The Dirac Equation

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0.1 Introduction

In non-relativistic quantum mechanics, wave functions are descibed by the time-dependent Schrodinger equation :

$$-\frac{1}{2m}\nabla^2\psi + V\psi = i\frac{\partial\psi}{\partial t} \tag{1}$$

This is really just energy conservation (kinetic energy ($\frac{p^2}{2m}$) plus potential energy (V) equals total energy (E)) with the momentum and energy terms replaced by their operator equivalents

$$p \to -i\nabla; E \to i\frac{\partial}{\partial t} \tag{2}$$

In relativistic quantum theory, the energy-momentum conservation equation is $E^2 - p^2 = m^2$ (note that we are working in the standard particle physics units where $\hbar = c = 1$). Proceeding with the same replacements, we can derive the *Klein-Gordon* equation :

$$E^2 - p^2 - m^2 \to -\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi - m^2 = 0 \tag{3}$$

In covariant notation this is

$$-\partial^\mu\partial_\mu\psi - m^2\psi = 0 \tag{4}$$

Suppose ψ is solution to the Klein-Gordon equation. Multiplying it by $-i\psi^*$ we get

$$i\psi^*\frac{\partial^2\psi}{\partial t^2} - i\psi^*\nabla^2\psi + i\psi^* m^2 = 0 \tag{5}$$

Taking the complex conjugate of the Klein-Gordon equation and multiplying by $-i\psi$ we get

$$i\psi\frac{\partial^2\psi^*}{\partial t^2} - i\psi\nabla^2\psi^* + i\psi m^2 = 0 \tag{6}$$

If we subtract the second from the first we obtain

$$\frac{\partial}{\partial t}\left[i\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)\right] + \nabla\cdot\left[-i\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)\right] = 0 \tag{7}$$

This has the form of an equation of continuity

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0 \tag{8}$$

with a probability density defined by

$$\rho = i\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) \tag{9}$$

and a probability density current defined by

$$\mathbf{j} = i\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \tag{10}$$

Now, suppose a solution to the Klein-Gordon equation is a free particle with energy E and momentum $\mathbf{p}$

$$\psi = Ne^{-ip_\mu x^\mu} \tag{11}$$

Substitution of this solution into the equation for the probability density yields

$$\rho = 2E|N|^2 \tag{12}$$

The probability density is proportional to the energy of the particle. Now, why is this a problem? If you substitute the free particle solution into the Klein-Gordon equation you get, unsurpisingly, the relation

$$E^2 - p^2 = m^2 \tag{13}$$

so the energy of the particle could be

$$E = \pm\sqrt{p^2 + m^2} \tag{14}$$

The fact that you have negative energy solutions is not that much of a problem. What is a problem is that the probability density is proportional to E. This implies a possibility for negative probability densities...which don't exist.

This (and some others) problem drove Dirac to think about another equation of motion. His starting point was to try to factorise the energy momentum relation. In covariant formalism

$$E^2 - p^2 - m^2 \rightarrow p^\mu p_\mu - m^2 \tag{15}$$

where p^μ is the 4-momentum : (E, p_x, p_y, p_z). Dirac tried to write

$$p^\mu p_\mu - m^2 = (\beta^\kappa p_\kappa + m)(\gamma^\lambda p_\lambda - m) \tag{16}$$

where κ and λ range from 0 to 3. This notation looks a bit confusing. It may help if we write both sides out long-hand. On the left-hand side we have

$$p^\mu p_\mu - m^2 = p_0^2 - \mathbf{p}\cdot\mathbf{p} = p_0^2 - p_1^2 - p_2^2 - p_3^2 - m^2 \tag{17}$$

On the right-hand side we have

$$(\beta^\kappa p_\kappa + m)(\gamma^\lambda p_\lambda - m) = (\beta^0 p_0 - \beta^1 p_1 - \beta^2 p_2 - \beta^3 p_3 + m)(\gamma^0 p_0 - \gamma^1 p_1 - \gamma^2 p_2 - \gamma^3 p_3 - m) \tag{18}$$

Expanding the right-hand side

$$(\beta^\kappa p_\kappa + m)(\gamma^\lambda p_\lambda - m) = \beta^\kappa \gamma^\lambda p_\kappa p_\lambda - m^2 + m\gamma^\lambda p_\lambda - m\beta^\kappa p_\kappa \tag{19}$$

