PX263 problems

*These problems are not for credit. You will get most from them by trying to solve them yourself first. Information that may prove useful in solving them is given at the end. I will keep adding to these as PX263 progresses. ** (***) indicates questions of above-average (way-above) difficulty; don't worry if you can't manage these. To start with there is a little more emphasis on electrostatics than in the lecture notes.*

1. These are questions to check that you are happy computing gradients and divergences primarily. Show that

   (a) $\nabla r = \hat{\boldsymbol{r}}$,

   (b) $\nabla x = \hat{\boldsymbol{x}}$,

   (c) $\nabla \phi = (r \sin\theta)^{-1}\, \hat{\boldsymbol{\phi}}$,

   (d) $\nabla r^n = n r^{n-1}\, \hat{\boldsymbol{r}}$,

   (e) $\nabla . \vec{\boldsymbol{r}} = 3$,

   (f) $\nabla . \hat{\boldsymbol{r}} = 2/r$,

   (g) $\nabla \times \vec{\boldsymbol{r}} = \vec{\boldsymbol{0}}$. (Why is this last result "obvious"?)

   In part (c), r, θ and ϕ are the usual spherical polar coordinates.

2. These are actually questions which require you to compute the curl of some vectors (Remember that an electrostatic field is conservative and can be written as the gradient of a potential. A check on whether a field is conservative is whether its curl vanishes $(\nabla \times (\nabla\phi)) = 0$, for all ϕ). Which, if any, of the following, represent electrostatic fields?

   (a) $\vec{\boldsymbol{E}} = z\, \hat{\boldsymbol{x}} + x\, \hat{\boldsymbol{y}} + y\, \hat{\boldsymbol{z}}$,

   (b) $\vec{\boldsymbol{E}} = -(y/r)\, \hat{\boldsymbol{x}} + (x/r)\, \hat{\boldsymbol{y}}$, where $r = \sqrt{x^2 + y^2}$,

   (c) $\vec{\boldsymbol{E}} = \dfrac{(3x^2 - r^2)}{r^5}\, \hat{\boldsymbol{x}} + \dfrac{3xy}{r^5}\, \hat{\boldsymbol{y}} + \dfrac{3xz}{r^5}\, \hat{\boldsymbol{z}}$, where $r = \sqrt{x^2 + y^2 + z^2}$,

   (d) $\vec{\boldsymbol{E}} = -(y/r^2)\, \hat{\boldsymbol{x}} + (x/r^2)\, \hat{\boldsymbol{y}}$, where $r = \sqrt{x^2 + y^2}$.

3. This question is included to provide an illustration of the divergence theorem. It asks you to compute the charge density from the electric field in a spherically symmetric system using Gauss's law in integral form and in differential form (Maxwell's first equation). A static, spherical distribution of electric charge produces an electric field at position $\vec{\boldsymbol{r}}$ relative to the centre of the symmetry given by

   $$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \alpha r^2\, \hat{\boldsymbol{r}},$$

   where the radius $r = |\vec{\boldsymbol{r}}|$, α is a constant and $\hat{\boldsymbol{r}}$ is the unit vector in the radial direction.

   Use Gauss's law in integral form to deduce the charge density as a function of radius.

   Confirm that the differential form of Gauss's law (one of Maxwell's equations) leads to the same result.

4. This question asks you to do the same type of calculation as the previous question but for a system with cylindrical coodinates.

   An electric field has strength $E = E_0\left(1 - \mathrm{e}^{-|z|/L}\right)$, where E_0 and L are constants, and points away from the plane $z = 0$. Derive expressions for

   (a) the electrostatic potential ψ, and

   (b) the charge density ρ.

5. Derive an expression for the electrostatic potential ψ (which you may assume exists) that corresponds to the electric field

$$\vec{\boldsymbol{E}} = \frac{(2x^2 - r^2)}{r^4}\,\hat{\boldsymbol{x}} + \frac{2xy}{r^4}\,\hat{\boldsymbol{y}},$$

   where $r = \sqrt{x^2 + y^2}$.

