Leptonic Mixings from a Variational Principle

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Outline of Talk

- Common Origin of Masses and Mixing in the Standard Model
- Jarlskog Covariance
- Historical Attempts to Constrain Masses and Mixings
- Towards a Theory of Masses and Mixings
- Some Examples
- The Future

Work done in collaboration with Bill Scott (RAL) - hep-ph/0508012.

Draws heavily on, and develops earlier work by Cecilia Jarlskog.

Flavour Physics: the Issues

- Why 3 families?

  I can't answer that.

- What are the unmeasured parameters?

  We are working on neutrino experiments to try to answer that.

- Why are the fermion mass spectra what they are?

  Hope eventually to answer this.

- Why are the fermion mixing spectra what they are?

  We suggest a simple way to address this.

  Claim some success, with the lepton mixings, at least.

Common Origin of Masses and Mixing in the SM

- In SM, they have common origin in fermion-Higgs "Yukawa couplings".
- Will focus on leptons – consider 3 families of leptonic $SU(2)$ doublets:

$$\psi_a = \begin{pmatrix} \nu_a \\ \ell_a^- \end{pmatrix}, \quad \psi_b = \begin{pmatrix} \nu_b \\ \ell_b^- \end{pmatrix}, \quad \psi_c = \begin{pmatrix} \nu_c \\ \ell_c^- \end{pmatrix}$$

  Each transforms among itself under $SU(2)$ weak transformations in usual way.

- Define also two three-component vectors in family-space:

$$\underline{\nu} = \left(\nu_a, \nu_b, \nu_c\right), \quad \underline{\ell} = \left(\ell_a^-, \ell_b^-, \ell_c^-\right),$$

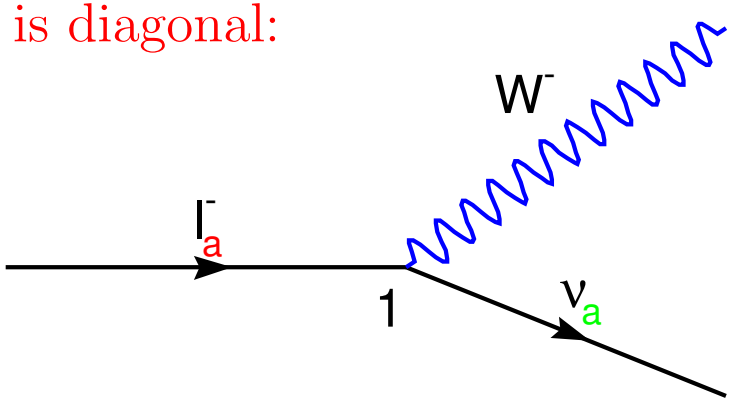
Origin of Masses and Mixing (contd.)

- Accommodate ν masses in SM with simplest extension: Dirac masses only $\rightarrow$ νSM (extension to Majarana neutrinos is on the "to do" list).
- After SSB, mass terms appear in Lagrangian (Dirac structure assumed):

$$\mathcal{L}_{mass} = \bar{\underline{\ell}}\, L\, \underline{\ell} + \bar{\underline{\nu}}\, N\, \underline{\nu} + H.C.$$

- L and N, are arbitrary, 3×3 Hermitian mass matrices.
- Charged-current weak interaction (same basis) is diagonal:

$$\mathcal{L}_{cc} = g(\bar{\underline{\ell}}\,.\,\underline{\nu})W^+ + H.C.$$



Origin of Masses and Mixing (contd.)

- Diagonalise L and N, $\rightarrow$ "physical" states.

$$\underline{\ell} \rightarrow \underline{\ell}^{\,m} = U_\ell \, \underline{\ell}\,, \qquad \underline{\nu} \rightarrow \underline{\nu}^{m} = U_\nu \, \underline{\nu}$$

