Leptonic Mixings from a Variational Principle

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Outline of Talk

- Flavour Physics - the Issues.
- Common Origin of Masses and Mixing in the Standard Model
- Historical Attempts to Constrain Masses and Mixings
- Jarlskog Covariance
- Flavour-Symmetric Jarlskog Invariants
- Covariant Constraints from Extremisation
- Some Examples
- The Future

Work done in collaboration with Bill Scott (RAL) - hep-ph/0508012.

Draws heavily on and extends the seminal contributions of Cecilia Jarlskog to flavour physics from 1985 $\longrightarrow$ 1988.

Parameters of the Standard Model

The Standard Model (minimally extended to accommodate neutrino masses, analogously to the other fermions) has ~ 25 free parameters:

- 3 gauge coupling constants - g_1, g_2, g_3
- 2 Higgs parameters - μ, λ
- 6 quark masses - m_d, m_s, m_b, m_u, m_c, m_t
- 6 lepton masses - m_e, m_μ, m_τ, m_1, m_2, m_3
- 3 quark mixing angles and a phase - θ_{us}, θ_{cb}, θ_{ub}, δ_q
- 3 leptonic mixing angles and another phase - θ_{12}, θ_{23}, θ_{13}, δ_l.

No accepted theory for predicting any of these (or relating them to each other).

There are, of course, many efforts (unsuccessful or unproven).

This is a new (and quite original) one.

Flavour Physics: the Issues

20/25 of SM parameters differentiate between the flavours or relate them to each other.

Most of the phenomena of the SM are parameterised by these quantities - mass-differences between fermions, branching fractions in weak decays, CP violation...

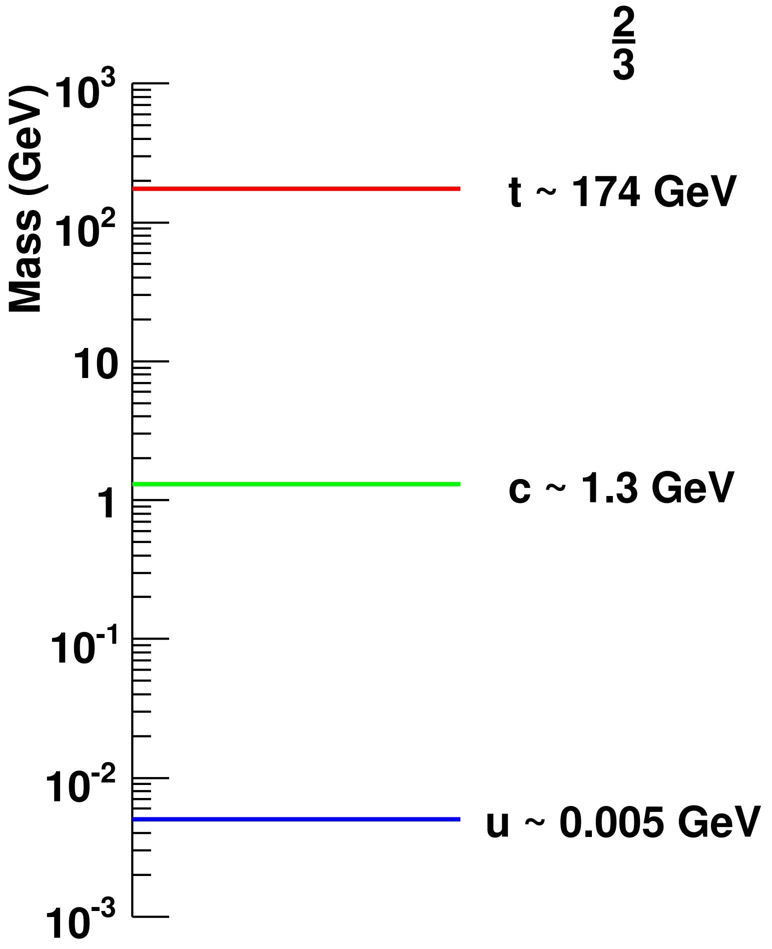
Gauge sector of SM has only 3 parameters (plus 2 Higgs couplings). SM is internally consistent with only one family of quarks and leptons.

Why 3 families? I can't answer that.

