Multiple phase estimation with Gaussian states[1]

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Quantum Roundabout, University of Nottingham
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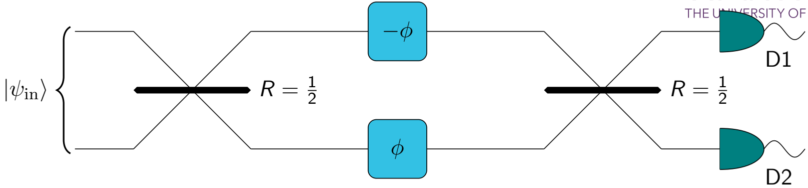
[1] Christos N. Gagatsos, Dominic Branford, and Animesh Datta. "Gaussian systems for quantum enhanced multiple phase estimation". In: *arXiv:1605.04819 [quant-ph]* (May 2016).

Measuring Quantum Things

- Phases $\exp(i\phi\sigma_z)$
- Reflectivitives $\exp(iR\sigma_x)$
- Indeed any (linear) optic network
- Noise, loss, decoherence etc.
- Squeezings and displacements

Mach-Zehnder Interferomtery

Some states are more equal than others



- Single photons, $|1, 0\rangle$: $\delta^2\phi \sim \frac{1}{N}$
- Coherent light, $|\alpha, 0\rangle$: $\delta^2\phi \sim \frac{1}{N}$

} Classical ⇔ Shot Noise Limit

- Holland-Burnett, $|N, N\rangle$: $\delta^2\phi \sim \frac{1}{N^2}$
- Squeezed vacuum, $|\xi, \xi\rangle$: $\delta^2\phi \sim \frac{1}{N^2}$

} Quantum ⇔ Heisenberg Limit

Lang & Caves PRL 111 173601 (2013); PRA 90 025802 (2014)

Cramér-Rao Bound (CRB)

Finite resources = finite precision

$$\delta^2\phi \geq \frac{1}{MF} \tag{1}$$

$$\mathrm{Cov}(\phi_i, \phi_j) \geq \frac{1}{M}\left(F^{-1}\right)_{ij} \tag{2}$$

$$F_{ij} = \int \mathrm{d}\mathbf{x} \frac{1}{P(\mathbf{x}|\phi)} \frac{\partial P(\mathbf{x}|\phi)}{\partial \phi_i} \frac{\partial P(\mathbf{x}|\phi)}{\partial \phi_j} \tag{3}$$

- Precision is lower bounded by the classical Fisher information (the Cramér-Rao bound)
- Minimise $\sum_i \delta^2\phi_i \Leftrightarrow$ Minimise $\mathrm{Tr}\left(F^{-1}\right)$
- Measurement dependent $\Rightarrow$ always (asymptotically) attainable

Quantum Cramér-Rao Bound (QCRB)

- CRB is measurement dependent ⇒ hard to calculate
- Quantum CRB is a measurement independent lower bound on the CRB

$$\mathrm{Cov}(\phi_i, \phi_j) \geq \frac{1}{M}\left(F^{-1}\right)_{ij} \geq \frac{1}{M}\left(H^{-1}\right)_{ij} \tag{4}$$

- Quantum Fisher Information matrix is expectation of covariance matrix of the parameter generators

Simultaneous Phase Estimation

Is there an advantage to be had beyond Heisenberg scaling?

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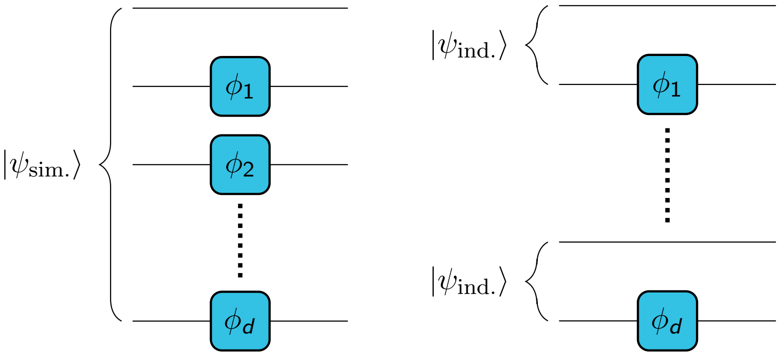

Fair comparison: $E(|\psi_{\text{sim.}}\rangle) = d \times E(|\psi_{\text{ind.}}\rangle)$

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Simultaneous Phase Estimation (Better Together)

What if I want to estimate a set of phases $\{\phi_i\}$?