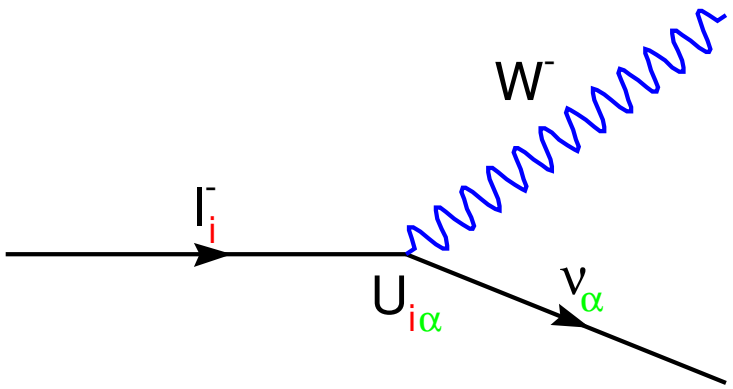
- Then

$$\mathcal{L}_{mass} \;\; = \;\; \underline{\bar{\ell}}^{\,m} D_\ell \, \underline{\ell}^{\,m} + \underline{\bar{\nu}}^{\,m} D_\nu \, \underline{\nu}^{m}$$

where $D_\ell \equiv U_\ell L U_\ell^\dagger$ and $D_\nu \equiv U_\nu N U_\nu^\dagger$ are diagonal mass matrices.

- Also

$$\mathcal{L}_{cc} \;\; = \;\; g(\underline{\bar{\ell}}^{\,m} U_{MNS} \, \underline{\nu}^{m}) W^+ + H.C.$$



where $U_{MNS} \equiv U_\ell U_\nu^\dagger$ is the lepton mixing matrix. NB. Analogous for quarks.

Jarlskog Covariance

- Instead of diagonalising, could have made arbitrary *simultaneous* transf.:

$$\underline{\ell} \to \underline{\ell}\,' = U\underline{\ell}\,, \qquad \underline{\nu} \to \underline{\nu}' = U\underline{\nu} \qquad \text{(ie. same } U\text{)}$$

- Then

$$\mathcal{L}_{mass} \;=\; \underline{\bar{\ell}}\,' L'\,\underline{\ell}\,' + \underline{\bar{\nu}}'\,N'\,\underline{\nu}'$$

  where $L' \equiv ULU^\dagger$ and $N' \equiv UNU^\dagger$ are new arbitrary Hermitian mass matrices.

- In this basis, charged current is:

$$\begin{aligned}\mathcal{L}_{cc} &= g(\underline{\bar{\ell}}\,' UU^\dagger\,\underline{\nu}')W^+ + H.C. \\ &= g(\underline{\bar{\ell}}\,'.\underline{\nu}')W^+ + H.C.\end{aligned}$$

  ie. still diagonal, but mass matrices have changed ($\equiv$ "Jarlskog-covariant")

- New basis looks like just as good a starting point...

...and Observables are Invariant:

- Now

$$\underline{\ell}^{\,m} = U_\ell\, U^\dagger \underline{\ell}\,', \qquad \underline{\nu}^{\,m} = U_\nu\, U^\dagger \underline{\nu}'$$

and

$$\mathcal{L}_{mass} \;=\; \underline{\bar{\ell}}^{\,m} D'_\ell\, \underline{\ell}^{\,m} + \underline{\bar{\nu}}^{\,m} D'_\nu\, \underline{\nu}^{m}$$

where $D'_\ell \equiv U_\ell U^\dagger L' U U^\dagger_\ell = D_\ell$ and $D'_\nu \equiv U_\nu U^\dagger N' U U^\dagger_\nu = D_\nu$ are unchanged.

- Also

$$\mathcal{L}_{cc} \;=\; g(\underline{\bar{\ell}}^{\,m} U'_{MNS}\, \underline{\nu}^{m}) W^+ + H.C.$$

where $U'_{MNS} \equiv U_\ell U^\dagger U U^\dagger_\nu = U_{MNS}$ is also unchanged.

- ie. Masses and mixing angles are "Jarlskog-invariant".
- NB. Is not a "symmetry" (that would be if $L' = L$), but an invariance of νSM

Historical Attempts to Constrain Fermion Masses and Mixings

- Inspired by phenomenological relationships in quark mixing, eg.

$$\tan\theta_C \simeq 0.23 \simeq \sqrt{\frac{m_d}{m_s}}\,; \qquad \tan\theta_{cb} \simeq 0.04 \simeq \sqrt{\frac{m_d}{m_b}}$$

  and common origin of masses and mixings, many attempts made to relate them.