   Specify a charge distribution that would cause such a field.

   (If you take the $\nabla \cdot \vec{\boldsymbol{E}}$ it should be zero except when $r \to 0$. So we'll have to use guesswork. Nothing depends on z, so the distribution is going to involve line charge(s). Look up the potential for a line charge. Imagine two lines of opposite charge a small distance s apart in, say, the x-direction and compute the corresponding potential. Write your answer as $s\partial\psi/\partial x$ and compare with your answer above.)

   Show that, in charge-free regions, each component of an electrostatic field satisfies Laplace's equation.

   Apply this result to a Coulomb-like potential $\psi = -1/r$, by interpreting the corresponding x-component, E_x, as a potential, i.e. what sort of potential does it represent?

6. Write down Maxwell's equations in free space and the continuity equation from memory. Check against your notes and mark yourself out of five.

7. ** How would you generalise Maxwell's equations in free space to allow for the (hypothetical) discovery of magnetic charges ("monopoles")?

   ***! If you want to try you could also look for an argument to the the amended Lorentz force law. [To work out the modified Lorentz force, consider the force between electric and magnetic charges, and apply Newton's third law.]

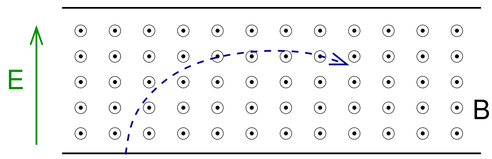
8. An infinite, dielectric medium of relative permittivity ϵ_r is uniform everywhere except for an empty spherical cavity of radius a. There are no free charges. Far from the hole, the electric field is uniform in the z-direction and of strength E_0. The question is to determine the exact fields and potentials inside and outside the spherical cavity.

   (a) Show that there are solutions to Laplace's equation in spherical polar coordinates of the form $\psi = r^n \cos\theta$, and determine possible values of the exponent n.

   (b) Show that one of the solutions to part (a) can describe the uniform field far from the cavity.

(c) By satisfying boundary conditions at the surface of the cavity and as $r \to \infty$, determine the electric potentials both inside and outside the spherical cavity. [Hint: consider potentials which are the superposition of the solutions of part (a), with different coefficients inside and outside the cavity.]

9. **



Two parallel plates separated by distance d have a voltage difference between them that leads to an electric field E as shown in the figure above. At the same time, a magnetic field B is applied perpendicular to E (in the figure it is directed out of the page). A particle of positive charge q and mass m is initially stationary on the lower plate and then starts to move under the influence of the electric and magnetic fields, as approximately indicated above.

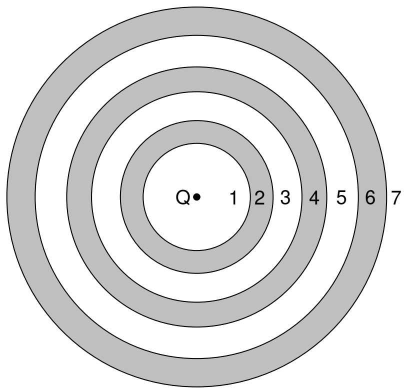
(a) Use the Lorentz force to write down equations of motion for the particle.

(b) Solve them to work out the nature of the particle's motion.

(c) Show that a sufficiently strong magnetic field will prevent the particle from reaching the upper plate. Derive an expression for the limiting value of B in terms of q, m, d and E.

10. Write down Maxwell's equations in matter in terms of $\vec{\boldsymbol{E}}$, $\vec{\boldsymbol{B}}$, $\vec{\boldsymbol{D}}$ and $\vec{\boldsymbol{H}}$, and the relations between $\vec{\boldsymbol{D}}$ and $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{H}}$ and $\vec{\boldsymbol{B}}$ involving $\vec{\boldsymbol{P}}$ and $\vec{\boldsymbol{M}}$. Do this from memory. Check against your notes and mark yourself out of six.