This should be equal to $p^\mu p_\mu - m^2$, so we need to get rid of the terms that are linear in p. We can do this just be choosing $\beta^\kappa = \gamma^\kappa$. Then

$$p^\mu p_\mu - m^2 = \gamma^\kappa \gamma^\lambda p_\kappa p_\lambda - m^2 \tag{20}$$

Now, since κ, λ runs from 0 to 3, the right hand side can be fully expressed by

$$\begin{aligned}
\gamma^\kappa \gamma^\lambda p_\kappa p_\lambda - m^2 &= (\gamma^0)^2 p_0^2 + (\gamma^1)^2 p_1^2 + (\gamma^2)^2 p_2^2 + (\gamma^3)^2 p_3^2 && (21)\\
&+ (\gamma^0\gamma^1 + \gamma^1\gamma^0) p_0 p_1 + (\gamma^0\gamma^2 + \gamma^2\gamma^0) p_0 p_2 && (22)\\
&+ (\gamma^0\gamma^3 + \gamma^3\gamma^0) p_0 p_3 + (\gamma^1\gamma^2 + \gamma^2\gamma^1) p_1 p_2 && (23)\\
&+ (\gamma^1\gamma^3 + \gamma^3\gamma^1) p_1 p_3 + (\gamma^2\gamma^3 + \gamma^3\gamma^2) p_2 p_3 - m^2 && (24)
\end{aligned}$$

This must be equal to

$$p_\mu p^\mu - m^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2 - m^2 \tag{25}$$

One could make the squared terms equal by choosing $(\gamma^0)^2 = 1$ and $(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2$=-1, but this would not eliminate the cross-terms. Dirac realised that what you needed was something which anticommuted : i.e. $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0$ if $\mu \neq \nu$. He realised that the γ factors must be 4x4 matrices, not just numbers, which satisfied the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{26}$$

where

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{27}$$

We also define, for later use, another γ matrix called $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. This has the properties (you should check for yourself) that

$$(\gamma^5)^2 = 1, \{\gamma^5, \gamma^\mu\} = 0 \tag{28}$$

We will want to take the Hermitian conjugate of these matrices at various times. The Hermitian conjugate of each matrix is

$$\gamma^{0\dagger} = \gamma^0 \qquad \gamma^{5\dagger} = \gamma^5 \qquad \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 = -\gamma \quad \text{for} \quad \mu \neq 0 \tag{29}$$

If the matrices satisfy this algebra, then you can factor the energy momentum conservation equation

$$p_\mu p^\mu - m^2 = (\gamma^\kappa p_\kappa + m)(\gamma^\lambda p_\lambda - m) = 0 \tag{30}$$

The Dirac equation is one of the two factors, and is conventionally taken to be

$$\gamma^\lambda p_\lambda - m = 0 \tag{31}$$

Making the standard substitution, $p_\mu \to i\partial_\mu$ we then have the usual covariant form of the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{32}$$

where $\partial_\mu = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, m is the particle mass and the γ matrices are a set of 4-dimensional matrices. Since these are matrices, ψ is a 4-element column matrix called a “bi-spinor”. This bi-spinor is *not* a 4-vector and doesn’t transform like one.

0.2 The Bjorken-Drell convention

Any set of four 4x4 matrices that obey the algebra above will work. The one we normally use includes the Pauli spin matrices. Recall that each component of the spin operator **S** for spin 1/2 particles is defined by its own Pauli spin matrix :

$$S_x = \frac{1}{2}\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{1}{2}\sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{1}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{33}$$

In terms of the Pauli spin matrices, the γ matrices have the form

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{34}$$

Each element of the matrices in Equations 34 are **2x2 matrices**. 1 denotes the 2 x 2 unit matrix, and 0 denotes the 2 x 2 null matrix.