- Best known is by H. Fritzsch from 1977 and 1978: mass matrices:

$$M^{\frac{2}{3}} = \begin{pmatrix} 0 & a & 0 \\ a^* & 0 & b \\ 0 & b^* & c \end{pmatrix}; \qquad M^{-\frac{1}{3}} = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha^* & 0 & \beta \\ 0 & \beta^* & \gamma \end{pmatrix}$$

- Predicts (after approximations) above θ_C relation, but also

$$\tan\theta_{cb} > \sqrt{\frac{m_s}{m_b}} - \sqrt{\frac{m_c}{m_t}} \simeq 0.09, \qquad \text{(now excluded)}.$$

How does Fritzsch ***Ansatz*** work?

- Mass matrices encode 10 pieces of information (6 masses and 4 mixing parameters) using only 6 magnitudes and one phase, $\Longrightarrow$ 3 constraints.
- Spawned many refinements, some phenomenologically successful.
- Modern models appeal to flavour symmetries to enforce zeroes or other "textures".
- Many build-in supersymmetry and/or complicated Higgs structures etc.
- However, generally assume that $\exists$ a "special" basis, in which the mass matrices take some "particular" form (which limits number of parameters to < 10).
- "Specialness" of this basis is never explained.

Towards a Theory of Masses and Mixings (ToMM)

Look to νSM for some principles on which to base a theory

- νSM as it stands has no preferred basis in which to write the lepton (and quark) mass matrices. A ToMM should respect this. If not, it should at least explain existence of a preferred basis.
- νSM as it now stands treats all families on an equal footing. A ToMM should respect this, ie. mass/mixing constraints should also be flavour-symmetric.

Flavour-Symmetric Jarlskog Invariants

- "Usual" observables are not flavour-symmetric. eg. the electron mass, m_e, (electron label) and U_{e3} (flavour and mass labels).
- Jarlskog showed how to write flavour-symmetric invariant observables.
- Examples are (all matrices in an arbitrary weak-basis, primes dropped):

$$\mathrm{Tr}(L) = m_e + m_\mu + m_\tau; \quad \mathrm{Tr}(L^2) = m_e^2 + m_\mu^2 + m_\tau^2; \quad \mathrm{Tr}(L^3) = m_e^3 + m_\mu^3 + m_\tau^3$$

  which form a complete set.

- All traces of polynomials of L and N are invariant (and FS), eg.:

$$\mathrm{Tr}(L'N') = \mathrm{Tr}(ULU^\dagger UNU^\dagger) = \mathrm{Tr}(ULNU^\dagger) = \mathrm{Tr}(LNU^\dagger U) = \mathrm{Tr}(LN)$$

  etc.

And for Mixing...

- From earlier: $U_{MNS} \equiv U_\ell U_\nu^\dagger$.
- Clearly, $U_\nu = U_\ell \implies U_{MNS} = I \implies$ no mixing.
- So, experiment $\implies U_\nu \neq U_\ell$: L and N are diagonalised by different U's.
- Hence,

$$[L, N] \equiv iC \neq 0$$

- Now, under a change of weak basis,

$$C \to C' = UCU^\dagger$$

  and likewise for any polynomial of C. As before, traces are invariant.

- However, $\mathrm{Tr}C = 0$ for any commutator.

  $\implies \exists$ only two indepenent commutator invariants for C: $\mathrm{Tr}C^2$ and $\mathrm{Tr}C^3$.

For Example

- Now,

$$\begin{aligned} \mathrm{Tr}\, C^3/3 &= \mathrm{Det}\, C \\ &= -2(m_\mu - m_\tau)(m_\tau - m_e)(m_e - m_\mu)(m_2 - m_3)(m_3 - m_1)(m_1 - m_2)J \end{aligned}$$

  is a pure-CP-violating observable. (J is twice the area of the unitarity triangle) – manifestly symmetric among all flavour and mass indices.

- $\mathrm{Tr}C^2$ is a pure-CP-conserving function of lepton masses and mixings (complicated to write out). Also manifestly symmetric among flavour and mass labels.

- Represent independent flavour-symmetric *measures* of mixing (they $\to 0$ as mixing $\to 0$).

Jarlskog-Covariant Constraints?