11.



A point electric charge Q is surrounded by three, uncharged spherical shells of inner and outer radii a_1, b_1, a_2, b_2, a_3, b_3, going from the innermost to the outermost shell, i.e. $a_1 < b_1 < a_2 < b_2 < a_3 < b_3$. In the figure above the shells are represents by the shaded annuli. The innermost and outermost shells are isotropic dielectrics with relative permittivities ϵ_1 and ϵ_2; the middle shell is a conductor.

(a) Derive expressions for the electric field strength as a function of radius r from the charge in regions 1 to 7. You should use Maxwell's equation $\nabla \cdot \vec{\boldsymbol{D}} = \rho_f$ and know that the answer in the conductor is zero (why?).

(b) Calculate the potential difference between the inside of the innermost shell and the outside of the outermost shell.

12. ***The Biot-Savart law can be written

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}') \times (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3} dV'.$$

It contains within it both the solenoidal condition and Ampère's law, as this question shows. Before starting this question, I advise reading through the 5 relations given under item (4) at the end of this sheet. Part a) asks you to show that this expression gives $\vec{\boldsymbol{B}}$ as the curl of something. This something is actually called the vector potential, denoted by $\vec{\boldsymbol{A}}$ and is the field that Maxwell used to set up the equations. This is discussed in Appendix D section 2.

(a) Show that the expression for $\vec{\boldsymbol{B}}$ may be written as

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\nabla \times \int_{V'} \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} dV'.$$

Here ∇ indicates derivatives with respect to the components of $\vec{\boldsymbol{r}}$ not $\vec{\boldsymbol{r}}'$.

(b) Hence show that $\nabla \cdot \vec{\boldsymbol{B}} = 0$.

(c) Show that

$$\nabla \times \vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\left[\nabla \int_{V'} \vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}') \cdot \nabla\left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) dV' - \int_{V'} \vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}')\nabla^2\left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) dV'\right].$$

(d) Hence show that $\nabla \times \vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{J}}$.

13. In the next two problems it is necessary to approximate how the electric and magnetic fields vary close to a point of interest. Considering just the x component of some vector field $\vec{\boldsymbol{W}}$, show that to first order in the small displacement $\vec{\boldsymbol{r}} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}}$, W_x near a point at position $\vec{\boldsymbol{r}}_0$ can be written

$$W_x(\vec{\boldsymbol{r}}_0 + \vec{\boldsymbol{r}}) = W_x(\vec{\boldsymbol{r}}_0) + x\,\partial_x W_x(\vec{\boldsymbol{r}}_0) + y\,\partial_y W_x(\vec{\boldsymbol{r}}_0) + z\,\partial_z W_x(\vec{\boldsymbol{r}}_0).$$

Here, the $\vec{\boldsymbol{r}}_0$ in the derivative terms indicates that the derivatives are to be evaluated at that point.

Hence justify the following more compact formulation of the same result:

$$\vec{\boldsymbol{W}}(\vec{\boldsymbol{r}}_0 + \vec{\boldsymbol{r}}) = \vec{\boldsymbol{W}}(\vec{\boldsymbol{r}}_0) + (\vec{\boldsymbol{r}} \cdot \nabla)\vec{\boldsymbol{W}}(\vec{\boldsymbol{r}}_0).$$

** Can you work out an expression for the expansion to higher-than-linear order in $\vec{\boldsymbol{r}}$?

14. Modelling an electric dipole as two equal-but-opposite charges of $\pm q$ separated by a vector $\vec{\boldsymbol{\ell}}$ from $-q$ to $+q$, such that $\vec{\boldsymbol{p}} = q\vec{\boldsymbol{\ell}}$, show that in the limit of small $\vec{\boldsymbol{\ell}}$, the force and torque on the dipole in an electric field $\vec{\boldsymbol{E}}$ are given by

$$\vec{\boldsymbol{F}} = (\vec{\boldsymbol{p}} \cdot \nabla)\vec{\boldsymbol{E}},$$
$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{p}} \times \vec{\boldsymbol{E}}.$$

[Hint: to calculate the force, use the linear order expansion given in the previous question; in the case of the torque, assume the field to be constant.]