0.3 The Probability Density and Current

In order to understand the probability density and probability flow we will want to derive an equation of continuity for the probability. The first step is to write the Dirac equation out longhand :

$$i\gamma^0\frac{\partial\psi}{\partial t} + i\gamma^1\frac{\partial\psi}{\partial x} + i\gamma^2\frac{\partial\psi}{\partial y} + i\gamma^3\frac{\partial\psi}{\partial z} - m\psi = 0 \tag{35}$$

We want to take the Hermitian conjugate of this :

$$[i\gamma^0\frac{\partial\psi}{\partial t} + i\gamma^1\frac{\partial\psi}{\partial x} + i\gamma^2\frac{\partial\psi}{\partial y} + i\gamma^3\frac{\partial\psi}{\partial z} - m\psi]^\dagger \tag{36}$$

Now, we must remember that the γ are matrices and that ψ is a 4-component column vector. Thus

$$\begin{aligned}
[\gamma^0\frac{\partial\psi}{\partial t}]^\dagger &= [\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{\partial\psi_1}{\partial t} \\ \frac{\partial\psi_2}{\partial t} \\ \frac{\partial\psi_3}{\partial t} \\ \frac{\partial\psi_4}{\partial t} \end{pmatrix}]^\dagger & (37) \\
&= \begin{pmatrix} \frac{\partial\psi_1}{\partial t} \\ \frac{\partial\psi_2}{\partial t} \\ \frac{\partial\psi_3}{\partial t} \\ \frac{\partial\psi_4}{\partial t} \end{pmatrix}^\dagger \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}^\dagger & (38) \\
&= \begin{pmatrix} \frac{\partial\psi_1^\dagger}{\partial t} & \frac{\partial\psi_2^\dagger}{\partial t} & \frac{\partial\psi_3^\dagger}{\partial t} & \frac{\partial\psi_4^\dagger}{\partial t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & (39) \\
&= \frac{\partial\psi^\dagger}{\partial t}\gamma^0 & (40)
\end{aligned}$$

Using the Hermitian conjugation properties of the γ matrices defined in the previous section we can write Equation 36 as

$$-i\frac{\partial\psi^\dagger}{\partial t}\gamma^{0\dagger} - i\frac{\partial\psi^\dagger}{\partial x}\gamma^{1\dagger} - i\frac{\partial\psi^\dagger}{\partial y}\gamma^{2\dagger} - i\frac{\partial\psi^\dagger}{\partial z}\gamma^{3\dagger} - m\psi^\dagger \tag{41}$$

which, as $\gamma^{\mu\dagger} = -\gamma^\mu$ for $\mu \neq 0$ means we can write this as

$$-i\frac{\partial\psi^\dagger}{\partial t}\gamma^0 - i\frac{\partial\psi^\dagger}{\partial x}(-\gamma^1) - i\frac{\partial\psi^\dagger}{\partial y}(-\gamma^2) - i\frac{\partial\psi^\dagger}{\partial z}(-\gamma^3) - m\psi^\dagger \tag{42}$$

We have a problem now - this is no longer covariant. That is, I would like to write this in the form $\psi^\dagger(-i\partial_\mu\gamma^\mu - m)$ where the derivative ∂_μ now operates to the *left*. I can't because the minus signs on the spatial components coming from the $-\gamma^\mu$ spoils the scalar product.

To deal with this we have to multiply the equation on the right by γ^0, since we can flip the sign of the γ matrices using the relationship $-\gamma^\mu\gamma^0 = \gamma^0\gamma^\mu$. Thus

$$-i\frac{\partial\psi^\dagger}{\partial t}\gamma^0\gamma^0 - i\frac{\partial\psi^\dagger}{\partial x}(-\gamma^1\gamma^0) - i\frac{\partial\psi^\dagger}{\partial y}(-\gamma^2\gamma^0) - i\frac{\partial\psi\dagger}{\partial z}(-\gamma^3\gamma^0) - m\psi^\dagger\gamma^0 \tag{43}$$

or

$$-i\frac{\partial\psi^\dagger}{\partial t}\gamma^0\gamma^0 - i\frac{\partial\psi^\dagger}{\partial x}(\gamma^0\gamma^1) - i\frac{\partial\psi^\dagger}{\partial y}(\gamma^0\gamma^2) - i\frac{\partial\psi\dagger}{\partial z}(\gamma^0\gamma^3) - m\psi^\dagger\gamma^0 \tag{44}$$