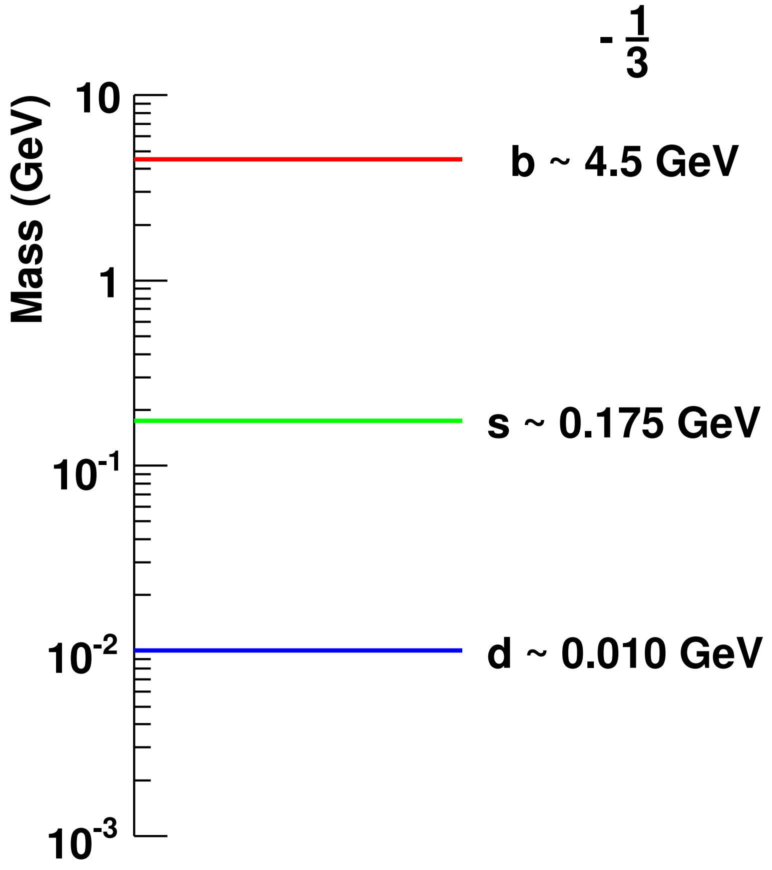
What are the unmeasured parameters? We are working on neutrino experiments to try to answer that.

Why Such Peculiar Quark Mass Spectra?