15. A loop of wire carrying a current I is placed in a magnetic field $\vec{\boldsymbol{B}} = \vec{\boldsymbol{B}}(\vec{\boldsymbol{r}})$.

(a) Starting from the expression for the Lorentz force acting on a particle, convince yourself that the force on an element of wire, $d\vec{\boldsymbol{r}}$, carrying a current, I, is

$$d\vec{\boldsymbol{F}} = I d\vec{\boldsymbol{r}} \times \vec{\boldsymbol{B}}.$$

Hint: The velocity of the charges is $\vec{\boldsymbol{J}}/\rho$.

(b) Show that the total force and torque acting on the loop are given by:

$$\vec{\boldsymbol{F}} = I \oint_C d\vec{\boldsymbol{r}} \times \vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}),$$
$$\vec{\boldsymbol{\tau}} = I \oint_C \vec{\boldsymbol{r}} \times (d\vec{\boldsymbol{r}} \times \vec{\boldsymbol{B}}(\vec{\boldsymbol{r}})),$$

where C denotes a circuit around the wire defined by $\vec{\boldsymbol{r}}$ the position vector of the line elements along the wire.

(c) Show that if $\vec{\boldsymbol{B}}$ is constant, then the total force is zero.

(d) Assume that the loop is flat and that it lies in the x–y plane. Show that

$$\begin{aligned}\vec{\boldsymbol{r}} \times (d\vec{\boldsymbol{r}} \times \vec{\boldsymbol{B}}) =& (B_y y\,dx - B_x y\,dy + B_z z\,dx)\hat{\boldsymbol{x}} + \\ & (B_z z\,dy - B_y x\,dx + B_x x\,dy)\hat{\boldsymbol{y}} + (-B_z x\,dx - B_z y\,dy)\hat{\boldsymbol{z}}.\end{aligned}$$

(e) Use the result of parts (b) and (c) to show that, if $\vec{\boldsymbol{B}}$ is constant, then

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{m}} \times \vec{\boldsymbol{B}},$$

where $\vec{\boldsymbol{m}}$ is the magnetic dipole moment of the loop.
[You will need to use Green's theorem once you have simplified the integrals in the expression for $\vec{\boldsymbol{\tau}}$. Green's theorem says that

$$\oint_C L\,dx + M\,dy = \int_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx\,dy,$$

where D is the domain in the xy-plane, bounded by C, on which L and M have continuous derivatives. It is a special case of Stokes's theorem.]

(f) ** Now consider the relation for the force for the case of spatially varying $\vec{\boldsymbol{B}}$. By expanding to first order in position, i.e. making the approximation

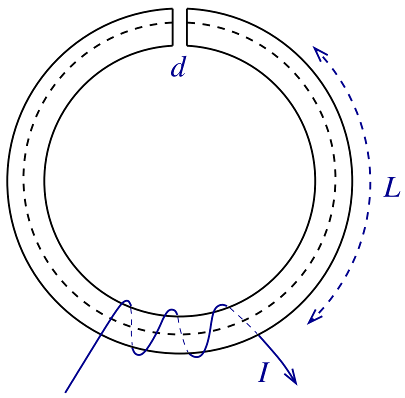
$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}_0 + \vec{\boldsymbol{r}}) = \vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}_0) + (\vec{\boldsymbol{r}} \cdot \nabla)\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}_0),$$

for small $\vec{\boldsymbol{r}}$ where the $\vec{\boldsymbol{r}}_0$ in the last term means that the values of any derivatives are those evaluated at $\vec{\boldsymbol{r}}_0$, show that

$$\vec{\boldsymbol{F}} = \nabla(\vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{B}}).$$

(You should again assume that the loop lies in the x–y plane.)