- Motivated by quasi-maximal nature of leptonic mixing, attempt to "explain" it by maximising (extremising) a flavour-symmetric Jarlskog invariant:

$$\mathcal{A} = \mathrm{Tr}[f(L, N)].$$

- $\Longrightarrow$ find values for which derivatives are zero.
- Q: Derivative wrt what? (mixing angles, masses?).
- A: Derivatives wrt Yukawas themselves:

$$\partial(\mathcal{A})/\partial L \equiv \partial_L(\mathcal{A}) = 0; \qquad \partial(\mathcal{A})/\partial N \equiv \partial_N(\mathcal{A}) = 0.$$

- Treats all indices symmetrically, and always $\rightarrow$ covariant matrix constraints!
- But which invariant? (We tried several!)

Eg. Consider $\mathbf{Tr}C^3$

- Extremisation conditions are:

$$\begin{aligned} \partial_L \,\mathrm{Tr}\, C^3/3 &= -i[N, C^2]^T = 0, \\ \partial_N \,\mathrm{Tr}\, C^3/3 &= +i[L, C^2]^T = 0. \end{aligned}$$

- Manifestly covariant constraints: both sides of equation are evidently form-invariant under an arbitrary transformation $L \to U L U^\dagger$, $N \to U N U^\dagger$.
- Would get same constraints doing the extremisation in any basis.
- Equations treat all flavour and mass labels symmetrically $\Longrightarrow$ satisfy our two principal requirements.
- Are general properties of our extremisation procedure.

Make a Modest Start

- Decide to extremise at *fixed* lepton masses.

  (Allowing them to vary is the ultimate goal)

- Extremisation with constraints $\Longrightarrow$ Lagrange multipliers.

- Outcome is to modify RHS of equations:

$$(\partial_L \,\mathrm{Tr}\, C^3/3)^T = -i[N, C^2] \quad = \quad \lambda_{L0} \,+\, \lambda_{L1}\, L \,+\, \lambda_{L2}\, L^2$$

$$(\partial_N \,\mathrm{Tr}\, C^3/3)^T = +i[L, C^2] \quad = \quad \lambda_{N0} \,+\, \lambda_{N1}\, N \,+\, \lambda_{N2}\, N^2$$

  where the arbitrary constants λ are the LMs. (Eqs. still Jarlskog-covariant).

- $\Longrightarrow$ 18 constraint eqns. but not all independent.

- They exactly constrain the mixings, while all masses are fixed to their exptl. values by LMs.

Solution for $\mathbf{Tr}C^3$

- Solution to constraint eqautions corresponds to trimaximal lepton mixing (an old friend):

$$U_{MNS} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^* \\ 1 & \omega^* & \omega \end{pmatrix}$$

  where $\omega = e^{2i\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, a cube-root of unity.

- $\Longrightarrow$ maximal CP violation.

- Suggestive, but excluded by experiment.

Try Instead $\mathbf{Tr}C^2$

- Constraints are now:

$$(-\partial_L \mathrm{Tr}\, C^2/2)^T = +[N,C] = \lambda_{L0} + \lambda_{L1}\, L + \lambda_{L2}\, L^2$$
$$(-\partial_N \mathrm{Tr}\, C^2/2)^T = -[L,C] = \lambda_{N0} + \lambda_{N1}\, N + \lambda_{N2}\, N^2.$$

- Get CP-conserving solution:

$$(|U_{MNS}|^2) = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} .33333 & .33333 & .33333 \\ .17079 & .16257 & .66663 \\ .49587 & .50409 & .00003 \end{pmatrix} \end{array} \simeq \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/6 & 1/6 & 2/3 \\ 1/2 & 1/2 & 0 \end{pmatrix} \end{array}.$$

  – also inconsistent with experiment.

- BUT it is a simple permutation of the (viable) tribimaximal mixing matrix!

What Works?

- Try a linear combination:

$$\mathcal{A} = \mathrm{Tr}C^2 + q\,\mathrm{Tr}(C.[L^2, N]),$$

  where q is a JI constant of dimensions m^{-1}.

- Only CP-conserving solutions.

- $\rightarrow$ a trajectory of mixing matrices, as a function of q (and unknown m_{ν_1}).

