Generalisations of Tri-bimaximal Mixing

and Flavour-symmetric Constraints

Paul Harrison*

University of Warwick

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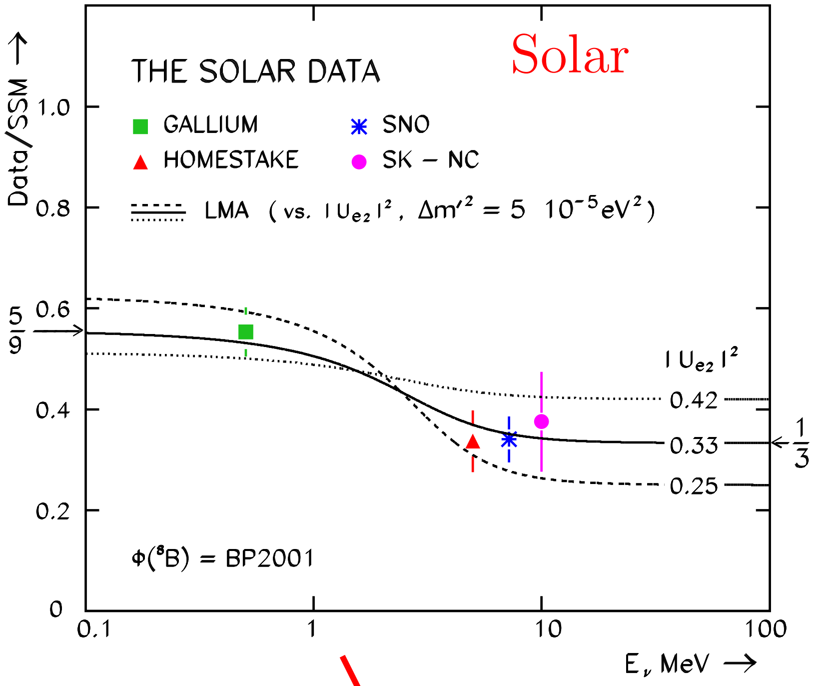
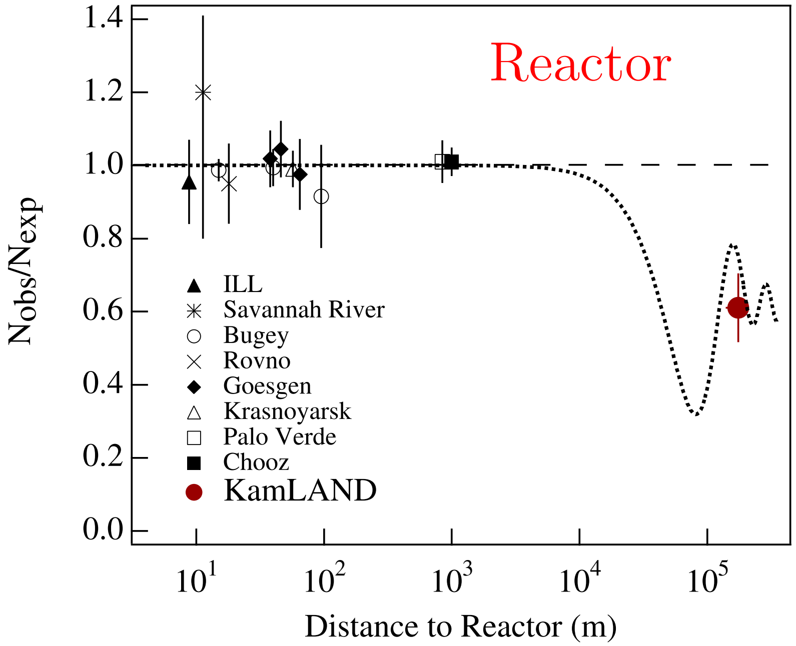
*In collaboration with Bill Scott, RAL