Defining the *adjoint spinor* $\overline{\psi} = \psi^\dagger\gamma^0$, we can rewrite this equation as

$$-i\frac{\partial\overline{\psi}}{\partial t}\gamma^0 - i\frac{\partial\overline{\psi}}{\partial x}(\gamma^1) - i\frac{\partial\overline{\psi}}{\partial y}(\gamma^2) - i\frac{\partial\overline{\psi}}{\partial z}(\gamma^3) - m\overline{\psi} \tag{45}$$

and finally we get the *adjoint Dirac equation*

$$\overline{\psi}(i\partial_\mu\gamma^\mu + m) = 0 \tag{46}$$

Now, we multiply the Dirac equation on the left by $\overline{\psi}$:

$$\overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{47}$$

and the adjoint Dirac equation on the right by ψ :

$$\overline{\psi}(i\partial_\mu\gamma^\mu + m)\psi = 0 \tag{48}$$

and we add them together, which eliminates the two mass terms

$$\overline{\psi}(\gamma^\mu\partial_\mu\psi) + (\overline{\psi}\partial_\mu\gamma^\mu)\psi = 0 \tag{49}$$

or, more succinctly,

$$\partial_\mu(\overline{\psi}\gamma^\mu\psi) = 0 \tag{50}$$

.

If we now identify the bit in the brackets as the *4-current* :

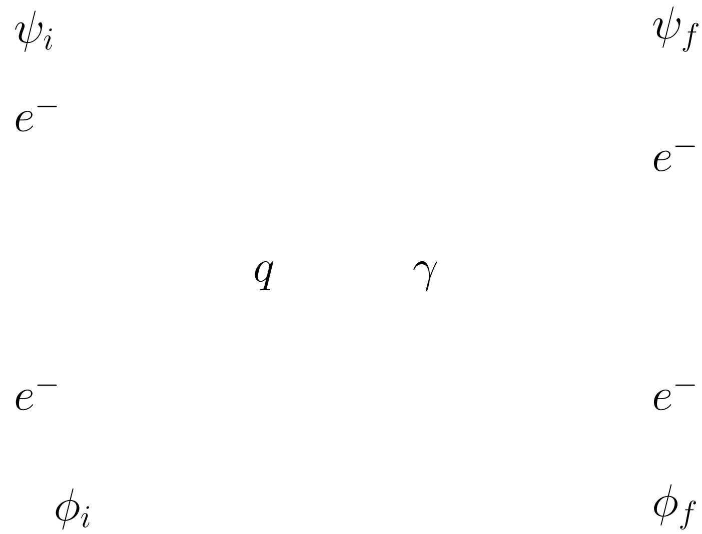
$$j^\mu = (\rho, \mathbf{j}) \tag{51}$$

where ρ is the probability density and $\mathbf{j}$ is the probability current, we can write the conservation equation as

$$\partial_\mu j^\mu = 0 \quad \text{with} \quad j^\mu = \overline{\psi}\gamma^\mu\psi \tag{52}$$

which is the covariant form for an equation of continuity. The probability density is then just $\rho = \overline{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi$ and the probability 3-current is $\mathbf{j} = \psi^\dagger\gamma^0\gamma^\mu\psi$.

This current is the same one which appears in the Feynman diagrams. It is called a Vector current, and is the current responsible for the electromagnetic interaction.



For the interaction in Figure 0.3, with two electromagnetic interactions, the matrix element is then

$$M = \frac{e^2}{4\pi}[\overline{\psi_f}\gamma^\mu\psi_i]\frac{1}{q^2}[\overline{\phi_f}\gamma^\nu\phi_i] \tag{53}$$

0.4 Examples

Now let's look at some solutions to the Dirac Equation. The first one we will look at is for a particle at rest.

0.4.1 Particle at rest

Suppose the fermion wavefunction is a plane wave. We can write this as

$$\psi(p) = e^{-ix_\mu p^\mu}u(p) \tag{54}$$

where $p_\mu = (E, -p_x, -p_y, -p_z)$ and $x^\mu = (t, x, y, z)$ and so $-ix_\mu p^\mu = -i(Et - \mathbf{x}\cdot\mathbf{p})$. This is just the phase for an oscillating plane wave.