- (Arbitrarily) choose $m_1 = 0$ (for simplicity).

- Look for solution with one trimaximally mixed mass eigenstate

  ie. $|U_{\alpha i}|^2 = 1/3, \forall\, \alpha$ and for some i (to fit phenomenology).

- Everything is now determined in terms of fixed ν mass-squared differences (we first need to find q to fit the known spectrum).

Solution

- First allow masses to float to examine properties of trajectory as fn. of q.
- Has 3 poles, each corresponding to a permutation of the tribimaximal form, and (in limit as we approach each pole) has a single ν mass degeneracy.
- As we approach "correct" pole, Δm^2_{21} tends to zero (not bad, cf. data). $\Longrightarrow$ smallness of $\sin\theta_{13}$ related to smallness of $\Delta m^2_{21}/\Delta m^2_{31}$.
- Of course, $\Delta m^2_{21}/\Delta m^2_{31}$ is known (roughly) which fixes q and the mixings:

$$(|U_{l\nu}|^2) = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} 0.662 & 0.333 & 0.005 \\ 0.219 & 0.333 & 0.448 \\ 0.120 & 0.333 & 0.547 \end{pmatrix} \end{array} \simeq \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{pmatrix} \end{array}.$$

– fully compatible with experiment.

What Have we Actually Done?

- Our approach is not an *extension* of the νStandard Model – there are no new particles, no new symmetries and no new interactions (boring!?)
- It works entirely *within* the existing νSM:
  - it relies on existing transformation properties of the νSM, merely requiring that new BSM constraints respect them.
  - it utilises existing degrees-of-freedom (the Yukawa coupling "constants"), merely treating them instead like dynamical variables (determined by a conventional variational principle).
- If anything, it is a *contraction* of the νSM – we reduce the number of free parameters in the lepton sector from $6 + 4$ to $6 + 1$.
- Though original in the context of flavour physics, our procedure builds on tried and trusted traditions of our field - looks like the principle of least action.

The Future

- We do not consider the successful invariant

$$\mathcal{A} = \mathrm{Tr}C^2 + q\,\mathrm{Tr}(C.[L^2, N])$$

  to be the last word.

  – does not accommodate CP violation

  – extremisation is performed at fixed masses.

- Effort currently underway to incorporate CP violation.

- Would like to find an invariant which can be extremised without using LMs

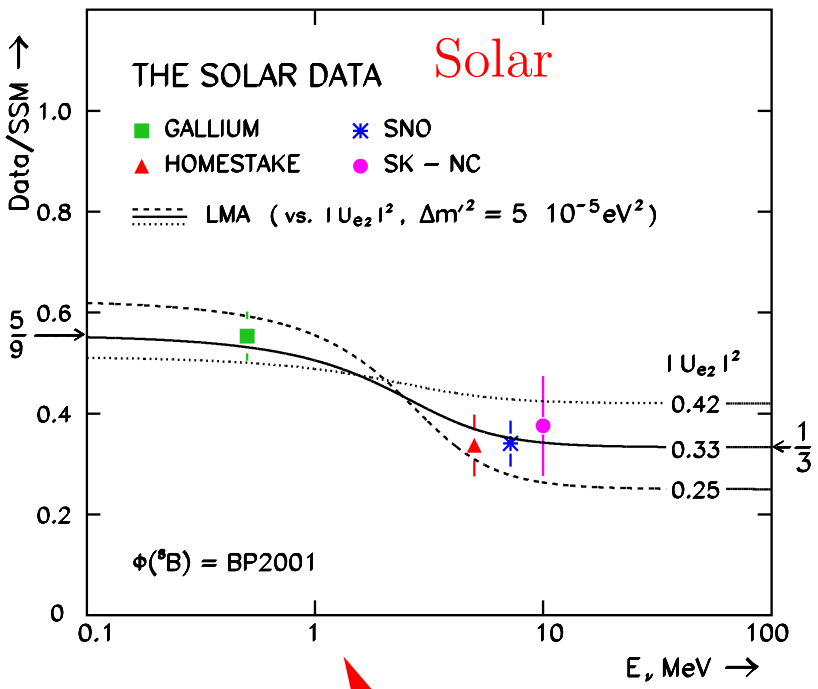
  $\rightarrow$ predict masses also.