16.



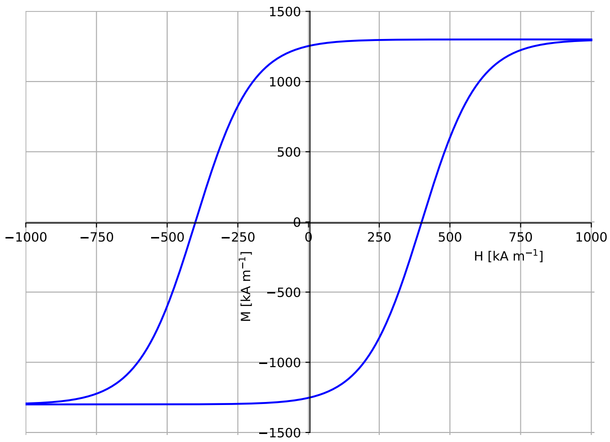
The figure shows a schematic picture of an electromagnet with N turns of a wire carrying current I wrapped around an iron core that has a length L and a toroidal shape but with a small air gap of width d. Show that

(a)
$$LH_I + dH_a = NI,$$
where H_I and H_a are the magnetic field strengths in the iron and air respectively, which you may assume to be the same at all points within the iron core for H_I and the air gap for H_a. [Hint: use Ampere's law adapted for matter, i.e. the integral form of $\nabla \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}}_f$.]

(b)
$$B_a = B_I,$$
where B_a and B_I are the magnetic flux densities in the air and iron. [Assume no lateral spreading of the field within the air gap.]

(c) Hence, assuming that within the iron $B = \mu_0 \mu_r H$, derive an expression for the magnetic flux density in the air gap. What is the maximum possible value of B?

17.



Permanent magnets: Suppose now a magnet of the same configuration as the previous question, except there is now no wire coil or electric current, and the material of the magnet does not satisfy $B = \mu_0\mu_r H$, but instead lies somewhere on the hysteresis curve of magnetisation M-vs-H shown above. Show that in this case

$$(L + d)H + dM = 0,$$

where H and M are the magnetic field strength and magnetisation in the material of the magnet.

Hence estimate graphically the magnetic flux density B for $L = 20\,\mathrm{cm}$ and $d = 5\,\mathrm{cm}$. [NB The units used for both axes of the plot are $\mathrm{kA\,m^{-1}}$.]

What is the maximum field this magnet could have, assuming that you could alter the width of the air gap?

18. When discussing EM waves in conductors, the current density was set proportional to the electric field, $\vec{\boldsymbol{J}}_f = g\vec{\boldsymbol{E}}$, neglecting the $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$ magnetic term. Is this is a fair assumption? Use the following to estimate the relative sizes of the two terms in the Lorentz force for copper: (1) $\vec{\boldsymbol{J}}_f = nq\vec{\boldsymbol{v}}$, where n is the density of charge carriers, q is the charge each carries, and $\vec{\boldsymbol{v}}$ is their mean velocity, (2) $g = 5.96 \times 10^7\,\Omega^{-1}\,\mathrm{m}^{-1}$ for Copper, (3) sensible assumptions for n and q, (4) relations between E and B for EM waves inside conductors, (5) assumed values for E and ω.

19. A coaxial cable consists of inner and outer cylindrical perfect conductors. The inner conductor has outer radius a while the outer conductor has inner radius b. A battery of voltage V is connected between the conductors at one end of the cable, while a resistance R is connected at the other end. Work out the electric and magnetic fields, E and H, inside the gap between the conductors and hence the Poynting vector. Hence calculate the total power carried by the Poynting flux in the cable.