For a particle at rest, the momentum term disappears : $-ix_\mu p^\mu = -i(Et)$. Furthur, since the momentum is zero, the spatial derivatives must be zero : $\frac{\partial\psi}{\partial x,y,z} = 0$. The Dirac equation therefore reads

$$i\gamma^0\frac{\partial\psi}{\partial t} - m\psi = 0 \tag{55}$$

or

$$i\gamma^0(-iE)u(p) - mu(p) = 0 \tag{56}$$

which gives us

$$E\gamma^0 u(p) = mu(p) \tag{57}$$

Now, u(p) is a 4-component bispinor, so properly expanding the gamma matrix in Equation 57, we get

$$E\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}=m\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}\tag{58}$$

This is an eigenvalue equation. There are four independent solutions. Two with energy $E=m$ and two with $E=-m$. The solutions are just

$$u_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\quad u_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\quad u_3=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\quad u_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\tag{59}$$

with eigenvalues m, m, -m and -m respectively.

The first two solutions can be interpreted as positive energy particle solutions with spin up and spin down. One can see this be checking if the function is an eigenfunction of the spin matrix : S_z.

$$S_zu_1=\frac{1}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\frac{1}{2}u_1\tag{60}$$

and similarly for u_2.

Note that we have yet to normalise these solutions. We won't either for the purposes of this discussion. What about these negative energy solutions? We are required to keep them (we can't just say that they are unphysical and throw them away) since quantum mechanics requires a complete set of states.

What happens if we allow the particle to have momentum?

0.4.2 Particle in Motion

Again, let's look for a plane wave solution

$$\psi(p)=e^{-ix_\mu p^\mu}u(p)\tag{61}$$

Plugging this into the Dirac equation means that we can replace the ∂_μ by p_μ and remove the oscillatory term to obtain the momentum-space version of the Dirac Equation

$$(\gamma^\mu p_\mu-m)u(p)=0\tag{62}$$

Notice that this is now purely algebraic and can be easily(!) solved

$$\begin{aligned}\gamma^\mu p_\mu-m &= E\gamma^0-p_x\gamma^1-p_y\gamma^2-p_z\gamma^3-m &(63)\\ &= \begin{pmatrix}1&0\\0&-1\end{pmatrix}E-\begin{pmatrix}0&\sigma\\-\sigma&0\end{pmatrix}\cdot\mathbf{p}-m\begin{pmatrix}1&0\\0&1\end{pmatrix} &(64)\\ &= \begin{pmatrix}(E-m)&-\sigma\cdot\mathbf{p}\\ \sigma\cdot\mathbf{p}&-(E+m)\end{pmatrix} &(65)\end{aligned}$$

where each element in this 2x2 matrix is actually a 2x2 matrix itself. In this spirit, let's write the 4-component bispinor solution as 2-component vector

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix} \tag{66}$$

then the Dirac Equation implies that

$$(\gamma^\mu p_\mu - m)u(p) = 0 \quad \Longrightarrow \quad \begin{pmatrix} (E-m) & -\sigma \cdot \mathbf{p} \\ \sigma \cdot \mathbf{p} & -(E+m) \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{67}$$

leading two coupled simultaneous equations

$$\begin{aligned} (\sigma \cdot \mathbf{p})u_B &= (E-m)u_A && (68) \\ (\sigma \cdot \mathbf{p})u_A &= (E+m)u_B && (69) \end{aligned}$$

Now, let's expand that $(\sigma \cdot \mathbf{p})$:

$$\begin{aligned} (\sigma \cdot \mathbf{p}) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x + \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix} p_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} && (70) \\ &= \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} && (71) \end{aligned}$$

and right back to the Dirac Equation

$$\begin{aligned} (\sigma \cdot \mathbf{p})u_B &= (E-m)u_A && (72) \\ (\sigma \cdot \mathbf{p})u_A &= (E+m)u_B && (73) \end{aligned}$$

gives

$$\begin{aligned} u_B &= \frac{(\sigma \cdot \mathbf{p})}{E+m} u_A && (74) \\ &= \frac{1}{E+m} \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} u_A && (75) \end{aligned}$$

Now, we just need to make choices for the form of u_A. Let's make the obvious choice and remember that u_A is a 2-component spinor so we need to specify two solutions:

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{76}$$

These give

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} \quad \text{and} \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \tag{77}$$

where N_1 and N_2 are normalisation factors we won't go into. Note that, if $p = 0$ these correspond to the $E > 0$ solutions we found for a particle at rest, which is good.