Charge $= +\frac{2}{3}$

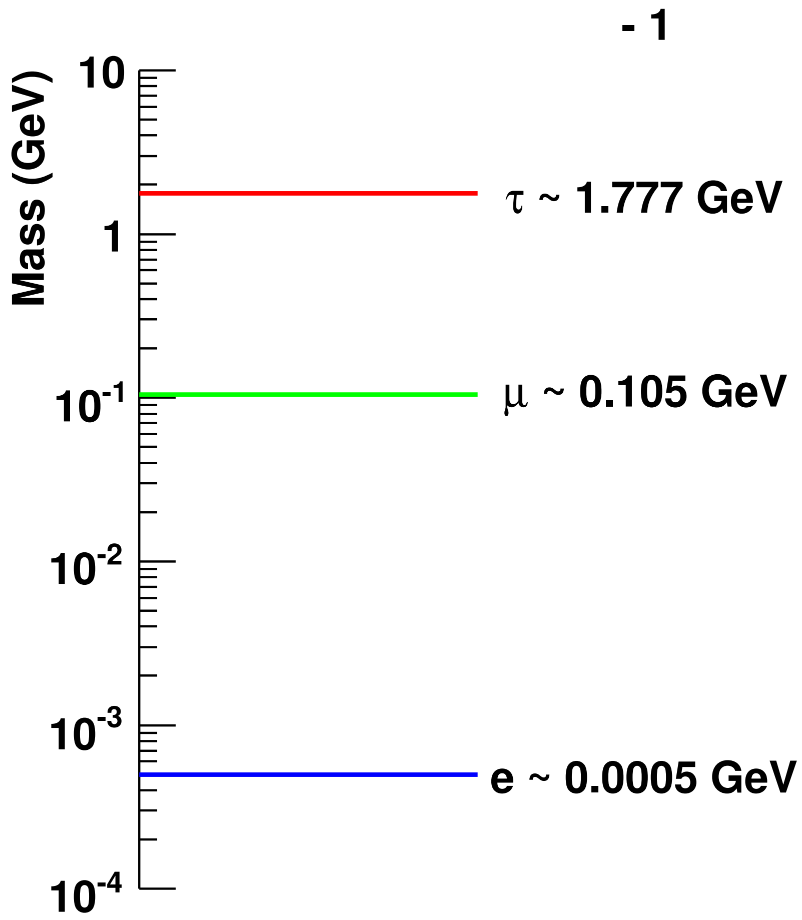


Charge $= -\frac{1}{3}$

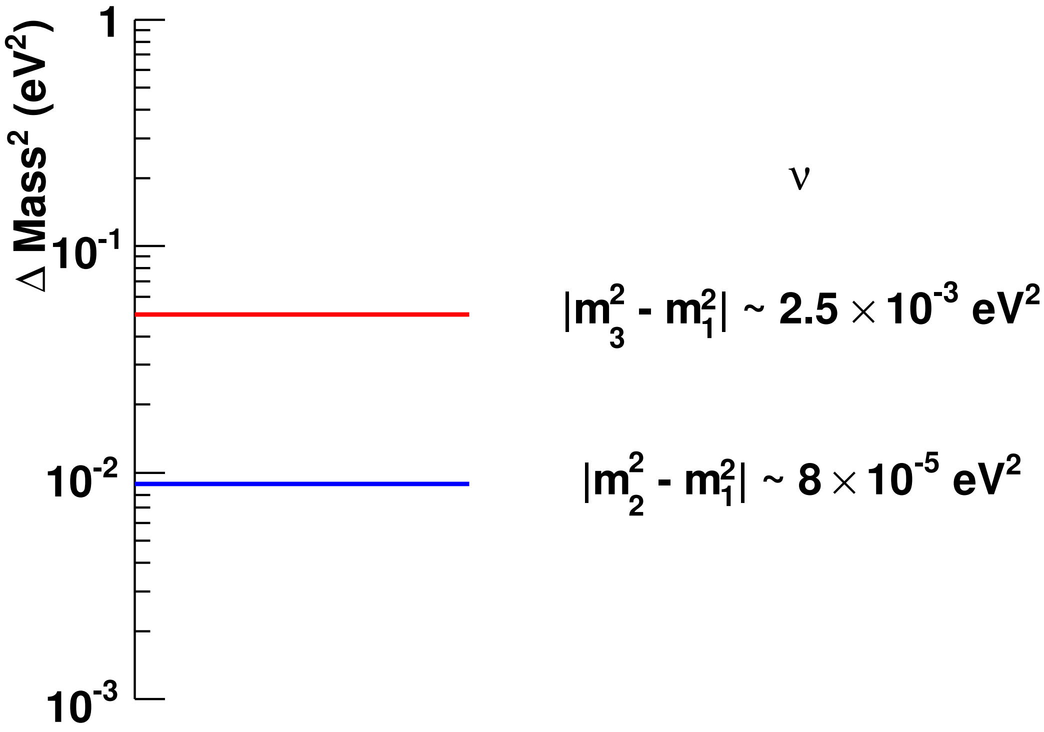


Why Such Peculiar Lepton Mass Spectra?

Charge $= -1$



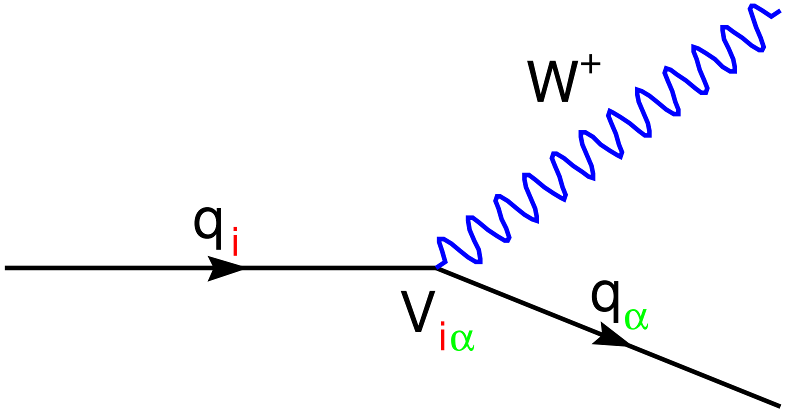
Charge $= 0$



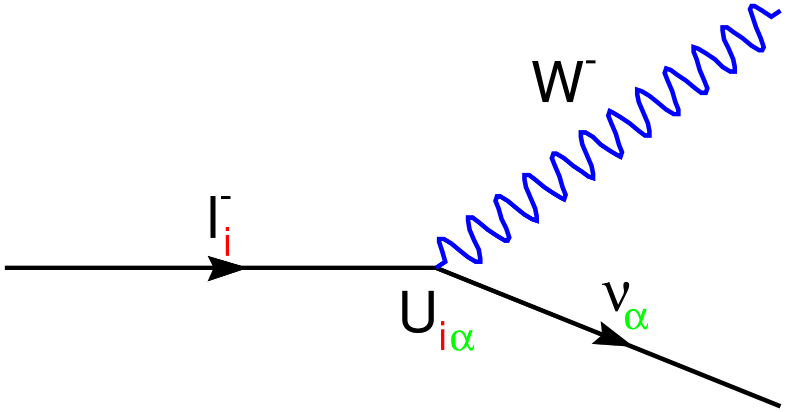
Why Such Peculiar Fermion Mixing Matrices?

MNS (U_{MNS}) and CKM (V_{CKM}) matrices parameterise couplings of mass eigenstates to W:

$$|V_{CKM}| \sim \begin{array}{c} \\ u \\ c \\ t \end{array} \begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{pmatrix} 0.974 & 0.224 & 0.004 \\ 0.224 & 0.974 & 0.04 \\ 0.01 & 0.04 & 0.999 \end{pmatrix} \end{array}$$



$$|U_{MNS}| \sim \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} 0.82 & 0.58 & < 0.17 \\ 0.41 & 0.58 & 0.71 \\ 0.41 & 0.58 & 0.71 \end{pmatrix} \end{array}$$



Common Origin of Masses and Mixing in the SM

Masses and mixings in the Standard Model have common origin in the "Yukawa couplings" of the fermions to the Higgs.