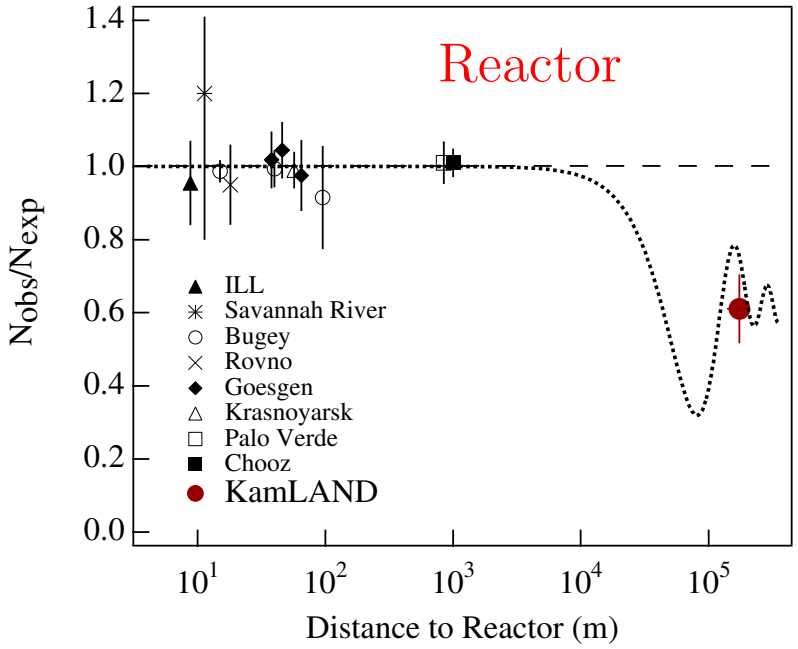
- Need to consider Majorana neutrinos and interaction with Seesaw mechanism

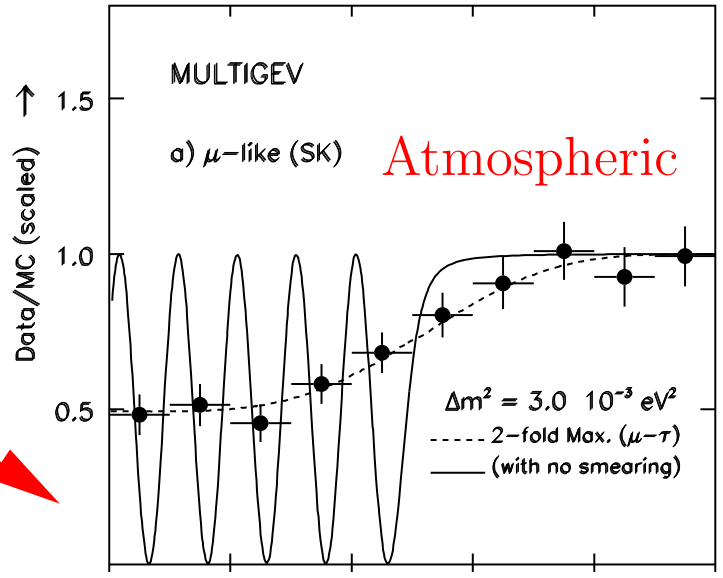
- Need to find successful application of our procedure to the quarks.

Back-up Slides

Summary of Neutrino Oscillation Data







	ν_1	ν_2	ν_3
e	0.82	0.58	< 0.17
μ	0.41	0.58	0.71
τ	0.41	0.58	0.71

$|U| \sim$

Nomenclature for Particular Bases

- "Mass basis" - (specified by the "m" superscript) is unique, up to a bunch of possible re-phasings (which I may come back to).
- "Weak (or gauge) basis" - Any basis in which the weak interaction is diagonal (eg. those specified by the un-superscripted fields, and/or by the "primed" fields). WB is not unique - different ones are all related by the U transformation. Any WB can be used to show gauge invariance, etc.
- "Flavour basis" - a particular choice of WB in which the charged lepton mass matrix, L', is diagonal (while N' is not). Corresponds to a special choice of U, and is unique (up to the definition of the phases of the charged lepton fields). It is the basis in which all 6 leptons carry one of the flavour indices e, μ, τ. It is the *conventional* basis.
- "Neutrino Mass basis" - the particular WB for which N' is diagonal and L' is not. Is not often used.