Repeating this for $u_B = \begin{pmatrix}1\\0\end{pmatrix}$ and $u_B = \begin{pmatrix}0\\1\end{pmatrix}$ which gives solutions u_3 and u_4

$$u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix} \tag{78}$$

and collecting them together

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \quad u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix} \tag{79}$$

All of which, if put back into the Dirac Equation, yields : $E^2 = p^2 + m^2$ as you might expect. Comparing with the $p = 0$ case we can identify u_1 and u_2 as positive energy solutions with energy $E = \sqrt{p^2 + m^2}$, and u_3, u_4 as negative energy solutions with energy $E = -\sqrt{p^2 + m^2}$.

How do we interpret these negative energy solutions? The conventional interpretation is called the "Feynman-Stuckelberg" interpretation : A negative energy solution represents a negative energy particle travelling backwards in time, or equivalently, a positive energy antiparticle going forwards in time.

With this interpretation we tend to rewrite the negative energy solutions to represent positive antiparticles. Starting from the $E < 0$ solutions

$$u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix} \tag{80}$$

Here we implicitly assume that E is negative. We define antiparticle states by just flipping the sign of E and p following the Feynman-Stuckelberg convention.

$$v_1(E,p)e^{-i(Et-\mathbf{x}\cdot\mathbf{p})} = u_4(-E,-p)e^{i(Et-\mathbf{x}\cdot\mathbf{p})} \tag{81}$$

$$v_2(E,p)e^{-i(Et-\mathbf{x}\cdot\mathbf{p})} = u_3(-E,-p)e^{i(Et-\mathbf{x}\cdot\mathbf{p})} \tag{82}$$

where $E = \sqrt{|p|^2 + m^2}$. Note that v_1 is associated with u_4 and v_2 is associated with u_3. We do this because the spin matrix, $S_{antiparticle}$, for anti-particles is equal to $-S_{particle}$ i.e. the spin flips for antiparticles as well and the spin eigenvalue for $v_1 = u_4$ is spin-up and $v_2 = u_3$ is spin-down.

In general u_1, u_2, v_1 and v_2 are *not* eigenstates of the spin operator (Check for yourself). In fact we should expect this since solutions to the Dirac Equation are, by definition, eigenstates of the Hamiltonian operator, $\hat{H}$, and the S_z does not commute with the Hamiltonian : $[\hat{H}, \hat{S}_z] \neq 0$, and hence we can't find solutions which are simultaneously solutions to S_z and $\hat{H}$. However if the z-axis is aligned with particle direction : $p_x = p_y = 0, p_z = \pm|p|$ then we have the following Dirac states

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{\pm|p|}{E+m} \\ 0 \end{pmatrix} \quad u_2 = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\mp|p|}{E+m} \end{pmatrix} \quad v_1 = N \begin{pmatrix} 0 \\ \frac{\mp|p|}{E+m} \\ 0 \\ 1 \end{pmatrix} \quad v_2 = N \begin{pmatrix} \frac{\pm|p|}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{83}$$

These *are* eigenstates of S_z

$$S_z u_1 = +\frac{1}{2}u_1 \qquad S_z^{anti} v_1 = -S_z v_1 = +\frac{1}{2}v_1 \tag{84}$$

$$S_z u_2 = -\frac{1}{2}u_2 \qquad S_z^{anti} v_2 = -S_z v_2 = -\frac{1}{2}v_2 \tag{85}$$

So we have found solutions of the Dirac Equation which are also spin eigenstates....but only if the particle is travelling along the z-axis.

0.4.3 Charge Conjugation

Classical electrodynamics is invariant under a change in the sign of the electric charge. In particle physics, this concept is represented by the "charge conjugation operator" that flips the signs of *all* the charges. It changes a particle into an anti-particle, and vice versa:

$$\hat{C}|p >= |\overline{p} > \tag{86}$$

. Application of $\hat{C}$ twice brings us back to the the orignal state : $\hat{C}^2 = 1$ and so eigenstates of $\hat{C}$ are ± 1. In general most particles are *not* eigenstates of $\hat{C}$. If $|p>$ were an eigenstate of $\hat{C}$ then

$$\hat{C}|p >= \pm|p >= |\overline{p} > \tag{87}$$

implying that only those particles which are their own antiparticles are eigenstates of $\hat{C}$. This leaves us with photons and the neutral mesons like π^0, η and ρ^0.