First we need to define some notation:

Will focus on leptons - consider 3 families of leptonic $SU(2)$ doublets:

$$\psi_a = \begin{pmatrix} \nu_a \\ \ell_a^- \end{pmatrix}, \quad \psi_b = \begin{pmatrix} \nu_b \\ \ell_b^- \end{pmatrix}, \quad \psi_c = \begin{pmatrix} \nu_c \\ \ell_c^- \end{pmatrix}$$

Each transforms among itself under $SU(2)$ weak transformations in usual way.

We define also two three-component vectors in family-space:

$$\underline{\nu} = \left(\nu_a, \nu_b, \nu_c\right), \quad \underline{\ell} = \left(\ell_a^-, \ell_b^-, \ell_c^-\right),$$

Origin of Masses and Mixing (contd.)

Will accommodate neutrino masses into SM with simplest extension, namely that neutrinos have Dirac masses.

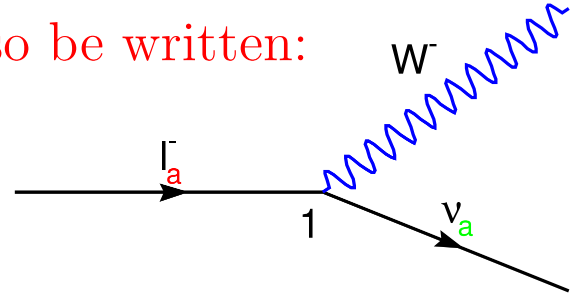
After spontaneous symmetry breaking, mass terms appear in the Lagrangian as follows (have simplified some details):

$$\mathcal{L}_{mass} = \underline{\bar{\ell}}\, L\, \underline{\ell} + \underline{\bar{\nu}}\, N\, \underline{\nu} + H.C.$$

The mass matrices, L and N, are arbitrary, 3×3 Hermitian matrices.

The charged-current weak interaction may also be written:

$$\mathcal{L}_{cc} = g(\underline{\bar{\ell}} \,.\, \underline{\nu})W^{+} + H.C.$$



It is “diagonal”: only leptons from the same doublet couple to each other.

Origin of Masses and Mixing (contd.)

We wish to diagonalise both matrices, in order to obtain the "physical" states.

So, we make the following unitary transformation on the fields:

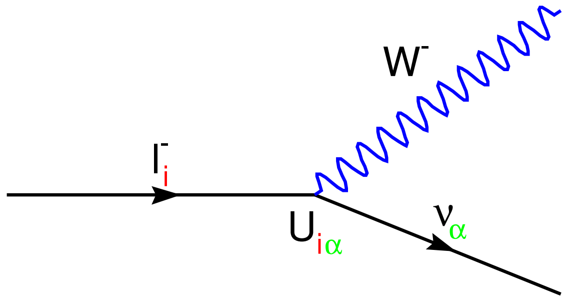
$$\underline{\ell} \to \underline{\ell}^{\,m} = U_\ell \, \underline{\ell} \,, \qquad \underline{\nu} \to \underline{\nu}^{m} = U_\nu \, \underline{\nu}$$

Then

$$\begin{aligned} \mathcal{L}_{mass} &= \bar{\underline{\ell}}^{\,m} \, U_\ell L U_\ell^\dagger \, \underline{\ell}^{\,m} + \bar{\underline{\nu}}^{\,m} \, U_\nu N U_\nu^\dagger \, \underline{\nu}^{m} \\ &= \bar{\underline{\ell}}^{\,m} D_\ell \, \underline{\ell}^{\,m} + \bar{\underline{\nu}}^{\,m} D_\nu \, \underline{\nu}^{m} \end{aligned}$$

where $D_\ell \equiv U_\ell L U_\ell^\dagger$ and $D_\nu \equiv U_\nu N U_\nu^\dagger$ are diagonal mass matrices. Also

$$\begin{aligned} \mathcal{L}_{cc} &= g(\bar{\underline{\ell}}^{\,m} \, U_\ell U_\nu^\dagger \, \underline{\nu}^{m}) W^+ + H.C. \\ &= g(\bar{\underline{\ell}}^{\,m} U_{MNS} \, \underline{\nu}^{m}) W^+ + H.C. \end{aligned}$$



where $U_{MNS} \equiv U_\ell U_\nu^\dagger$ is the MNS lepton mixing matrix. NB. Analogous for quarks.

Historical Attempts to Constrain Fermion Masses and Mixings

Inspired by some phenomenological relationships in quark mixing, eg.

$$\tan\theta_C \simeq 0.23 \simeq \sqrt{\frac{m_d}{m_s}}; \qquad \tan\theta_{cb} \simeq 0.04 \simeq \sqrt{\frac{m_d}{m_b}}$$

and the known common origin of masses and mixings, many attempts have been made to relate them.

