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Gaussian systems for quantum enhanced multiple phase estimation[1]

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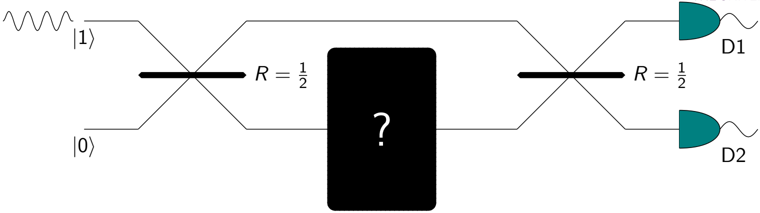
Postgraduate Seminars, University of Warwick (2nd June 2016)

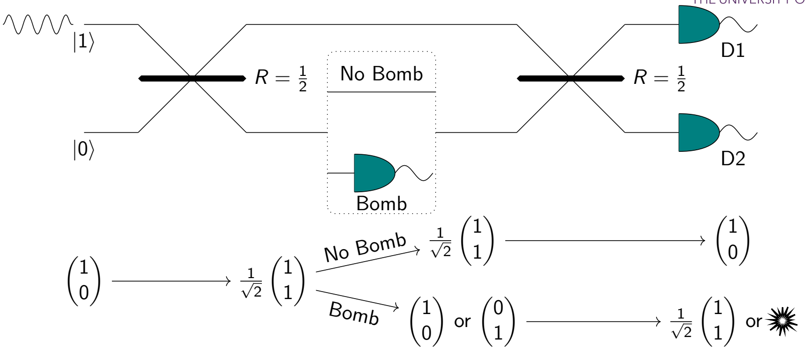
[1] Christos N. Gagatsos, Dominic Branford, and Animesh Datta. "Gaussian systems for quantum enhanced multiple phase estimation". In: *arXiv:1605.04819 [quant-ph]* (May 2016).

The Quantum Bomb Detector



Inside the black box may or may not be a bomb. The bomb will be detonated if it detects even a single photon.

The Quantum Bomb Detector



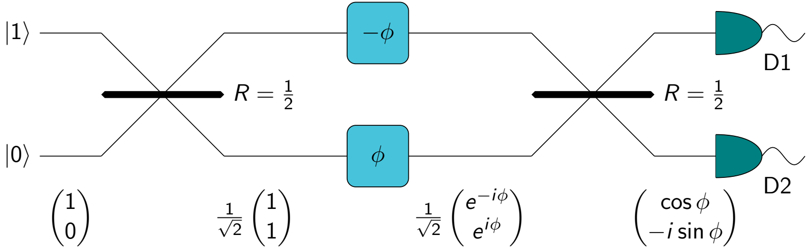
D2 clicks → we detect the bomb without blowing it up.

Detecting More Than Bombs

- Phases $\exp(i\phi\sigma_z)$
- Reflectivitives $\exp(iR\sigma_x)$
- Passive unitaries generally
- Phase diffusion and decoherences
- Squeezings and displacements

Mach-Zehnder Interferomtery

Single phase estimation

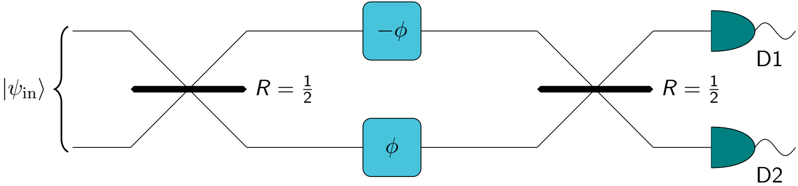


- $P(\mathrm{D1}) = \cos^2\phi$
- $P(\mathrm{D2}) = \sin^2\phi$

$$\tilde{\phi} = \cos^{-1}\left(\sqrt{\frac{N_{\mathrm{D1}}}{N_{\mathrm{D1}}+N_{\mathrm{D2}}}}\right)$$

Mach-Zehnder Interferomtery

Some states are more equal than others



- Single photons, $|1, 0\rangle$: $\delta^2\phi \sim \frac{1}{N}$
- Coherent light, $|\alpha, 0\rangle$: $\delta^2\phi \sim \frac{1}{N}$

} Classical ⇔ Shot Noise Limit

- Holland-Burnett, $|N, N\rangle$: $\delta^2\phi \sim \frac{1}{N^2}$
- Squeezed vacuum, $|\xi, \xi\rangle$: $\delta^2\phi \sim \frac{1}{N^2}$

} Quantum ⇔ Heisenberg Limit

Cramér-Rao Bound (CRB)

Finite resources = finite precision

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$$\mathrm{Cov}(\phi_i, \phi_j) \geq \frac{1}{M} \left(F^{-1}\right)_{ij} \tag{1}$$

$$F_{ij} = \int \mathrm{d}\mathbf{x} \frac{1}{P\left(\mathbf{x}|\phi\right)} \frac{\partial P\left(\mathbf{x}|\phi\right)}{\partial \phi_i} \frac{\partial P\left(\mathbf{x}|\phi\right)}{\partial \phi_j} \tag{2}$$