Individual

Estimating d phases with $E_{\mathrm{Tot.}}$, precision is $\sim \frac{d^3}{E^2_{\mathrm{Tot.}}}$

$$2^{-\frac{d}{2}}\left(|N0\rangle + |0N\rangle\right)^{\otimes d}$$

Simultaneous (Fixed Number States)

Estimating d phases with $E_{\mathrm{Tot.}}$, precision is $\sim \frac{d^2}{E^2_{\mathrm{Tot.}}}$

$$\frac{1}{\mathcal{N}}\left(|N0\cdots0\rangle + |0N0\cdots0\rangle + |0\cdots0N\rangle\right)$$

Improvement scales with the number of phases

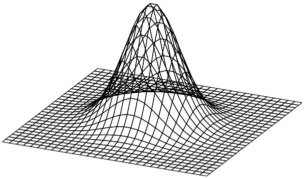
Humphreys et al. PRL 111 070403 (2013)

Gaussian States

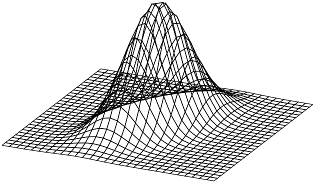
- States with gaussian quasiprobability distributions
- Single mode pure gaussian state $|\alpha, \xi\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle$
- Coherent (laser) light and thermal light both gaussian states
- Fully described by a covariance matrix σ and displacement vector $\vec{d}$

Gaussian States (Wigner functions)

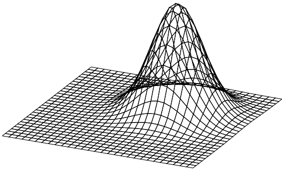
Phase space representations



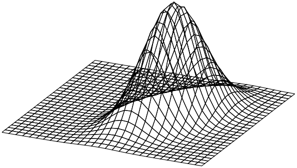
Vacuum



Squeezed vacuum

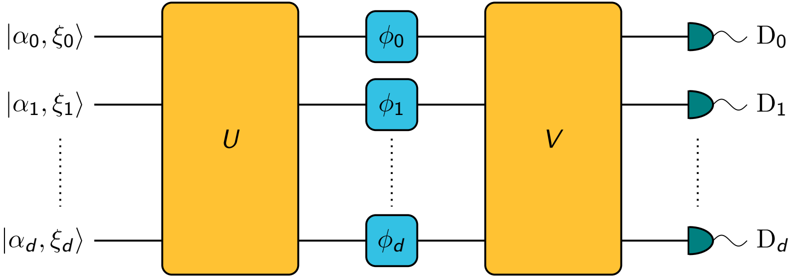


Coherent state



Squeezed coherent state

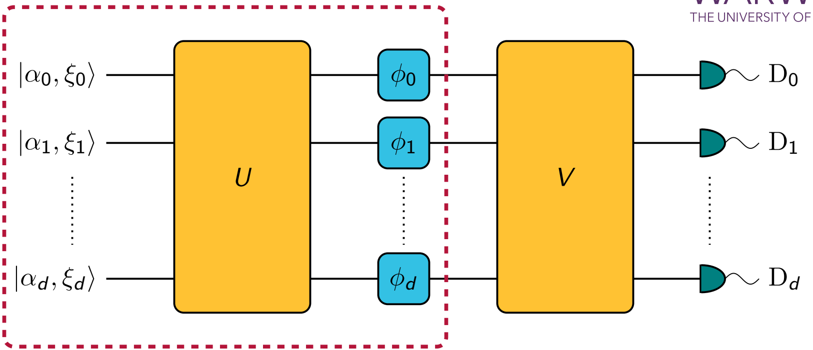
Simultaneous Phase Estimation



$d + 1$ gaussian states $\rightarrow$ general mixing unitary (beam splitter network) $\rightarrow$ phases $\rightarrow$ detection scheme

Consider the measurement independent bound given by the QFI

Simultaneous Phase Estimation



$d+1$ gaussian states $\rightarrow$ general mixing unitary (beam splitter network) $\rightarrow$ phases $\rightarrow$ detection scheme

Consider the measurement independent bound given by the QFI

Measuring linear optics

- Parameters encoded by $e^{i\sum_j \phi_j \hat{H}_j}$
- Choice of $\{\hat{H}_j\}$ matters

Jarzyna & Demkowicz-Dobrzański PRA 85 011801 (2012)

- No measurable phase with a single mode ('global phase')

Rewrite the Hamiltonian

Global phase is unknown $\Leftrightarrow$ Unitaries which evolve isolated systems have indefinite determinant

$$H = \sum_{i=0}^{d} \phi_i \hat{n}_i = \bar{\phi}\hat{n}_{\mathrm{T}} + \sum_{i=1}^{d} \varphi_i (\hat{n}_i - \hat{n}_0) \tag{5}$$

$\varphi_i = \phi_i - \bar{\phi}$. Discard the unmeasurable $\bar{\phi}$, we're left with traceless generators—estimation of remaining parameters is estimation of some $U \in \mathrm{SU}(n)$

Calculating the QFI

Abstract expressions for gaussian states

Monras arXiv:1303.3682; Gao & Lee Eur. Phys. J. D 68 347 (2014); Šafránek et al. NJP 17 073016 (2015)

