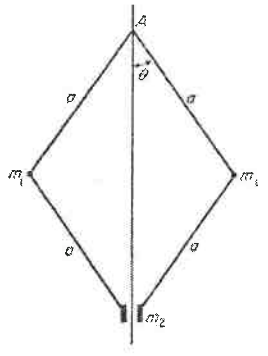
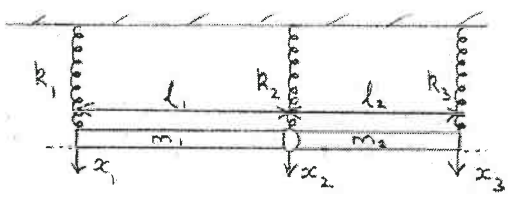
Write down the Lagrangian for each of the following systems.

1. Two masses M and m are connected by a massless, inextensible string hanging vertically over a massless, frictionless pulley.

2. A particle m_2 is free to move along a vertical axis. The whole system shown in the following figure rotates about this axis with a constant angular velocity Ω. The two masses, m_1, are connected by massless straight rods of length a to the upper pivot point A and the mass m_2. The angle between the rods and the vertical is θ. Hinges at the point A and at the masses m_1 allow θ to vary.



3. Consider the system shown in the figure below. It consists of two straight rods of mass m_1 and m_2 and lengths l_1 and l_2 that are joined together by a massless, frictionless hinge of negligible size. The free end of rod m_1, the hinge, and the free end of rod m_2 are to attached to springs of spring constants k_1, k_2 and k_3, respectively. The other ends of the springs are attached to fixed positions. The system is initially at equilibrium, with the two rods lying along a straight horizonal line.

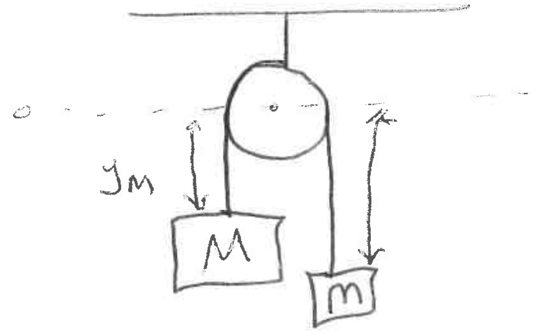


4. A uniform spherical ball of mass m and radius R rolls without slipping down a wedge of mass m and angle α, which itself can slide without friction on a horizontal table.

5. A cart of mass M rolls without slipping on a flat surface. Inside the cart, for mysterious reasons, someone has attached a mass, m, to one end of a spring with spring constant k and rest-length, l. The other end of the spring is attached to a pivot which allows it to rotate.

6. A pendulum is constructed from a mass m connected to a massless rigid rod of length l. The rod is attached to a point on the circumference of a ring of radius R. The pendulum is free to swing in the vertical plane, which is also the plane of the ring. The ring is free to rotate around its centre of rotation.

Q1



Degrees of freedom: - pulley is frictionless so don't need to worry about rotation.

- position of masses:

$$M: \quad y = -y_M$$
$$m: \quad y = -(l - y_M)$$

$\Rightarrow$ 1 d.o.f : y_M

Positions and speeds: mass M: $y = -y_m \Rightarrow V_M = -\dot{y}_M$

m: $y = -(l - y_m) \Rightarrow V_m = \dot{y}_M$

Kinetic Energies: $\frac{1}{2} M V_M^2 + \frac{1}{2} m V_m^2 = \frac{1}{2} M \dot{y}_M^2 + \frac{1}{2} m \dot{y}_M^2$

$$T = \frac{1}{2}(M+m)\cancel{V_m}\,\dot{y}_M^2$$

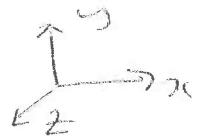
Potential energies: Use level of rotation axis as reference.
Then, PE of masses is only from gravity

$$V = PE = -Mg y_m - mg(\cancel{-}(l - y_m))$$
$$= \cancel{+ (M+m) g y_m + mgl} \; -(M-m) g y_M - mgl$$

