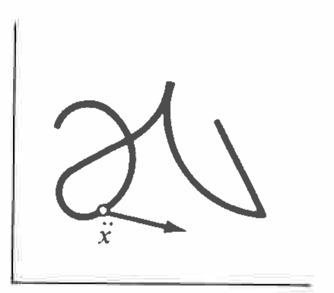
Quantum Phenomena Problems

1. * **Warming Up**

(a) Is it possible for a particle on a plane to have the trajectory shown in the figure if the particle is subject to finite forces? Is it possible for the acceleration vector to have the value shown? If the answer is no in either case justify your answer.



(b) Give two examples of wave motion that you encounter every day. In each case identify the physical quantities that are varying as a function of time and position?

(c) Which of the following are possible wavelengths (in cm) for oscillations of a vibrating wire fixed at both ends if the wire is 24 cm long:

$$8, 9, 12, 16, 24, 36, 48, 72 \ ?$$

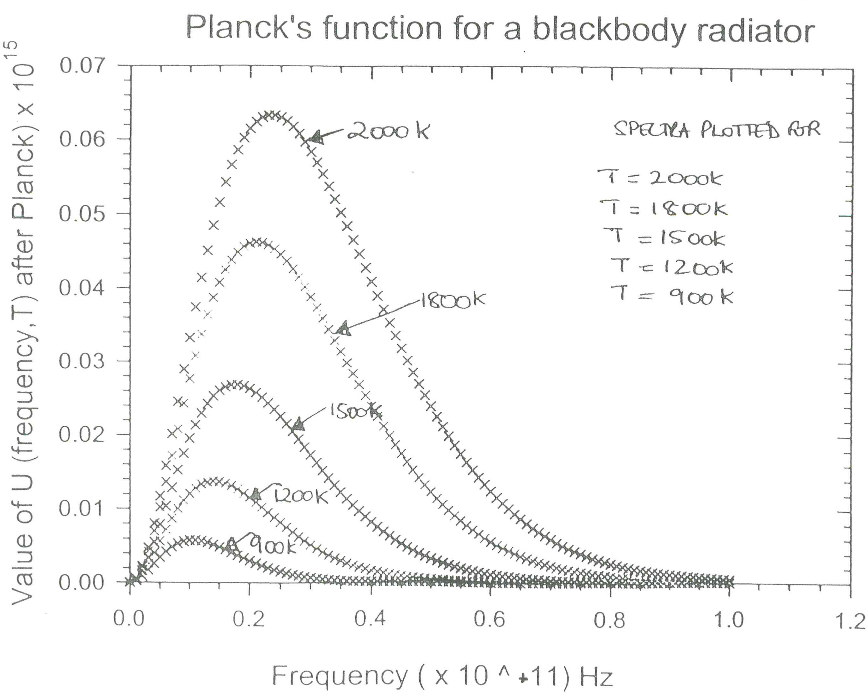
(d) Planck suggested that the energy E of a photon with frequency ν is given by

$$E = h\nu.$$

Deduce the units and dimensions of Planck's constant h. Give another important physical quantity which has the same dimensions. (Remember dimensions are things like length, mass, time, charge and combinations of them. Units are things like kilograms, seconds, metres, Coulombs, Joules, Volts, *etc* and combinations of them.)

2. **Blackbody Radiation**

(a)



The figure shows the blackbody spectrum of a body at 5 different temperatures T. Estimate the frequency ν_m, at which the radiation emitted is a maximum at each temperature. Hence show that T/ν_m is a constant and determine the value of the constant.

(b) The Rayleigh-Jeans formula for the rate at which energy should be radiated by a black body at temperature T at wavelengths between λ and $\lambda + d\lambda$ is

$$I(\lambda, T)\, d\lambda = A\frac{c\, kT\, d\lambda}{\lambda^4}$$

where c is the speed of light, k is Boltzmann's constant and A is the body's area. Compute the corresponding formula for the rate at which the body radiates energy at angular frequencies between ω and $\omega + d\omega$. (Remember that $\omega = 2\pi\nu$ and $d\lambda = (d\lambda/d\omega)\, d\omega$.)

QUANTUM PHENOMENA—EXAMPLES 2

Charge on electron: 1.6×10^{-19} C
Mass of electron: 9.1×10^{-31} kg
Mass of proton: 1.67×10^{-27} kg
Speed of Light: 3×10^{8} ms^{-1}
Planck's constant: $\hbar = 1.05458 \times 10^{-34} Js$;
Rydberg constant $R = \frac{1}{2}\frac{\mathrm{e}^2}{4\pi\epsilon_0 a} = 13.6\,\mathrm{eV}$, Bohr radius $a_0 = \frac{\hbar^2 4\pi\epsilon_0}{me^2}$
$h = 6.626 \times 10^{-34} Js$

3. * **Photoelectric and Compton Effects**

(a) The maximum energy of photoelectrons from aluminium is $3.68\,\mathrm{eV}$ for radiation of wavelength $2000\,\text{Å}$ ($1\text{Å} = 10^{-10}$m) and $1.44\,\mathrm{eV}$ for radiation of $3130\,\text{Å}$. Use these data to calculate Planck's constant. Check your result against the quoted value shown above.

(b) Compute the Compton wavelength for a proton using the formula given in the lectures and the data given above.

What is the maximum possible change in wavelength for a photon incident on a stationary proton? If the photon has an initial energy of 100MeV, what is the wavelength of the incident photon and what proportion of its initial energy does the scattered photon have, if the change of wavelength is a maximum? How much energy has the proton absorbed (give the answer in MeV)?