Best known is by H. Fritzsch from 1977 and 1978: Mass matrices:

$$M^{\frac{2}{3}} = \begin{pmatrix} 0 & a & 0 \\ a^* & 0 & b \\ 0 & b^* & c \end{pmatrix}; \qquad M^{-\frac{1}{3}} = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha^* & 0 & \beta \\ 0 & \beta^* & \gamma \end{pmatrix}$$

Predicts (after some reasonable approximations) above θ_C relation, but also

$$\tan\theta_{cb} > \sqrt{\frac{m_s}{m_b}} - \sqrt{\frac{m_c}{m_t}} \simeq 0.09,$$

now excluded.

Historical Attempts (Contd.)

How does Fritzsch *Ansatz* work?

Mass matrices encode 10 pieces of information - 6 masses and 4 mixing parameters.

Ansatz uses only 6 magnitudes and one phase, $\implies$ 3 constraints.

Fritzsch's *Ansatz* has spawned a host of immitators, some who can do better in terms of predictions.

Although initially, just a mathematical device, modern models appeal to "flavour symmetries" to enforce the zeroes.

Many build-in supersymmetry, and/or extra dimensions, complicated Higgs structures etc.

However, all seem to assume the same basic premise, that $\exists$ a "special" basis, in which the mass matrices take some "special" form which limits the number of parameters to fewer than 10.

Jarlskog Covariance

Instead of diagonalising, we could have made the following *simultaneous* arbitrary unitary transformation:

$$\underline{\ell} \rightarrow \underline{\ell}' = U\underline{\ell}\,, \qquad \underline{\nu} \rightarrow \underline{\nu}' = U\underline{\nu}$$

Then

$$\begin{aligned}\mathcal{L}_{mass} &= \underline{\bar{\ell}}'\,ULU^\dagger\,\underline{\ell}' + \underline{\bar{\nu}}'\,UNU^\dagger\,\underline{\nu}' \\ &= \underline{\bar{\ell}}'L'\,\underline{\ell}' + \underline{\bar{\nu}}'N'\,\underline{\nu}'\end{aligned}$$

where $L' \equiv ULU^\dagger$ and $N' \equiv UNU^\dagger$ are new arbitrary Hermitian mass matrices. In this basis, the charged current is:

$$\begin{aligned}\mathcal{L}_{cc} &= g(\underline{\bar{\ell}}'\,UU^\dagger\,\underline{\nu}')W^+ + H.C. \\ &= g(\underline{\bar{\ell}}'.\,\underline{\nu}')W^+ + H.C.\end{aligned}$$

In terms of new fields, CC is still diagonal, but mass matrices have changed.

Jarlskog Invariance

New basis (with transformed mass matrices) looks like just as good a starting point

But what about the observables (masses and mixing matrix elements)?

Now:

$$\underline{\ell}^{\,m} = U_\ell\, U^\dagger \underline{\ell}^{\,\prime}, \qquad \underline{\nu}^{m} = U_\nu\, U^\dagger \underline{\nu}'$$

and

$$\mathcal{L}_{mass} \;=\; \bar{\underline{\ell}}^{\,m} D'_\ell\, \underline{\ell}^{\,m} + \bar{\underline{\nu}}^{\,m} D'_\nu\, \underline{\nu}^{m}$$

where $D'_\ell \equiv U_\ell U^\dagger L' U U_\ell^\dagger = D_\ell$ and $D'_\nu \equiv U_\nu U^\dagger N' U U_\nu^\dagger = D_\nu$ are unchanged. Also

$$\begin{aligned}
\mathcal{L}_{cc} &= g(\bar{\underline{\ell}}^{\,m}\, U_\ell U^\dagger U U_\nu^\dagger\, \underline{\nu}^{m})W^+ + H.C. \\
&= g(\bar{\underline{\ell}}^{\,m} U'_{MNS}\, \underline{\nu}^{m})W^+ + H.C.
\end{aligned}$$

where $U'_{MNS} \equiv U_\ell U^\dagger U U_\nu^\dagger = U_{MNS}$ is also unchanged.

Summary of Jarlskog Co(In)variance

Given two mass matrices L and N which, when diagonalised, give correct mass eigenvalues and mixing angles, can always make *simultaneous* unitary transformations on both and use resulting matrices. All observables (eigenvalues and mixing matrix elements) are invariant under this transformation.

[Related to issue of extra degrees of freedom in the mass matrices:
Well known that only 6 masses and 4 mixing parameters are observable, but two distinct (Hermitian, 3×3) mass matrices require 18 parameters to be specified. 8 degrees of freedom are therefore unobservable and the mass matrices cannot be unique. Extra d.o.f.s correspond to the arbitrary parameters in the unknown transformation matrix U.]

NB. Is not a "symmetry" - that would be if $L' = L$.

Nomenclature for Particular Bases

- "Mass basis" - (specified by the "m" superscript) is unique, up to a bunch of possible re-phasings (which I may come back to).
- "Weak (or gauge) basis" - Any basis in which the weak interaction is diagonal (eg. those specified by the un-superscripted fields, and/or by the "primed" fields). WB is not unique - different ones are all related by the U transformation. Any WB can be used to show gauge invariance, etc.
- "Flavour basis" - a particular choice of WB in which the charged lepton mass matrix, L', is diagonal (while N' is not). Corresponds to a special choice of U, and is unique (up to the definition of the phases of the charged lepton fields). It is the basis in which all 6 leptons carry one of the flavour indices e, μ, τ. It is the *conventional* basis.
- "Neutrino Mass basis" - the particular WB for which N' is diagonal and L' is not. Is not often used.