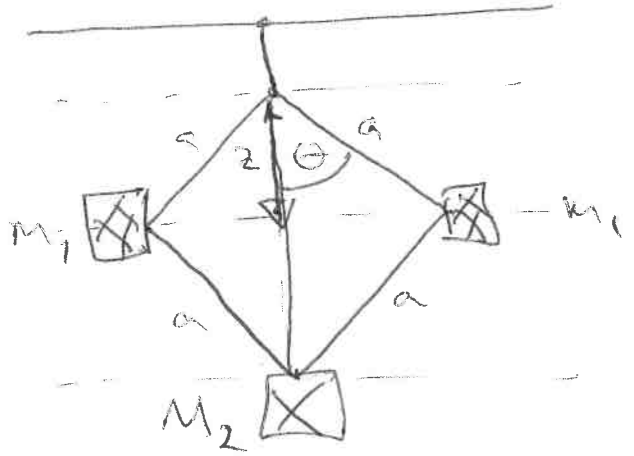
Lagrangian: $L = T - V = \frac{1}{2}(M+m)\dot{y}_m^2 + (M - m) g y_m + mgl$

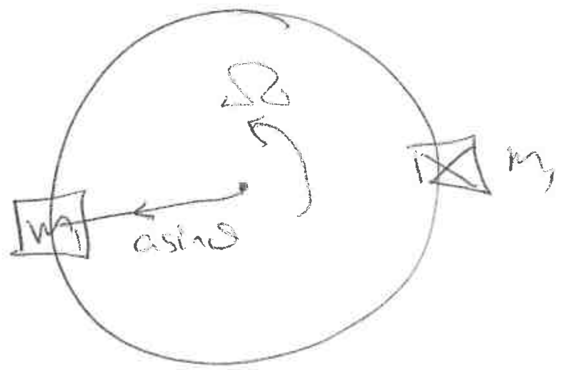
Q2

Side View



Top View





DOF: We have 3 bodies with coordinates:

m_1: $(\pm x_1, -y_1, \pm z_1)$

m_2: $(0, -y_2, 0)$

However x_1, y_1 are defined by θ and the length of the rods, a, which is constant

z_1 is determined by Ω, given x_1

∴ DOF: θ, Ω

Position of masses: m_1: $x_1 = \pm a\sin\theta$

$y_1 = \cancel{\$} - a\cos\theta$

m_2: $y_2 = -2a\cos\theta$

Speeds: $V_{x_1} = \dot{x}_1 = \frac{d}{dt}(\pm a\sin\theta) = \pm a\dot{\theta}\cos\theta$

$V_{y_1} = \dot{y}_1 = \frac{d}{dt}(-a\cos\theta) = a\dot{\theta}\sin\theta$

Transverse plane $V_{t_2} = \dot{y}_2 = \frac{d}{dt}(-2a\cos\theta) = -2a\dot{\theta}\sin\theta$

8 Top view: $V_{m_1,\Omega} = V_{m_2} = r\omega = a\sin\theta\,\Omega$

Kinetic Energies: ~~$T = \frac{1}{2} m_1 (V_{m_1}^2) 2\frac{1}{2}$~~

m_1: $T_{m_1} = \frac{1}{2} m_1 (V_{x_1}^2 + V_{y_1}^2 + V_{m_1,\Omega}^2)$

$= \frac{1}{2} m_1 (a^2\dot\theta^2\cos^2\theta + a^2\dot\theta^2\sin^2\theta + a^2\sin^2\theta\,\Omega^2)$

m_2: $T_{m_2} = \frac{1}{2} m_2 V_m^2 = \frac{1}{2} m_2 V_{x_2}^2 = \frac{1}{2} m_2 (2a\dot\theta\sin\theta)^2$

Total KE: $T = 2 \times T_{m_1} + T_{m_2}$

$= m_1 (a^2\dot\theta^2\cos^2\theta + a^2\dot\theta^2\sin^2\theta + a^2\sin^2\theta\,\Omega^2)$

$+ \frac{1}{2} m_2 (2a\dot\theta\sin\theta)^2$

$= m_1 (a^2\dot\theta^2 + a^2\sin^2\theta\,\Omega^2) + 2m_2 a^2\dot\theta^2\sin^2\theta$