4. **Bohr Model of a Harmonic Atom**

Consider a particle moving at angular frequency ω around a circular orbit of radius r in a potential

$$V(r) = \frac{kr^2}{2},$$

where the particle has mass m and k is a constant. By balancing the force due to the potential and the acceleration towards the centre of the circular orbit, show that $\omega = \sqrt{k/m}$.

If the angular momentum is quantised in integer multiples of $\hbar$, *ie,* $L = n\hbar$, find the allowed values for the radius of the orbit in this 'harmonic atom'. Compute the corresponding energy level of the particle in the orbit as a function of n. What spectral lines would be visible in the emission from such an atom? (Your answer is best written in terms of the constant ω.)

5. * **Bohr Model of the H Atom**

(a) In the derivation of the Bohr model of the H atom in the lectures we assumed that the electron could be treated non-relativistically. Check whether this is a fair assumption.

(b) If, in the Bohr model, the electron were circling a nucleus with charge Ze what would the radius of the orbit with $n = 1$ and what would the lowest energy for the electron be? Give your answers in units of the Bohr radius, a_0, and the Rydberg constant R respectively.
(You should be able to do this without having to repeat the calculation from the lectures. Just set up the original equations, identify what has changed and then deduce what the corresponding change in the two quantities has to be.)

QUANTUM PHENOMENA—EXAMPLES 3

6. **de Broglie Wavelength**
Compute the de Broglie wavelength of:

(a) A dust particle of radius $r = 10^{-6}$m, density 1000 kg m^{-3} moving with velocity 0.01ms^{-1}. Comment on whether the investigation of the motion of such particles could be used to prove/disprove the de Broglie hypothesis.

(b) An electron with kinetic energy 10eV. Why does your answer suggest the possibility of electron diffraction experiments from crystals?

7. ***Time-dependent Schrödinger Equation**
The (unnormalised) solutions to Schrödinger's equation for a particle of mass m trapped in a 1D box between $x = 0$ and $x = L$ can be shown to be the ψ_n,

$$\psi_n = e^{-i\omega(k_n)t} \sin k_n x, \quad k_n = \frac{n\pi}{L}, \quad \omega(k_n) = \frac{\hbar^2 k_n^2}{2m},$$

and linear combinations of the ψ_n. Here the $\{n\}$ are positive integers.

If the wavefunction is

$$\psi = a_n \psi_n,$$

use the normalisation condition to compute the constant a_n.

If the wavefunction is

$$\psi = A(\psi_1 + \psi_2)$$

show that the probability density has a component which oscillates as a function of time and find the angular frequency of oscillation.

(The multiple angle formula

$$2 \sin\theta \sin\phi = \cos(\theta - \phi) - \cos(\theta + \phi)$$

may help you.)

QUANTUM PHENOMENA—EXAMPLES 4

Charge on electron: 1.6×10^{-19} C
Mass of electron: 9.1×10^{-31} kg
Speed of Light: 3×10^{8} ms^{-1}
Gravitational constant G = 6.673×10^{-11} m^3 kg^{-1} s^{-2}
Planck's constant: $\hbar = 1.05458 \times 10^{-34} Js \qquad h = 6.626 \times 10^{-34} Js$

8. **Heisenberg's Uncertainty Principle**
The spreading of wavepackets is an almost universal phenomenon. It happens whenever the group velocity is not constant for all relevant momenta (the group velocity $v_g = d\omega(k)/dk$). Components at one end of the spread of momenta in the wavepacket move at a different speed to those at the other.

Electrons with a mean kinetic energy E are described by a wave packet of width Δx.

(a) Estimate the uncertainty in momentum of the wavepacket?
(b) What is the average group velocity of the wavepacket?
(c) Estimate the uncertainty in its group velocity (use $m\Delta v_g = \Delta p/m$).
(d) Assuming that the parts of the wavepacket with momenta $p \pm \Delta p/2$ travel at group velocities $v_g \pm \Delta v_g/2$, estimate the width of the wavepacket after it has travelled a distance s.
(e) For the case $E = 13.6$ eV, $\Delta x = 1$ mm and $s = 10^4$ km estimate the new width of the wavepacket.

9. * **Semi-infinite Well**
A particle moves in the following potential

$$V(x) = \begin{cases} \infty & \text{for} \quad x < 0; \\ 0 & \qquad\; 0 \le x \le L; \\ V_0 & \qquad\; L < x. \end{cases}$$

(a) Sketch $V(x)$ as a function of x.
(b) Show that for a particle with energy $E < V_0$ the (unnormalised) stationary state wavefunctions are of the form:

$$\psi(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ A \sin kx & \qquad 0 \le x \le L \\ \alpha \exp -\kappa(x - L) & \qquad L < x \end{cases}$$

Here A and α are constants. Why can there be no term of the form $\beta \exp \kappa(x - L)$ in the region $x > L$?
(c) The boundary conditions at $x = L$ are that ψ and the derivative $d\psi/dx$ are continuous. Use this to show that

$$\tan kL = -\frac{k}{\kappa} = -\frac{k}{\sqrt{\tilde{V}_0 - k^2}},$$

where $\tilde{V}_0 = 2mV_0/\hbar^2$. This is a transcendental equation and so cannot be solved analytically.

(d) Assuming that $V_0' \gg k^2$ for the solutions of interest (so that κ may be taken to be independent of k), sketch $y = \tan kL$ and $y = -k/\kappa$ against k using the same set of axes for both curves and identify the values of k corresponding to solutions. Show that $k = n\pi/L$ (n integer) are approximate solutions in this limit.

(**Not for credit**)

(e) Indicate how you should amend your diagram, if you cannot assume that $\tilde{V}_0 \gg k^2$.

10. **The Planck Length**

Estimate the distance at which the classical theory of gravity (general relativity) breaks down as a result of quantum effects.

[Ans: $\sim 10^{-35}$ m]