20. In lectures, the following equation was derived for imaging at a single spherical surface
$$\frac{n_1}{u} + \frac{n_2}{v} = \frac{n_2 - n_1}{R},$$
where the centre of the spherical surface lay to the right of the surface itself. Show that if instead the centre of curvature lay to the left of the surface, the right-hand side of this equation switches sign.

21. In geometrical optics, the quantity $1/f$ where f is the focal length of a lens is referred to as its "power". Show that two thin lenses of focal lengths f_1 and f_2, placed right next to each other, act as a single lens with a total power equal to the sum of the powers of each lens $= 1/f_1 + 1/f_2$.

22. A lens is made of glass with refractive index n that varies linearly with wavelength as characterised by the derivative, $dn/d\lambda$. Derive an expression for the derivative of the power of the lens with respect to wavelength, i.e. $d(1/f)/d\lambda$ in terms of f, n and $dn/d\lambda$.

23. An "achromat" is a lens made by joining lenses made from different glasses having different values of n and $dn/d\lambda$ (see previous question). The aim is to make a lens that has non-zero power $1/f \neq 0$ but at the same time the power is achromatic, that is it does *not* vary with wavelength $d(1/f)/d\lambda = 0$. Use the results of the previous two questions to write down conditions that ensure that this will be the case.

24. In lectures, the lensmaker's equation and thin lens equation were derived by considering phase. An alternative approach is to consider the geometry of refraction and Snell's law. Use this approach to derive the equation

$$\frac{n_1}{u} + \frac{n_2}{v} = \frac{n_2 - n_1}{R},$$

for a single spherical surface from which the other results were deduced. Make use of small angle approximations (i.e. assume paraxial rays).

25. Consider an object being imaged onto a detector by a lens (similar to a camera or human eye). Sketch a ray diagram with the object in focus and also another object somewhat closer to the lens which is out of focus. If everything is fixed, apart from the diameter of the lens, convince yourself the out-of-focus object becomes more blurred as the lens diameter increases.

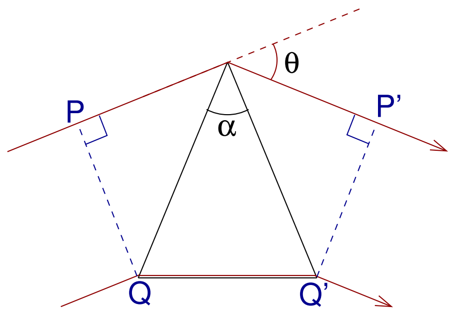
26. The simplest of all imaging systems is a pin-hole camera, which is simply a small hole and a screen some distance away from the hole. Light from the far side of the hole with respect to the screen propagates in a straight line through the hole to hit the screen, giving a one-to-one mapping of the external scene onto the screen. If the hole is too large, the image becomes blurred because of the geometry, but if it is too small, diffraction will dominate. Labelling the angular resolution set by these two effects as θ_{geom} and θ_{diff}, the total resolution of a pin-hole camera, θ_{pin}, is given by

$$\theta_{\text{pin}}^2 = \theta_{\text{geom}}^2 + \theta_{\text{diff}}^2$$

Use this and expressions for θ_{geom} and θ_{diff} to show that there is an optimum diameter for the hole that gives minimum θ_{pin} and derive an expression for it in terms of the wavelength of light λ, and the distance between the pin-hole and screen d.

With no fancy optics or need to focus, what's not to like about a pin-hole camera?

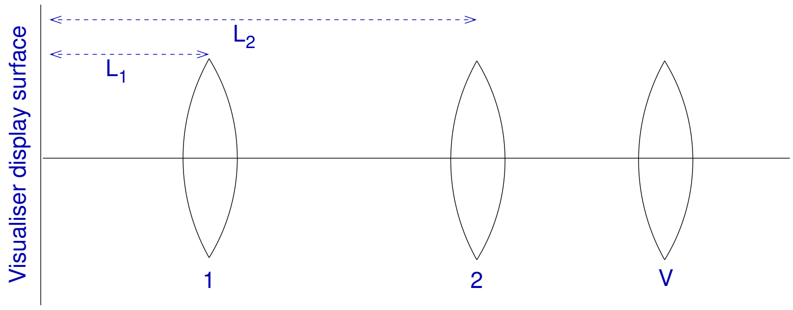
27.