Our Principles for Constraining Masses and Mixings

Any theory of flavour which constrains masses and mixings should respect certain principles:

- Jarlskog demonstrated that the SM as it stands has no preferred basis in which to write the lepton (and quark) mass matrices. The mathematical expressions of *beyond* SM constraints should therefore be Jarlskog-covariant. If they are not, then there is a preferred basis, and we know of no physics which should prefer one basis over another.

- The SM as it now stands treats all families on an equal footing. This should continue to be the case, ie. mass/mixing constraints should be flavour-symmetric.

At lower priority, mass/mixing constraints should treat charged leptons and neutrinos symmetrically. This would appear to be in line with $SU(2)$ gauge symmetry, although because this is broken by weak hypercharge, there is perhaps scope to relax this (as we will need, in fact).

Flavour-Symmetric Jarlskog Invariants

Conventional observables are not, in themselves, flavour-symmetric. eg. the electron mass, m_e, carries the electron flavour label and U_{e3}, for example, carries the obvious flavour and mass labels.

Jarlskog also showed, however, how to write flavour-symmetric invariant observables.

Examples are (all matrices in an arbitrary weak-basis, primes dropped):

$$\mathrm{Tr}(L) = m_e + m_\mu + m_\tau; \quad \mathrm{Tr}(L^2) = m_e^2 + m_\mu^2 + m_\tau^2; \quad \mathrm{Tr}(L^3) = m_e^3 + m_\mu^3 + m_\tau^3$$

which form a complete set and are obviously invariant:

$$\begin{aligned} \mathrm{Tr}(L') &= \mathrm{Tr}(ULU^\dagger) = \mathrm{Tr}(LU^\dagger U) = \mathrm{Tr}(L); \\ \mathrm{Tr}(L'^2) &= \mathrm{Tr}(ULU^\dagger ULU^\dagger) = \mathrm{Tr}(UL^2U^\dagger) = \mathrm{Tr}(L^2) \end{aligned}$$

etc.

And for Mixing...

We derived earlier that $U_{MNS} \equiv U_\ell U_\nu^\dagger$.

Clearly, if we had $U_\nu = U_\ell$, then $U_{MNS} = I$ and there would be no mixing.

Hence, non-zero mixing $\implies U_\nu \neq U_\ell$ and L and N are diagonalised by different unitary matrices, ie. they are not simultaneously diagonalisable.

Undergraduate quantum mechanics tells us that if two matrices are not diagonalised by the same operator, then they do not commute.

Hence, a signature of non-zero mixing must be that the commutator

$$iC = [L, N] \neq 0$$

Clearly, under a change of weak basis, C transforms according to $C \to C' = UCU^\dagger$ and any polynomial of C tranforms likewise. As before, traces are invariant.

However, $\mathrm{Tr}C = 0$ for any commutator. Therefore only two indepenent commutator invariants can be formed from C: $\mathrm{Tr}C^2$ and $\mathrm{Tr}C^3$.

Continued...

It turns out that $\mathrm{Tr}C^3$ is a pure-CP-violating observable:

$$\begin{aligned}\mathrm{Tr}\, C^3/3 &= \mathrm{Det}\, C \\ &= -2(m_\mu - m_\tau)(m_\tau - m_e)(m_e - m_\mu)(m_2 - m_3)(m_3 - m_1)(m_1 - m_2)J\end{aligned}$$

where J is twice the area of the unitarity triangle. It is manifestly symmetric among all flavour indices and all mass indices.

$\mathrm{Tr}C^2$ is a pure-CP-conserving observable, and is a somewhat more complicated (but also manifestly symmetric among flavour and mass labels) function of lepton masses and mixing invariants.

If mixing is zero, then both these invariants are zero. Hence, they represent different flavour-symmetric *measures* of mixing.

Jarlskog-Covariant Constraint on the Mixing

Motivated by quasi-maximal nature of the leptonic mixings, were motivated to try to "explain" this by maximising (more formally, extremising) a flavour-symmetric Jarlskog invariant, $\mathcal{A} = \mathrm{Tr}[f(L, N)]$.

Extremisation means finding values for which derivatives of invariant is zero.

Derivative wrt what? (mixing angles, masses?).

Doh! Derivatives wrt elements of matrices themselves, of course:

$$\partial_L(\mathcal{A}) = \partial(\mathcal{A})/\partial L = 0; \qquad \partial_N(\mathcal{A}) = \partial(\mathcal{A})/\partial N = 0.$$

Treats all indices symmetrically, and leads inevitably to Jarlskog-covariant matrix constraints!

We re-discovered some established results on differentiating scalar functions of matrices with respect to their elements.