- Precision is lower bounded by the classical Fisher information (the Cramér-Rao bound)
- Best measurement is when $\sum_i \delta^2 \phi_i$ is minimised $\Leftrightarrow$ when $\mathrm{Tr}\left(F^{-1}\right)$ is minimised
- Measurement dependent $\Rightarrow$ always attainable

Quantum Cramér-Rao Bound (QCRB)

- CRB is measurement dependent $\Rightarrow$ hard to calculate

- Quantum CRB is a measurement independent lower bound on the CRB

$$\mathrm{Cov}(\phi_i, \phi_j) \geq \frac{1}{M} \left(F^{-1}\right)_{ij} \geq \frac{1}{M} \left(H^{-1}\right)_{ij} \tag{3}$$

Gaussian States

- Gaussian states are gaussian

- Single mode pure gaussian state $|\alpha, \xi\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle$

- Coherent (laser) light and thermal light both gaussian states

Gaussian States

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  *in terms of their characteristic function/Wigner function/Husimi Q function/Glauber P funct...

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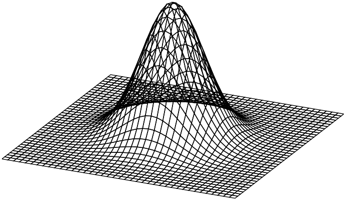
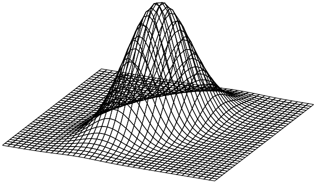
Gaussian States (Wigner functions)

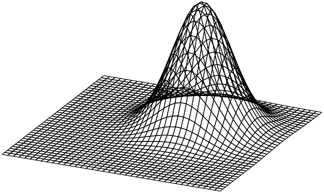
Phase space representations



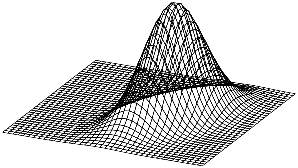
Vacuum



Squeezed vacuum



Coherent state



Squeezed coherent state

Simultaneous Phase Estimation (Better Together)

What if I want to estimate a set of phases $\{\phi_i\}$?

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Individual

Estimating d phases with $E_{\mathrm{Tot.}}$, precision is $\sim \frac{d^3}{E_{\mathrm{Tot.}}^2}$

$$2^{-\frac{d}{2}} \left(|N0\rangle + |0N\rangle \right)^{\otimes d}$$

Simultaneous (Fixed Number States)

Estimating d phases with $E_{\mathrm{Tot.}}$, precision is $\sim \frac{d^2}{E_{\mathrm{Tot.}}^2}$

$$\frac{1}{\mathcal{N}} \left(|N0 \cdots 0\rangle + |0N0 \cdots 0\rangle + |0 \cdots 0N\rangle \right)$$

Improvement scales with the number of phases[2]

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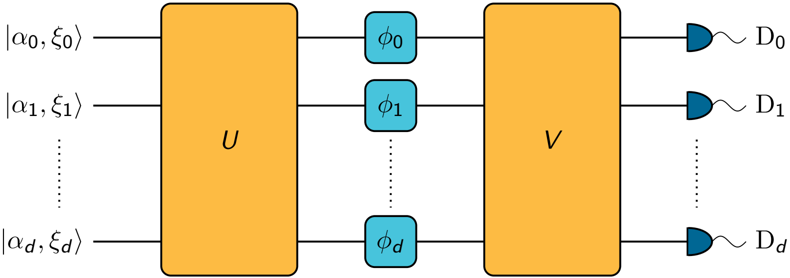
[2]Peter C. Humphreys et al. "Quantum Enhanced Multiple Phase Estimation". In: *Physical Review Letters* 111.7 (Aug. 2013), p. 070403. DOI: 10.1103/PhysRevLett.111.070403.

Multiple Phase Estimation

(Multiple=Simultaneous)

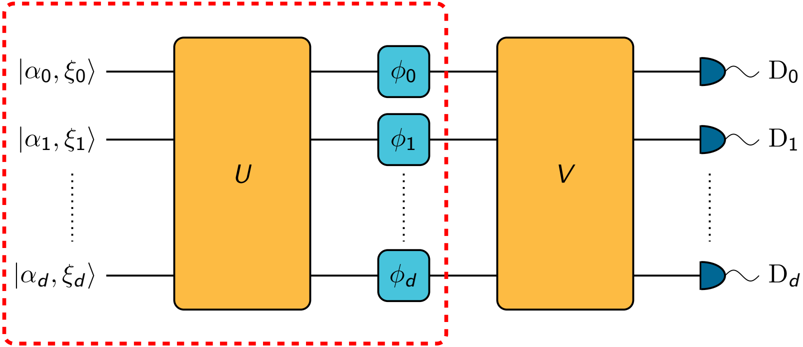


$d+1$ gaussian states $\rightarrow$ general mixing unitary $\rightarrow$ phases $\rightarrow$ detection scheme