Uhlmann fidelity for gaussian states

Banchi et al. PRL 115 260501 (2015)

For general states with the generator variances

Paris Int. J. Quantum Inf. 07 125 (2009); Baumgratz & Datta PRL 116 030801 (2016)

Quantum Fisher Information

For phases QFI is expectation of covariance of number operators

$$H_{ij} = \left\langle \hat{G}_i \hat{G}_j \right\rangle - \left\langle \hat{G}_i \right\rangle \left\langle \hat{G}_j \right\rangle \tag{6}$$
$$= \mathrm{Cov}(\hat{n}_i, \hat{n}_j) + \mathrm{Var}(\hat{n}_0) - \mathrm{Cov}(\hat{n}_i, \hat{n}_0) - \mathrm{Cov}(\hat{n}_j, \hat{n}_0)$$

Covariance matrix of $\{\hat{n}_i\}$ is

$$C_{\vec{n}} = -\mathbb{1} + \sum_{2\times 2 \text{ Matrices}} \sigma_U \circ \left(\frac{1}{2}\sigma_U + \frac{4}{\hbar} \vec{d}_U \vec{d}_U^T \right) \tag{7}$$

H_{ij} can be constructed entirely from $C_{\vec{n}}$

The Analytics Are Painful

Want to have an analytically manageable system, time for simplifying assumptions

- Surely some symmetry, set $\xi_i = \xi, \forall i$
- Want to use an interferometer, take $U \in \mathrm{SO}(d+1)$

$$\mathrm{Tr}\left(H_{ij}^{-1}\right) = \sum_{i=1}^{d} \frac{1}{\eta_i} - \left(\sum_{i=0}^{d} \frac{1}{\eta_i}\right)^{-1} \sum_{i=1}^{d} \frac{1}{\eta_i^2} \tag{8}$$

$$\eta_i = 2\sinh^2(2|\xi|) + 4e^{-2|\xi|}(\Re\alpha_i{}')^2 + 4e^{2|\xi|}(\Im\alpha_i{}')^2 \tag{9}$$

$$\alpha_i' = \sum_{j=0}^{d} U_{ij}\alpha_j \tag{10}$$

Optimisation

- Minimise $\mathrm{Tr}\left(H_{ij}^{-1}\right) \Leftrightarrow$ maximise η_i
- η_i maximised when all energy in ith mode contributes to squeezing $|\alpha, \zeta\rangle \rightarrow |0, \xi\rangle$
- Squeezed vacuum is best

$$\mathrm{Tr}\left(H^{-1}\right) = \frac{d^2(d+1)}{8E_{\mathrm{Tot.}}} \frac{1}{d+1+E_{\mathrm{Tot.}}} \tag{11}$$

So how does this match up?

Simultaneous vs Individual Estimations

Didn't that last equation go as $\frac{d^3}{E^2_{\mathrm{Tot.}}}$?

So how does this match up?

Simultaneous vs Individual Estimations

Didn't that last equation go as $\frac{d^3}{E_{\mathrm{Tot.}}^2}$?

$\mathrm{Tr}\left(H^{-1}\right) = \frac{d^2(d+1)}{8E_{\mathrm{Tot.}}} \frac{1}{d+1+E_{\mathrm{Tot.}}}$

Yes, yes it did ☹

What's the simultaneous advantage with Gaussian states?

$$R = \frac{\mathrm{Tr}\left(H_{\mathrm{Sim.}}^{-1}\right)}{\mathrm{Tr}\left(H_{\mathrm{Ind.}}^{-1}\right)} = \frac{(d+1)(E_{\mathrm{Tot.}}+2d^2)}{2d(E_{\mathrm{Tot.}}+d+1)} \quad (12)$$

In the limit $E_{\mathrm{Tot.}} \gg d$, $R \rightarrow \frac{1}{2}$

In Conclusion

- Precision bounds and optimal states for estimating an arbitrary number of parameters with gaussian states
- Gaussian states do not exhibit an $\mathcal{O}(d)$ advantage for the purposes of multiple phase estimation
- What advantage there is is technically attainable

What next?

- Did we ask an experimentally useful question?
- How do we attain the advantage?
- What happens when we acknowledge the existence of noise?
- Why did Gaussian states not see an $O(d)$ improvement?

Attainability of bounds

- QCRB needs suitable POVM to saturate CRB
- QCRB - Single parameter attainable through eigenstates of SLD

  Braunstein & Caves PRL 72 3439 (1994)

- Multiple parameters - $[L_i, L_j] \neq 0$ in general $\Leftrightarrow$ no obvious POVM to attain QFI (as no common eigenbasis)
- Exception for pure states $\mathrm{Tr}\left(\rho\left[L_i, L_j\right]\right) = 0$ sufficient for existence of a POVM

  Matsumoto J. Phys. A 35 3111 (2002)