The figure shows a prism of apex angle α, with light entering and leaving symmetrically, a configuration that leads to the minimum angular deflection θ. The figure shows two wavefronts PQ and P'Q'. The optical path travelled by the rays between these two wavefronts is the same, thus O.P.(PP') equals O.P.(QQ'). Use this to show that

$$n \sin\left(\frac{\alpha}{2}\right) = \sin\left(\frac{\alpha}{2} + \frac{\theta}{2}\right).$$

Hence derive an expression for θ in terms of n and α in the limit of small angles.

28. 

Suppose that I want to demonstrate magnification using the visualiser during lectures. I don't want the visualiser to try to focus on the lenses or my hands during the demonstration, so I have to switch off the autofocus. Therefore any lens or lenses I insert must give a virtual image located on the visualiser display surface, because then the image will also be in focus on the visualiser.

(a) Show that no single lens inserted between the visualiser's objective ("V" in the figure above) and the display surface will work for this demonstration.

(b) Show that a two lens configuration as illustrated with $f_1 = 5\,\text{cm}$, $L_1 = 7\,\text{cm}$, $f_2 = 10\,\text{cm}$ and $L_2 = 32.126\,\text{cm}$ will work as desired in terms of the location of the image seen by the visualiser.

(c) Calculate the linear magnification of the final image compared to the object in this case.

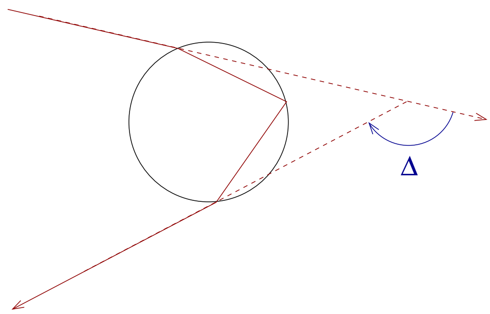
29. Show that the telescope drawn in lectures with two convex lenses will have inverted images.

30. **Both anti-reflection and dielectric mirror coatings are made with quarter-wave layers. Show that, rather counter-intuitively, *half-wave* layers have *no* effect on reflectivity, i.e. the reflectance is the same as if they were not there. Assume normal incidence and account for *all* reflections from the interfaces, not just the first two that we considered when discussing anti-reflection layers. You should obtain a geometric series, which you need to sum to prove the result claimed.

31. We considered optical coatings in normal incidence. At non-normal incidences, the individual reflection and transmission coefficients at each interface change as one would expect according to Fresnel's relations. However the phase change for the ray to traverse the thin layer (medium 2) and back, is more subtle. At normal incidence we used $\Delta\phi = 2k_2 d$. For non-normal incidence, you might guess that it would become $\Delta\phi = 2k_2 d/\cos t$, i.e. longer, where t is the angle of transmission through the 1–2 interface. Show instead that it becomes shorter, $\Delta\phi = 2k_2 d \cos t$.

32. In the consideration of the FP etalon the maximum amplitude, we showed that the maximum transmitted amplitude (which occurs whenever $\sin\phi = 0$) was $t^2 t'^2/(1 - r^2)^2$. Show that, in the absence of losses in the medium, this is equal to 1. If materials are lossy how qualitatively would this change the theory?

33.



**Rainbows form when light is scattered by spherical drops of water ($n = 1.33$). The primary rainbow is formed by rays that enter the drops, are reflected once internally, and then leave the drop. You may sometimes see secondary rainbows from rays that are reflected twice internally as striking double rainbows. Depending upon the angle of incidence, rays can be deflected through a wide range of angles, Δ, (see figure), all the way up to straight in and straight back ($\Delta = 180°$), but there is a minimum value for Δ where multiple close-by rays exhibit the same or similar deviations causing a pile-up that we see as a bright arc, i.e. the rainbow.