Potential Energies: $PE_{m_1} = -m_1 g z = -m_1 g a\cos\theta \times 2$

$PE_{m_2} =$ ~~$2 \times PE_{m_1} = -2m_2 g z\cos\theta$~~

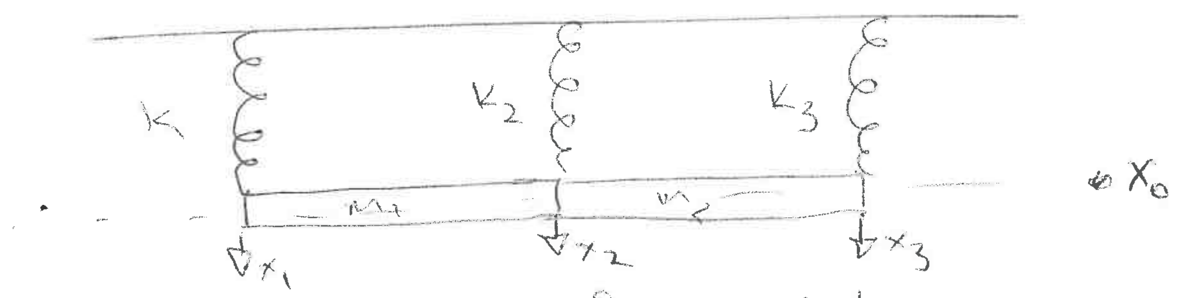
$= -2m_2 g a\cos\theta$

Lagrangian: $L = T - V = m_1 (a^2\dot\theta^2 + a^2\sin^2\theta\,\Omega^2) + 2m_2 a^2\dot\theta^2\sin^2\theta$

$+ 2ag(m_1 + m_2)\cos\theta$

#

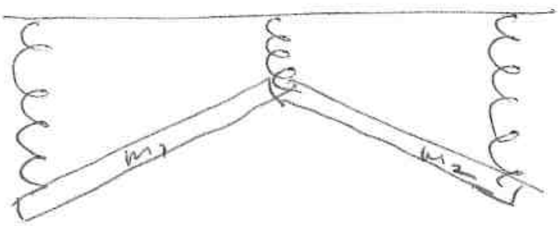
3)



let equilibrium position be reference point.

The masses can move up and down, and can also rotate around their COM

eg.



So we will need to take the rotation into account in the kinetic energy

<u>Degrees of freedom</u>: x_1, x_2, x_3 = position of springs from equilibrium

<u>Position</u>: The <u>ends</u> of rods can be at different positions, but the body as a whole will move with a single velocity. So, lets assume that the mass of the rods are concentrated at their COM.

For a given set of (x_1, x_2, x_3) extensions, the position of the COM of the rods are

$$m_1: \quad \frac{(x_0+x_1)+(x_0+x_2)}{2} = X_0 + \frac{x_1+x_2}{2}$$

$$m_2: \quad \frac{(x_0+x_2)+(x_0+x_3)}{2} = X_0 + \frac{x_2+x_3}{2}$$

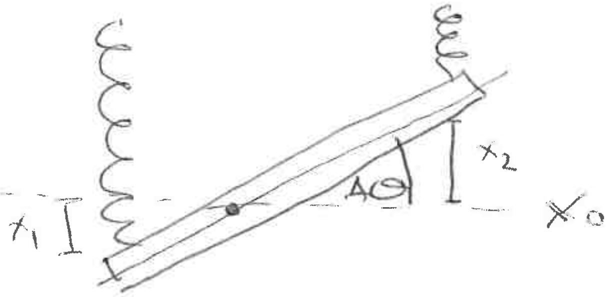
The translational speed of the COM is

$$m_1: \quad V_{m_1} = \left(\dot{X_0 + \frac{x_1+x_2}{2}}\right) = \left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)$$

$$m_2: \quad V_{m_2} = \left(\dot{X_0 + \frac{X_2+X_3}{2}}\right) = \left(\frac{\dot{x}_2 + \dot{x}_3}{2}\right)$$

<u>Translational KE</u> : $\frac{1}{2} m_1 \left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 + \frac{1}{2} m_2 \left(\frac{\dot{x}_2 + \dot{x}_3}{2}\right)^2$