$\hat{C}$ changes a particle spinor into an anti-particle spinor. The relevant operation is

$$\psi \to \psi_C = \hat{C}\gamma^0\psi^* \tag{88}$$

with $\hat{C} = i\gamma^2\gamma^0$.

We can apply this to, say, the u_1 solution to the Dirac Equation. if $\psi = u_1 e^{-i(Et-\mathbf{x}\cdot\mathbf{p})}$, the $\psi_C = i\gamma^2\psi^* = i\gamma^2 u_1^* e^{i(Et-\mathbf{x}\cdot\mathbf{p})}$

$$i\gamma^2 u_1^* = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}^* = N_1 \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} = v_1 \tag{89}$$

or, in full, $\hat{C}(u_1 e^{-i(Et-\mathbf{x}\cdot\mathbf{p})}) = v_1 e^{i(Et-\mathbf{x}\cdot\mathbf{p})}$.

Hence charge conjugation changes the particle eigenspinors into their respective anti-particle spinors.

0.4.4 Helicity

The fact that we can find spin eigenvalues for states in which the particles are travelling along the spin-direction indicates that the quantity we need is not *spin* but *helicity*. The helicity is defined as the *projection of the spin along the direction of motion*:

$$\hat{h} = \mathbf{\Sigma}\cdot\hat{\mathbf{p}} = 2\mathbf{S}\cdot\hat{\mathbf{p}} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}\cdot\hat{\mathbf{p}} \tag{90}$$

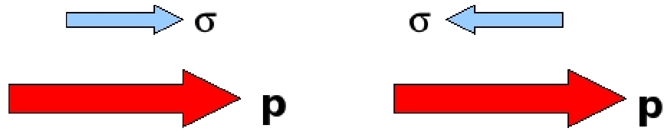


Figure 1: The two helicity states. On the left the spin vector is aligned in the direction of motions of the particle. This is a **right-handed** helicity state and has helicity +1. On the right the spin vector is antiparallel to the particle momentum. This is a **left-handed** helicity state with helicity -1.

and has eigenvalues equal to +1 (called *right-handed* where the spin vector is aligned in the same direction as the momentum vector) or -1 (called *left-handed* where the spin vector is aligned in the opposite direction as the momentum vector), corresponding to the diagrams in Figure1.

It can be shown that the helicity *does* commute with the Hamiltonian and so one can find eigenstates that are simultaneously states of helicity and the Hamiltonian.

The problem, and it is a *big* problem, is that helicity is *not* Lorentz invariant in the case of a massive particle. If the particle is massive it is possible to find an inertial reference frame in which the particle is going in the opposite direction. This does not change the direction of the spin vector, so the helicity can change sign.

The helicity is Lorentz invariant only in the case of massless particles.

0.4.5 Chirality

We'd rather have operators which are Lorentz invariant, than commute with the Hamiltonian. In general wave functions in the Standard Model are eigenstates of a Lorentz invariant quantity called the *chirality*. The chirality operator is γ^5 and it does not commute with the Hamiltonian. Due to this, it is somewhat difficult to visualise. The best picture I can get comes from the definition : something is *chiral* if it's mirror image does not superimpose on itself. Think of your left hand. It's mirror image (from the point of view of someone in that mirror looking back at you) is actually a right hand. It and it's mirror image cannot be superimposed andit is therefore an intrinsically chiral object.

In the limit that $E >> m$, or that the particle is massless, the chirality is identical to the helicity. For a massive particle this is no longer true.

In general the eigenstates of the chirality operator are

$$\gamma^5 u_R = +u_R \quad \gamma^5 u_L = -u_L \quad \gamma^5 v_R = -v_R \quad \gamma^5 v_L = +v_L \tag{91}$$

, where we define u_R and u_L are right- and left-handed chiral states. We can define the projection operators

$$P_L = \frac{1}{2}(1-\gamma^5) \quad P_R = \frac{1}{2}(1+\gamma^5) \tag{92}$$

such that P_L projects outs the left-handed chiral particle states and right-handed chiral *anti*-particle states. P_R projects out the right-handed chiral particle states and left-handed chiral anti-particle states. The projection operators have the following properties :

$$P_{L,R}^2 = P_{L,R}; \quad P_R P_L = P_L P_R = 0; \quad P_R + P_L = 1 \tag{93}$$