Continued...

Question was, which invariant? (We tried several!)

As an example, consider $\mathrm{Tr}C^3$. Extremisation conditions are:

$$\begin{aligned}\partial_L \,\mathrm{Tr}\, C^3/3 &= -i[N, C^2]^T = 0,\\ \partial_N \,\mathrm{Tr}\, C^3/3 &= +i[L, C^2]^T = 0.\end{aligned}$$

Manifestly Jarlskog-covariant (ie. basis-independent) constraints: both sides of the equation are evidently form-invariant under an arbitrary $U(3)$ transformation $L \to U L U^\dagger$, $N \to U N U^\dagger$. We would obtain the same constraints performing the extremisation in any basis.

The equations obviously treat all flavour and mass labels symmetrically and therefore satisfy our two principal requirements.

This is a general property of our extremisation procedure.

Finally, L and N are treated symmetrically in this case.

Make a Modest Start

Before proceding, decide to extremise at *fixed* lepton masses.

(Allowing them to vary is the ultimate goal, but wanted a modest start.)

Extremisation with constraints $\implies$ Lagrange multipliers.

Outcome is to modify RHS of equations:

$$
\begin{aligned}
(\partial_L \,\mathrm{Tr}\, C^3/3)^T = -i[N, C^2] \quad &= \quad \lambda_{L0} \;+\; \lambda_{L1}\, L \;+\; \lambda_{L2}\, L^2 \\
(\partial_N \,\mathrm{Tr}\, C^3/3)^T = +i[L, C^2] \quad &= \quad \lambda_{N0} \;+\; \lambda_{N1}\, N \;+\; \lambda_{N2}\, N^2
\end{aligned}
$$

where the arbitrary constants λ are the LMs. (Eqs. still Jarlskog-covariant).

NB. 18 constraint eqns. but not all independent. Should exactly fix the mixings to some values, while all masses are fixed to known values.

Solution Method

Use covariance to choose any basis in which to solve. Choose flavour basis (L is diagonal, N is not) - off diagonal elements of ∂_L Eqn. $= 0$.

Further choose phase convention in which imaginary parts of N are all of equal modulus:

$$N = \begin{array}{c|c} & \begin{array}{ccc} e & \mu & \tau \end{array} \\ \begin{array}{c} e \\ \mu \\ \tau \end{array} & \begin{pmatrix} a & z+id & y-id \\ z-id & b & x+id \\ y+id & x-id & c \end{pmatrix} \end{array}$$

Variables a, b, c, x, y, z and d determine the three neutrino masses and the four mixing parameters

Solution for $\mathbf{Tr}\boldsymbol{C}^{\mathbf{3}}$

Solution is given by

$$a = b = c \qquad \text{and} \qquad x = y = z$$

with a, x and d fixed by the ν masses.

Corresponds to trimaximal lepton mixing (an old friend):

$$U_{MNS} \;=\; \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^* \\ 1 & \omega^* & \omega \end{pmatrix}$$

where $\omega = e^{2i\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, a cube-root of unity.

This corresponds (unsurprisingly) to maximal CP violation. Suggestive, but excluded by experiment.

Try Instead $\mathbf{Tr}\boldsymbol{C}^{\mathbf{2}}$

Constraints are now:

$$(-\partial_L \mathrm{Tr}\, C^2/2)^T = +[N, C] \;=\; \lambda_{L0} \;+\; \lambda_{L1}\, L \;+\; \lambda_{L2}\, L^2$$

$$(-\partial_N \mathrm{Tr}\, C^2/2)^T = -[L, C] \;=\; \lambda_{N0} \;+\; \lambda_{N1}\, N \;+\; \lambda_{N2}\, N^2.$$

Unsurprisingly, here we get a CP-conserving solution:

$$x = \pm\sqrt{(a-b)(c-a)E} \qquad E = \frac{(m_e - m_\mu)(m_\tau - m_e)}{(m_e + m_\mu - 2m_\tau)(m_\tau + m_e - 2m_\mu)}$$

$$y = \pm\sqrt{(b-c)(a-b)M} \qquad M = \frac{(m_\mu - m_\tau)(m_e - m_\mu)}{(m_\mu + m_\tau - 2m_e)(m_e + m_\mu - 2m_\tau)}$$

$$z = \pm\sqrt{(c-a)(b-c)T} \qquad T = \frac{(m_\tau - m_e)(m_\mu - m_\tau)}{(m_\tau + m_e - 2m_\mu)(m_\mu + m_\tau - 2m_e)}.$$

with a, b, c fixed by the ν masses.

Solution for $\mathbf{Tr}\boldsymbol{C}^{\mathbf{2}}$

Interestingly, the observed ν mass-squared difference hierarchy is correctly obtained setting $\frac{b-a}{a-c} = 1$ ($\Longrightarrow$ $\frac{\Delta m_{21}^2}{\Delta m_{31}^2} = 0.03$).