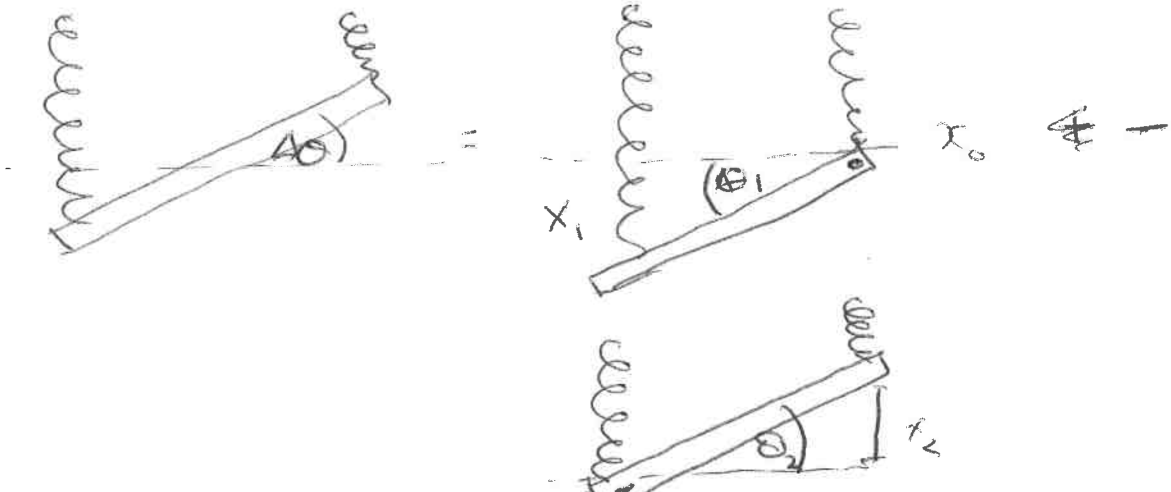
The bodies can rotate as well! The axis of rotation depends on the values of x_i



The angular speed is $\omega = \dfrac{\Delta\theta}{\Delta t}$

$\Delta\theta$ is made up of the angle formed when only one spring is extended. In this case the radius of rotation is the length of the rod

eg.



$$\Delta\theta = \theta_1 - \theta_2 \simeq \frac{X_1}{\ell} - \frac{x_2}{\ell}$$

where we approximate x_i as the arc lengths $(x_i \simeq r\theta)$

$\therefore$ Angular speed is: $\omega = \frac{\Delta\theta}{\Delta t} = \frac{(x_1 - x_2)}{\Delta t\, l}$

but $\frac{x_1}{\Delta t} = \dot{x}_1$ and $\frac{x_2}{\Delta t} = \dot{x}_2$

$$\therefore \quad \omega = \frac{(\dot{x}_1 - \dot{x}_2)}{l}$$

<u>Hence rotational KE:</u> ~~$\frac{1}{2} I$~~

m_1: $\frac{1}{2} I_1 \omega_1^2 = \frac{1}{2} I_1 \left(\frac{\dot{x}_1 - \dot{x}_2}{l}\right)^2$

m_2: $\frac{1}{2} I_2 \omega_2^2 = \frac{1}{2} I_2 \left(\frac{\dot{x}_2 - \dot{x}_3}{l}\right)^2$

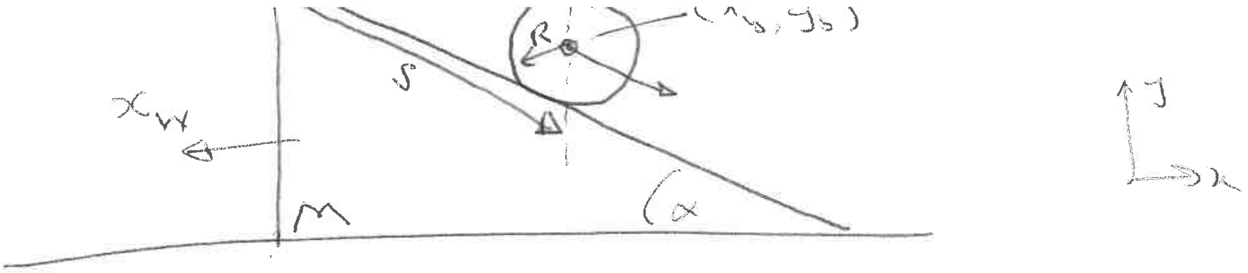
$\therefore$ <u>Total KE:</u> $\frac{1}{2} m_1 \left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 + \frac{1}{2} m_2 \left(\frac{\dot{x}_2 + \dot{x}_3}{2}\right)^2 + \frac{1}{2} I_1 \left(\frac{\dot{x}_1 - \dot{x}_2}{l_1}\right)^2 + \frac{1}{2} I_2 \left(\frac{\dot{x}_2 - \dot{x}_3}{l_2}\right)^2$

<u>Potential Energy:</u> Treat mass as being concentrated in COM ~~PE~~ Potential is ~~the~~ is the spring, since the system has zero PE at equilibrium.

$$\therefore PE = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_3 x_3^2$$

$$\therefore L = T - V = \frac{1}{2} m_1 \left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 + \frac{1}{2} m_2 \left(\frac{\dot{x}_2 + \dot{x}_3}{2}\right)^2 + \frac{1}{2} I_1 \left(\frac{\dot{x}_1 - \dot{x}_2}{l_1}\right)^2 + \frac{1}{2} I_2 \left(\frac{\dot{x}_2 - \dot{x}_3}{l_2}\right)^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 x_2^2 - \frac{1}{2} k_3 x_3^2 \; !$$

(4)



As ball rolls down incline, ~~it~~ the normal reaction force will act on the wedge ~~to~~ & will force it to the left.