Solution Method

- Covariance allows to choose any basis in which to solve $\Longrightarrow$ take flavour basis
- L is diagonal (N not) $\Longrightarrow$ off-diagonal elements of ∂_L Eqn. $= 0$.
- Further, choose phases such that imaginary parts of N all have equal moduli:

$$N = \begin{pmatrix} a & z & y \\ z & b & x \\ y & x & c \end{pmatrix} + id \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

- Variables a, b, c, x, y, z and d encode the three neutrino masses and the four mixing parameters

Try Instead $\mathbf{Tr}C^2$

Constraints are now:

$$(-\partial_L \mathrm{Tr}\, C^2/2)^T = +[N,C] \;=\; \lambda_{L0} \,+\, \lambda_{L1}\, L \,+\, \lambda_{L2}\, L^2$$

$$(-\partial_N \mathrm{Tr}\, C^2/2)^T = -[L,C] \;=\; \lambda_{N0} \,+\, \lambda_{N1}\, N \,+\, \lambda_{N2}\, N^2.$$

Get CP-conserving solution:

$$x = \pm\sqrt{(a-b)(c-a)E} \qquad E = \frac{(m_e - m_\mu)(m_\tau - m_e)}{(m_e + m_\mu - 2m_\tau)(m_\tau + m_e - 2m_\mu)}$$

$$y = \pm\sqrt{(b-c)(a-b)M} \qquad M = \frac{(m_\mu - m_\tau)(m_e - m_\mu)}{(m_\mu + m_\tau - 2m_e)(m_e + m_\mu - 2m_\tau)}$$

$$z = \pm\sqrt{(c-a)(b-c)T} \qquad T = \frac{(m_\tau - m_e)(m_\mu - m_\tau)}{(m_\tau + m_e - 2m_\mu)(m_\mu + m_\tau - 2m_e)}.$$

with a, b, c fixed by the ν masses.

Solution for $\mathbf{Tr}C^2$

Interestingly, the observed ν mass-squared difference hierarchy is correctly obtained setting $\frac{b-a}{a-c} = 1$ ($\Longrightarrow$ $\frac{\Delta m^2_{21}}{\Delta m^2_{31}} = 0.03$).

The mixing matrix $\rightarrow$

$$(|U_{MNS}|^2) = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} .33333 & .33333 & .33333 \\ .17079 & .16257 & .66663 \\ .49587 & .50409 & .00003 \end{pmatrix} \end{array} \simeq \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/6 & 1/6 & 2/3 \\ 1/2 & 1/2 & 0 \end{pmatrix} \end{array}.$$

which is inconsistent with experiment.

BUT it is a simple permutation of the (viable) tribimaximal mixing matrix!

What Next?

We tried the obvious next step, ie. a linear combination:

$$\mathcal{A} = i\mathrm{Tr}C^3 + r\,\mathrm{Tr}C^2.$$

where r is a JI constant of dimensions m^2.

The solution gives a trajectory of mixing matrices, as a function of r (and the unknown offset of the neutrino masses).

Has some very interesting properties (see below), including viable mixing matrices and CP violation.

However, so far, have failed to find a solution which simultaneously fits observed masses and mixings.

Complete characterisation of solution space is work in progress.

Complete Set of Flavour-Symmetric, Invariant Measures of Mixing

$\exists$ four observables of mixing $\Longrightarrow$ must be possible to find four independent flavour-symmetric invariant measures of mixing. We extended the set as follows:

Commutators of powers of L and N are independent measures of mixing:

$$C_{ij} \equiv [L^i, N^j].$$

Complete set is obtained for $i, j \leq 2$ (higher powers of L and N may be reduced to functions of lower powers and their traces, using the characteristic equation).

While these are not invariants, it is easy to construct (CP-conserving) invariants by analogy with $\mathrm{Tr}C_{11}^2$ considered already:

$$Q_{mn} \equiv \mathrm{Tr}C_{ij}.C_{kl}$$

where $m = i + k - 1$ and $n = j + l - 1$. There are exactly 9 of these, any four of which are independent.