The mixing matrix $\rightarrow$

$$
(|U_{MNS}|^2) \; = \; \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \!\!\! \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} .33333 & .33333 & .33333 \\ .17079 & .16257 & .66663 \\ .49587 & .50409 & .00003 \end{pmatrix} \end{array} \simeq \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \!\!\! \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/6 & 1/6 & 2/3 \\ 1/2 & 1/2 & 0 \end{pmatrix} \end{array} .
$$

which is inconsistent with experiment.

BUT it is a simple permutation of the (viable) tribimaximal mixing matrix!

What Next?

We tried the obvious next step, ie. a linear combination:

$$\mathcal{A} = i\mathrm{Tr}C^3 + r\,\mathrm{Tr}C^2.$$

where r is a JI constant of dimensions m^2.

The solution gives a trajectory of mixing matrices, as a function of r (and the unknown offset of the neutrino masses).

Has some very interesting properties (see below), including viable mixing matrices and CP violation.

However, so far, have failed to find a solution which simultaneously fits observed masses and mixings.

Complete characterisation of solution space is work in progress.

Complete Set of Flavour-Symmetric, Invariant Measures of Mixing

$\exists$ four observables of mixing $\implies$ must be possible to find four independent flavour-symmetric invariant measures of mixing. We extended the set as follows:

Commutators of powers of L and N are independent measures of mixing:

$$C_{ij} \equiv [L^i, N^j].$$

Complete set is obtained for $i, j \leq 2$ (higher powers of L and N may be reduced to functions of lower powers and their traces, using the characteristic equation).

While these are not invariants, it is easy to construct (CP-conserving) invariants by analogy with $\mathrm{Tr} C_{11}^2$ considered already:

$$Q_{mn} \equiv \mathrm{Tr} C_{ij}.C_{kl}$$

where $m = i + k - 1$ and $n = j + l - 1$. There are exactly 9 of these, any four of which are independent.

What Works?

Returning to our extended set of commutator invariants, we tried another linear combination:

$$\mathcal{A} = Q_{11} + q\,Q_{21} = \mathrm{Tr}C_{11}^2 + q\,\mathrm{Tr}[C_{11}.C_{21}], \tag{1}$$

where q is a JI constant of dimensions m^{-1}.

This will have CP-conserving solutions.

The solution gives a trajectory of mixing matrices, as a function of q (and the unknown offset of the neutrino masses).

We (arbitrarily) choose $m_1 = 0$ (for simplicity) and look for a solution (to fit the phenomenology) with the ν_2 mass eigenstate trimaximally mixed,

ie. $|U_{\alpha 2}|^2 = 1/3, \forall\,\alpha$.

For fixed neutrino masses-squared differences, everything is now determined. We simply need to find q to fit the neutrino spectrum and the rest of the mixing matrix follows.

Solution for $\mathbf{Tr}\boldsymbol{C_{11}^2} + \boldsymbol{q}\,\mathbf{Tr}[\boldsymbol{C_{11}.C_{21}}]$

Allowing masses to float, trajectory (as function of q) has interesting properties:

It has 3 poles, each of which corresponds to a permutation of the tribimaximal form, and (in the limit as we approach each pole) has a single ν mass degeneracy.

As we approach the "correct" pole, Δm_{21}^2 tends to zero (not bad, cf. data). $\Longrightarrow$ smallness of $\sin\theta_{13}$ related to smallness of $\Delta m_{21}^2/\Delta m_{31}^2$.

Of course, $\Delta m_{21}^2/\Delta m_{31}^2$ is known (approximately) which fixes q and the mixing matrix:

$$(|U_{l\nu}|^2) = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} 0.662 & 0.333 & 0.005 \\ 0.219 & 0.333 & 0.448 \\ 0.120 & 0.333 & 0.547 \end{pmatrix} \end{array} \simeq \begin{array}{c} \\ e \\ \mu \\ \tau \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{pmatrix} \end{array}.$$

which is consistent with experiment.

The Future

We do not consider the successful invariant $\mathrm{Tr}C_{11}^2 + q\,\mathrm{Tr}[C_{11}.C_{21}]$ to be the last word: does not accommodate CP violation and extremisation is performed at fixed masses.

First job is to complete the analysis of the linear combination of $\mathrm{Tr}C^2$ and $\mathrm{Tr}C^3$, to look for viable CP-violating solutions.

Would like to find an invariant which can be extremised without fixing masses, to predict them also.

Need to understand how our procedure can be applied successfully to the quarks.

Discussion

This contribution is not an *extension* of the Standard Model - there are no new particles, no new symmetries and no new interactions.

If anything, it is a *contraction* of the SM. It relies on an existing covariance (JC) postulating a new principle, (extremisation of a JI) which provides constraints which reduce the number of free parameters in the Model.

Though original in the context of flavour physics, our extremisation procedure builds on treid and trusted traditions of our field - it looks like the priciple of least (extremal) action.

Perhaps our JI (expressed in appropriate dimensions) could be added to the SM action itself. Variations wrt the mass matrices result in new "constraint equations" which determine them.