DOF: Position of wedge ; x_w

Position of ball ; $(x_b, y_b) \Rightarrow$ defined by S, α with only S variable

$\Rightarrow$ Generalised coords : x_w, S

Positions : Wedge : $x = -x_w$ ← position of wedge relative to origin at $t=0$

Ball : $x_b = S\cos\alpha - x_w$

$y_b = S\sin\alpha$

Speeds : $V_w = -\dot{x}_w$

$V_{b,x} = \dot{S}\cos\alpha - \dot{x}_w$

$V_{b,y} = \dot{S}\sin\alpha$

Kinetic Energies : We have the wedge translating, the ball com ~~of~~ translating ~~&~~ the rotation of the ball

$$T = T_w + T_{ball} + T_{rotation}$$

$$T_w = \frac{1}{2} m \dot{x}_w^2$$

$$T_{ball} = \frac{1}{2} m (V_{b,x}^2 + V_{b,y}^2)$$

$$= \frac{1}{2} m \left[(\dot{s} \cos\alpha - \dot{x}_w)^2 + \dot{s}^2 \sin^2\alpha \right]$$

$$= \frac{m}{2} \left[\dot{s}^2 + \dot{x}_w^2 - 2 \dot{s} \dot{x}_w \cos\alpha \right]$$

$$T_{rotation} = \frac{1}{2} I_{ball} \omega^2$$

Since ball rolls without slipping: $\omega = \frac{V_b}{R}$

$$\therefore T_{rotation} = \frac{1}{2} I_{ball} \left(\frac{V_b}{R} \right)^2$$ ← velocity of ball with respect to the wedge

$$= \frac{1}{2} \left(\frac{1}{2} m R^2 \right) \left(\frac{\dot{s}}{R} \right)^2$$

$$= \frac{1}{4} m \dot{s}^2$$

$$\therefore T = \frac{m}{2} \left[\dot{s}^2 + \dot{x}_w^2 - 2 \dot{s} \dot{x}_w \cos\alpha \right] + \frac{1}{4} m \dot{s}^2$$

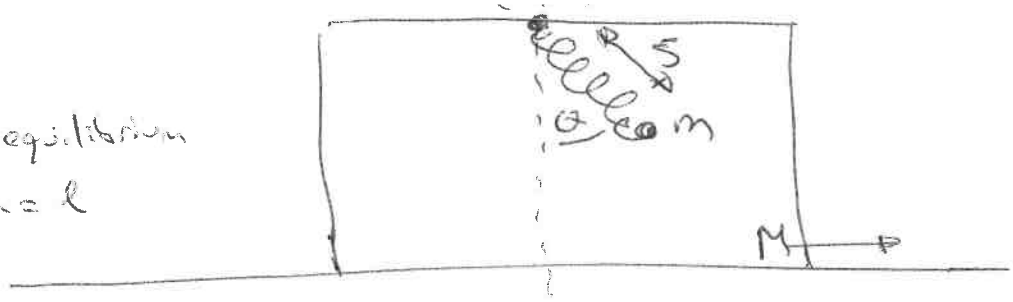
Potential Energy $\quad PE_{block} = 0$

$$PE_{ball} = - mgs \sin\alpha$$

$$\therefore L = T - V = \frac{m}{2} \left[\dot{s}^2 + \dot{x}_w^2 - 2 \dot{s} \dot{x}_w \cos\alpha \right] + \frac{1}{4} m \dot{s}^2 + mgs \sin\alpha$$

(5)

Spring equilibrium length = l



DOF - Position of cart
- position of pendulum: defined by angle of pendulum to vertical and extension of pendulum along the spring axis