(a) Why is the internal reflection clearly not a "total internal" reflection?

(b) Calculate the minimum deflection angle Δ for a single internal reflection.

(c) Do the same as above for the secondary rainbow.

(d) Try to explain (i) why rainbows appear as arcs, (ii) where one should look for them on the sky, (iii) why the region between primary and secondary rainbows appears dark, (iv) the ordering of colours in each type.

(e) Why would you *not* easily see a rainbow near midday in the summer in the UK? How about the winter?

(f) ***Calculate the polarisation induced by the path of the figure, expressing your answer as a percentage $= 100(I_{max} - I_{min})/(I_{max} + I_{min})$ where I_{min} and I_{max} are the minimum and maximum intensities of orthogonal polarisations with the E-vector in the plane and perpendicular to the plane of the figure. Assume that the ray is initially unpolarised.

(g) Show that "diamond rain", as hypothesised for Jupiter and Saturn, if it came in spherical form (unlikely), would not exhibit a primary rainbow. (Refractive index of diamond = 2.42.)

To see much more about rainbows, including many nice pictures, discussion of diffraction phenomena they exhibit and some of the answers to this question, I recommend the website https://www.atoptics.co.uk/bows.htm

Useful information:

1. In the following f and $\vec{\boldsymbol{W}}$ represent arbitrary scalar and vector fields, and ∇ and ∇' should be taken to act upon $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{r}}'$ independently.

$$\begin{aligned}
\nabla \times (f\vec{\boldsymbol{W}}) &= f\nabla \times \vec{\boldsymbol{W}} + \vec{\boldsymbol{W}} \times \nabla f, && (1)\\
\nabla \times (\nabla \times \vec{\boldsymbol{W}}) &= \nabla(\nabla \cdot \vec{\boldsymbol{W}}) - \nabla^2 \vec{\boldsymbol{W}}, && (2)\\
\nabla \left[f(|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|)\right] &= -\nabla' \left[f(|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|)\right], && (3)\\
\nabla \left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) &= -\frac{(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3}, && (4)\\
\nabla^2 \left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) &= -4\pi\delta(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'). && (5)
\end{aligned}$$

2. If a vector field in spherical polar coordinates is written as $\vec{\boldsymbol{W}} = W_r\hat{\boldsymbol{r}} + W_\theta\hat{\boldsymbol{\theta}} + W_\phi\hat{\boldsymbol{\phi}}$, then its divergence is given by

$$\nabla \cdot \vec{\boldsymbol{W}} = \frac{1}{r^2}\frac{\partial(r^2 W_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(W_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial W_\phi}{\partial\phi}.$$

3. In spherical polar coordinates, Laplace's equation is given by

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = 0.$$

4. Green's theorem:

$$\oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

5. In the following f and $\vec{\boldsymbol{W}}$ represent arbitrary scalar and vector fields, and ∇ and ∇' should be taken to act upon $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{r}}'$ independently.

$$\begin{aligned}
\nabla \times (f\vec{\boldsymbol{W}}) &= f\nabla \times \vec{\boldsymbol{W}} + \vec{\boldsymbol{W}} \times \nabla f, && (6)\\
\nabla \times (\nabla \times \vec{\boldsymbol{W}}) &= \nabla(\nabla \cdot \vec{\boldsymbol{W}}) - \nabla^2 \vec{\boldsymbol{W}}, && (7)\\
\nabla \left[f(|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|)\right] &= -\nabla' \left[f(|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|)\right], && (8)\\
\nabla \left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) &= -\frac{(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3}, && (9)\\
\nabla^2 \left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) &= -4\pi\delta(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'). && (10)
\end{aligned}$$