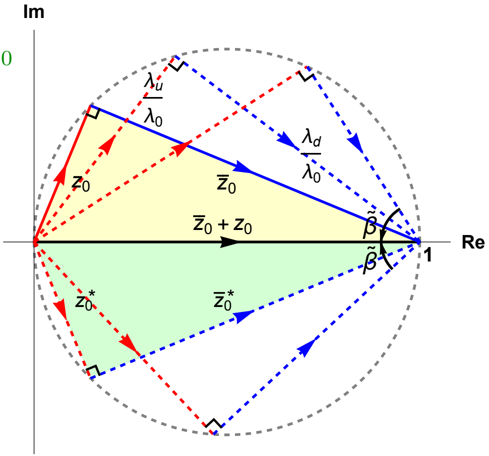
$$\tilde\beta \rightarrow \tilde\beta - \frac{\pi}{4} \qquad (*)$$

- Iff $\tilde\beta = \frac{\pi}{8}$, the result is just a CP transformation



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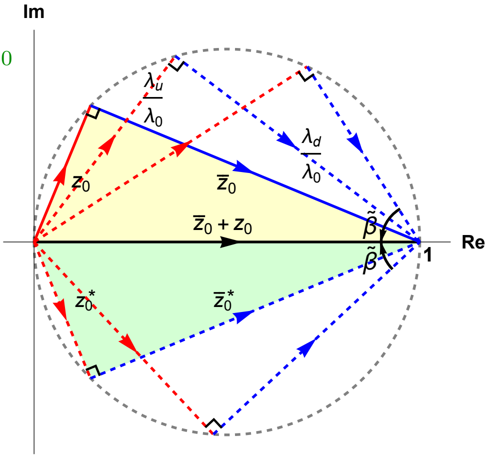

- The symmetry to fix $\beta_0 = \frac{\pi}{8}$ is transformation $(*)$ followed by CP transformation

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- Proposed geometric-hierarchical MM texture
- Mass hierarchy "slopes" are related to UT sides
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- Precise predictions of UT angles:
  - $\alpha - \frac{\pi}{2} = (1.30 \pm 0.02)^\circ$ c.f. $(1.6 \pm 1.4)^\circ$ (exp)
  - $\beta - \frac{\pi}{8} = (-0.2 \pm 0.1)^\circ$ c.f. $(0.1 \pm 0.4)^\circ$ (exp)
  - $\gamma - \frac{3\pi}{8} = (-1.1 \pm 0.1)^\circ$ c.f. $(-1.8 \pm 1.3)^\circ$ (exp)

Backup Slides

Isospin Reflection Symmetry?

- Can re-write texture:

$$M_q^{HS} \equiv n_q \begin{pmatrix} c' \lambda_q^4 & b' \boldsymbol{\lambda_q^3} & 0 \\ b' \boldsymbol{\lambda_q^{*3}} & b' \lambda_q^2 & A_0 \boldsymbol{\lambda_q^2} \\ 0 & A_0 \boldsymbol{\lambda_q^{*2}} & 1 \end{pmatrix} \pm d\, \lambda_q^4 I$$

  $b' \simeq b$. Still get good fit to data.
- First (leading) matrix solely responsible for quark mass differences and mixing parameters.
- Second (small) matrix is I_z-dependent "pedestal" on quark masses. Symmetric under a generation-$SU(3)$ symmetry.
- All coefficients (λ_0, A_0, b', c', d) symmetric under isospin reflection operator $u \leftrightarrow d$.
- Symmetry broken (only) by $\boldsymbol{\lambda_q}$, n_q and the sign of d.

Analytic NLO Solutions: 1) Mixing Parameters

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- We give here the algebraic NLO solutions of the texture:

$$\lambda = \lambda_0\left(1+\tfrac{1}{4b}f_\lambda {\lambda_0}^2\right)+\mathcal{O}({\lambda_0}^5)$$
$$A = A_0\left[1+\tfrac{1}{4b}(3b^2-2b\rho_0-2f_\lambda){\lambda_0}^2\right]+\mathcal{O}({\lambda_0}^4)$$
$$\overline{\rho} = \rho_0\left\{1+\tfrac{1}{4b}\left[(1+\sqrt{2})f_\rho+\sqrt{2}f_b\right]{\lambda_0}^2\right\}+\mathcal{O}({\lambda_0}^4)$$
$$\overline{\eta} = \eta_0\left\{1+\tfrac{1}{4b}\left[f_\rho+2b(1-5b)+4\rho_0 f_b\right]{\lambda_0}^2\right\}+\mathcal{O}({\lambda_0}^4)$$

where $f_\lambda = 3({A_0}^2+\overline{c})-5b+\sqrt{2}\,\delta_c$, $\overline{c}=\frac{1}{2}(c_d+c_u)$, $\delta_c = c_d - c_u$, $f_\rho = -2({A_0}^2+\overline{c})+7b^2-2\delta_c$ and $f_b = b+\delta_c$.

- NLO corrections above, as fractions of LO terms are respectively: -5.8×10^{-3}, $+2.6\%$, $+3.6\%$ and -1.8% (using fitted param values from table).

Analytic NLO Solutions: 2) Mass Ratios

- For the quark mass ratios, we find:

$$\frac{m_1^q}{m_2^q} = -{\lambda_q}^2(1-r_q)\left\{1+\left[r_A\frac{(2-r_q)}{(1-r_q)}-2\right]{\lambda_q}^2\right\}+\mathcal{O}({\lambda_q}^6)$$
$$\frac{m_2^q}{m_3^q} = b{\lambda_q}^2\left[1+(1-r_A){\lambda_q}^2\right]+\mathcal{O}({\lambda_q}^6),$$

  where $r_q = \frac{c_q}{b}$ and $r_A = \frac{{A_0}^2}{b}$.

- NLO corrections to mass ratios m_c/m_t, m_s/m_b, m_u/m_c, m_d/m_s as fractions of LO terms are (resp.) -4.3×10^{-3}, -2.5%, $+4.3\%$, and $+4.5\%$ (using fitted param values from table).
- All results compatible with full numerical results reported in table.