Positions: ~~Cart~~ Cart : $x = x_c$

Mass : $x = x_c + s\sin\theta$

$y = -s\cos\theta$

Speeds: Cart : $V_c = \dot{x}_c$

Mass : $V_y = \dot{y} = -\dot{s}\cos\theta + s\dot{\theta}\sin\theta$

$V_x = \dot{x} = \dot{X}_c + \dot{s}\sin\theta + s\dot{\theta}\cos\theta$

KE: $T = \frac{1}{2}M\dot{x}_c^2 + \frac{1}{2}m(V_x^2 + V_y^2)$

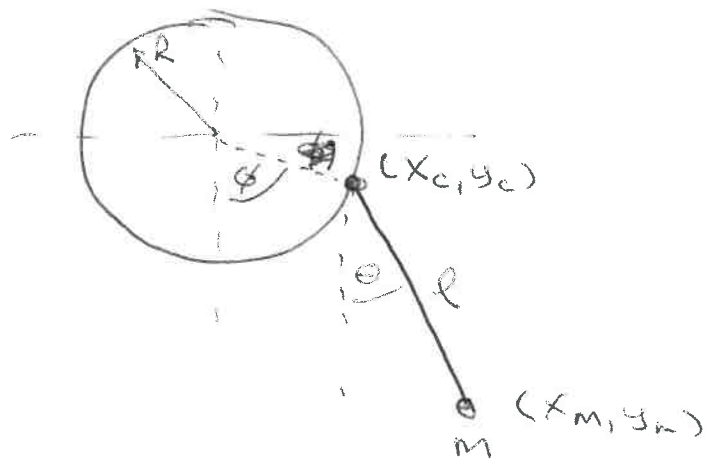
$= \frac{1}{2}M\dot{x}_c^2 + \frac{1}{2}m\left[(\dot{x}_c + \dot{s}\sin\theta + s\dot{\theta}\cos\theta)^2 + (s\dot{\theta}\sin\theta - \dot{s}\cos\theta)^2\right]$

Potential Energy: Mass has gravitational PE and spring has simple harmonic potential energy

$$PE = -mgs\cos\theta - \frac{1}{2}k(l-s)^2$$

Lagrangian: ~~T~~ $\mathcal{L} = T - PE$ [can be bothered writing it again]

(6)



DOF: Position of contact point : (x_c, y_c)
" " mass : (x_m, y_m)

x_c, y_c defined by radius of loop and angle of loop (R, ϕ)

x_m, y_m defined by (R, ϕ) and (l, θ)

$R + l$ are constant $\Rightarrow$ generalised coordinates are : θ, ϕ

Positions : $x_c = R\sin\phi \quad , \quad y_c = -R\cos\phi$

$$x_m = x_c + l\,\overset{\sin}{\cancel{\cos}}\,\theta = R\sin\phi + l\,\overset{\sin}{\cancel{\cos}}\,\theta$$

$$y_m = y_c - l\cos\theta = -R\cos\phi - l\cos\theta$$

Speed of mass, m : $V_{x,m} = \dot{x}_m = R\dot{\phi}\cos\phi + l\dot{\theta}\cos\theta$

$$V_{y,m} = \dot{y}_m = R\dot{\phi}\sin\phi + l\dot{\theta}\sin\theta$$

Kinetic Energies : Two types : transverse energy from movement of mass, m.
rotational energy from rotation of loop

$$KE(\text{mass}, m) = \frac{1}{2} m V_m^2 = \frac{1}{2} m \left(V_{x,m}^2 + V_{y,m}^2\right)$$

$$= \frac{1}{2} m \left[(R\dot{\phi}\cos\phi + l\dot{\theta}\cos\theta)^2 + (R\dot{\phi}\sin\phi + l\dot{\theta}\sin\theta)^2 \right]$$

$$= (\text{after a while}) \; \frac{1}{2} m \left[R^2\dot{\phi}^2 + l^2\dot{\theta}^2 + 2Rl\dot{\theta}\dot{\phi} \times \cos(\theta - \phi) \right]$$

$$KE(\text{rotation of ring}) = \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} (MR^2)(\dot{\phi}^2)$$

$$\therefore KE(\text{total}) = \frac{m}{2}\left[R^2 \dot{\phi} + l^2\dot{\theta}^2 + 2Rl\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right] + \frac{M}{2} R^2 \dot{\phi}^2$$

Potential energy: Total PE of ring is zero if reference point is center of rotation

PE of mass is $PE = -mgh$

$$= -mg(y_m)$$

$$= -mg\left[R\cos\phi + R\cos\theta\right]$$

Lagrangian: $L = T - V$

$$= \frac{m}{2}\left[R^2\dot{\phi}^2 + l^2\dot{\theta}^2 + 2Rl\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right] + \frac{M}{2} R^2\dot{\phi}^2 + mgR\left[\cos\phi + \cos\theta\right]$$