Any spinor can be written in terms of it's left- and right-handed chiral states:

$$\psi = (P_R + P_L)\psi = P_R\psi + P_L\psi = \psi_R + \psi_L \tag{94}$$

Chirality holds an important place in the standard model. Let's take a standard model probability current

$$\overline{\psi}\gamma^\mu\phi \tag{95}$$

We can decompose this into it's chiral states

$$\begin{aligned}
\overline{\psi}\gamma^\mu\phi &= (\overline{\psi_L} + \overline{\psi_R})\gamma^\mu(\phi_L + \phi_R) \\
&= \overline{\psi_L}\gamma^\mu\phi_L + \overline{\psi_L}\gamma^\mu\phi_R + \overline{\psi_R}\gamma^\mu\phi_L + \overline{\psi_R}\gamma^\mu\phi_R
\end{aligned}$$

Now, $\overline{\psi_L} = \psi_L^\dagger\gamma^0 = \psi^\dagger\frac{1}{2}(1-\gamma^5)\gamma^0 = \psi^\dagger\gamma^0\frac{1}{2}(1+\gamma^5) = \overline{\psi}\frac{1}{2}(1+\gamma^5) = \overline{\psi}P_R$ where I have used the fact that $\gamma^{5\dagger} = \gamma^5$ and $\gamma^5\gamma^0 = -\gamma^0\gamma^5$. Similarly $\overline{\psi_R} = \overline{\psi}P_L$.

Using this, it is easy to show that the terms $\overline{\psi_L}\gamma^\mu\phi_R$ and $\overline{\psi_R}\gamma^\mu\phi_L$are equal to 0:

$$\begin{aligned}
\overline{\psi_L}\gamma^\mu\phi_R &= \overline{\psi}P_R\gamma^\mu P_R\phi && (96)\\
&= \overline{\psi}\frac{1}{2}(1+\gamma^5)\gamma^\mu\frac{1}{2}(1+\gamma^5)\phi && (97)\\
&= \overline{\psi}\frac{1}{4}(1+\gamma^5)(\gamma^\mu + \gamma^\mu\gamma^5)\phi && (98)\\
&= \overline{\psi}\frac{1}{4}(1+\gamma^5)(\gamma^\mu - \gamma^5\gamma^\mu)\phi && (99)\\
&= \overline{\psi}\frac{1}{4}(1+\gamma^5)(1-\gamma^5)\gamma^\mu\phi && (100)\\
&= \overline{\psi}\frac{1}{4}(1+\gamma^5-\gamma^5-(\gamma^5)^2)\gamma^\mu\phi && (101)\\
&= 0 && (102)
\end{aligned}$$

since $(\gamma^5)^2 = 1$ and $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$. Similarly for the other cross term, $\overline{\psi_R}\gamma^\mu\phi_L$.

Hence,

$$\overline{\psi}\gamma^\mu\phi = \overline{\psi_L}\gamma^\mu\phi_L + \overline{\psi_R}\gamma^\mu\phi_R \tag{103}$$

So left-handed chiral particles couple only to left-handed chiral fields, and right-handed chiral fields couple to right-handed chiral fields.

One must be very careful with how one interprets this statement. What it does *not* say is that there are left-handed chiral electrons and right-handed chiral electrons which are distinct particles. Useful though it is when describing particle interactions, chirality is not conserved in the propagation of a free particle. In fact the chiral states ϕ_L and ϕ_R do not even satisfy the Dirac equation. Since chirality is not a good quantum number it can evolve with time. A massive particle starting off as a completely left handed chiral state will soon evolve a right-handed chiral component. By contrast, helicity *is* a conserved quantity during free particle propagation. Only in the case of *massless* particles, for which helicity and chirality are identical and are conserved in free-particle propagation, can left- and right-handed particles be considered distinct. For neutrinos this mostly holds.

0.5 What you should know

- Understand the covariant form of the Dirac equation, and know how to manipulate the γ matrices.
- Know where the definition of a current comes from.
- Understand the difference between helicity and chirality.
- Be able to manipulate gamma matrices.

0.6 Futhur reading

Griffiths Chapter 7, Sections 7.1, 7.2 and 7.3 Griffiths Chapter 10, Section 10.7