Consider the measurement independent bound given by the QFI

Multiple Phase Estimation

(Multiple=Simultaneous)



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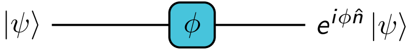
Global Phases

You can't measure everything

One mode + one phase = zero observables



Rewrite the Hamiltonian

Global phase is unknown $\Leftrightarrow$ Unitaries which evolve isolated systems have indefinite determinant

$$H = \sum_{i=0}^{d} \phi_i \hat{n}_i = \bar{\phi}\hat{n}_{\mathrm{T}} + \sum_{i=1}^{d} \varphi_i(\hat{n}_i - \hat{n}_0) \qquad (4)$$

$\varphi_i = \phi_i - \bar{\phi}$. Discard the unmeasurable $\bar{\phi}$, we're left with traceless generators—estimation of remaining parameters is estimation of some $U \in \mathrm{SU}(n)$

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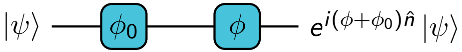

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Quantum Fisher Information

For phases QFI is expectation of covariance of number operators

$$H_{ij} = \left\langle \hat{G}_i \hat{G}_j \right\rangle - \left\langle \hat{G}_i \right\rangle \left\langle \hat{G}_j \right\rangle \tag{5}$$
$$= \mathrm{Cov}(\hat{n}_i, \hat{n}_j) + \mathrm{Var}(\hat{n}_0) - \mathrm{Cov}(\hat{n}_i, \hat{n}_0) - \mathrm{Cov}(\hat{n}_j, \hat{n}_0)$$

Covariance matrix of $\{\hat{n}_i\}$ is

$$C_{\vec{n}} = -\mathbb{1} + \sum_{2\times 2 \text{ Matrices}} \sigma_U \circ \left(\frac{1}{2}\sigma_U + \frac{4}{\hbar}\vec{d}_U \vec{d}_U^T \right) \tag{6}$$

H_{ij} can be constructed entirely from $C_{\vec{n}}$

The Analytics Are Painful

Want to have an analytically manageable system, time for simplifying assumptions

- Surely some symmetry, set $\xi_i = \xi, \forall i$
- Want to use an interferometer, take $U \in \mathrm{SO}(d+1)$

$$\mathrm{Tr}\left(H_{ij}^{-1}\right) = \sum_{i=1}^{d} \frac{1}{\eta_i} - \left(\sum_{i=0}^{d} \frac{1}{\eta_i}\right)^{-1} \sum_{i=1}^{d} \frac{1}{\eta_i^2} \tag{7}$$

$$\eta_i = 2\sinh^2(2|\xi|) + 4e^{-2|\xi|}(\Re\alpha_i{}')^2 + 4e^{2|\xi|}(\Im\alpha_i{}')^2 \tag{8}$$

$$\alpha_i' = \sum_{j=0}^{d} U_{ij}\alpha_j \tag{9}$$

Optimisation

- Minimise $\mathrm{Tr}\left(H_{ij}^{-1}\right) \Leftrightarrow$ maximise η_i
- η_i maximised when all energy in ith mode contributes to squeezing $|\alpha, \zeta\rangle \to |0, \xi\rangle$
- Squeezed vacuum is best

$$\mathrm{Tr}\left(H^{-1}\right) = \frac{d^2(d+1)}{8E_{\mathrm{Tot.}}} \frac{1}{d+1+E_{\mathrm{Tot.}}} \tag{10}$$

So how does this match up?

Simultaneous vs Individual Estimations

Didn't that last equation go as $\frac{d^3}{E_{\mathrm{Tot.}}^2}$?

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Simultaneous vs Individual Estimations

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Yes, yes it did ☹

What's the simultaneous advantage with Gaussian states?

$$R = \frac{\mathrm{Tr}\left(H_{\mathrm{Sim.}}^{-1}\right)}{\mathrm{Tr}\left(H_{\mathrm{Ind.}}^{-1}\right)} = \frac{(d+1)(E_{\mathrm{Tot.}}+2d^2)}{2d(E_{\mathrm{Tot.}}+d+1)} \tag{11}$$

In the limit[3] $E_{\mathrm{Tot.}} \gg d$, $R \to \frac{1}{2}$

[3] We're using $\hbar = 1$ units

In Conclusion

- Quantum mechanics allows us to do better sensing
- Gaussian states do not exhibit an unbounded advantage for the purposes of multiple phase estimation
- What advantage there is is technically attainable

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- Quantum mechanics allows us to do better sensing
- Gaussian states do not exhibit an unbounded advantage for the purposes of multiple phase estimation
- What advantage there is is technically attainable
- In reality, the how is still an open question

What Next?

- Did we ask an experimentally useful question?
- How do we attain the advantage?
- What happens when we acknowledge the existence of noise?