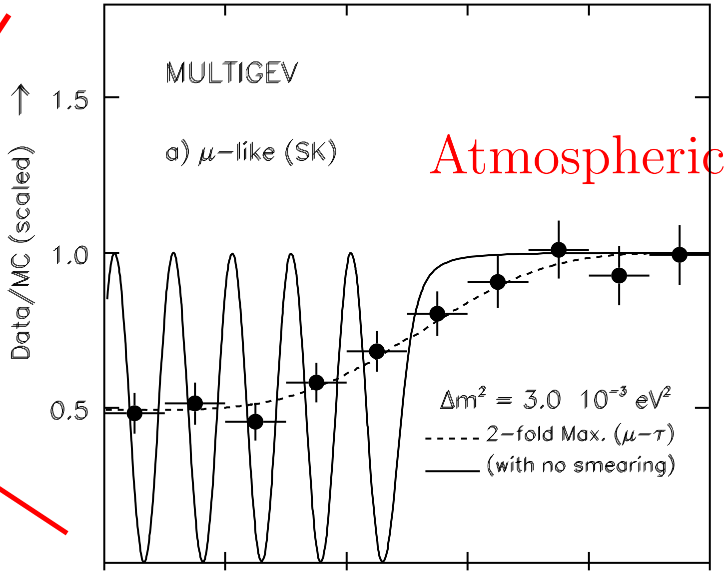
Outline of Talk

- Summary of ν Oscillation Data
- Tri-Bimaximal Mixing
- Symmetries of Tri-Bimaximal Mixing
- Generalisations of Tri-Bimaximal Mixing
- A New Flavour-symmetric Mixing Observable
- Discussion and Summary

Summarising $|U_{MNS}|^2$ from ν Oscillation Data







$$
|U_{MNS}|^2 = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array}
\begin{array}{c}
\begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\
\begin{pmatrix}
\sim \frac{2}{3} & 0.31 \pm 0.04 & < 0.013 \\
\sim \frac{1}{6} & \sim \frac{1}{3} & 0.50 \pm 0.11 \\
\sim \frac{1}{6} & \sim \frac{1}{3} & \sim \frac{1}{2}
\end{pmatrix}
\end{array}
$$

NB. 68% CL!

TriBimaximal Mixing is Leading Approximation

ν_2 is $\sim$ trimaximally mixed; ν_3 is $\sim$ bimaximally mixed:

$$(|U_{MNS}|^2) \simeq \begin{array}{c c} & \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{array}{c} e \\ \mu \\ \tau \end{array} & \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{pmatrix} \end{array} .$$

Corresponding neutrino mass matrix (squared) in the flavour basis, ie. $M_\nu^2 = U D_\nu^2 U^\dagger$ where $D_\nu^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$, is given by:

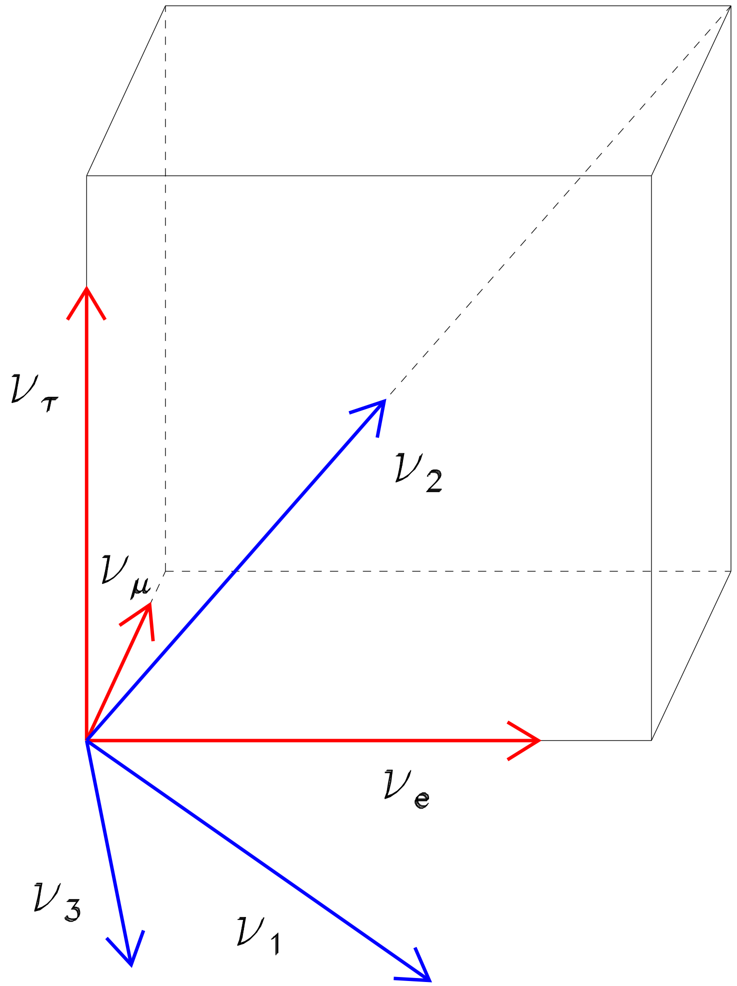
$$M_\nu M_\nu^\dagger = \begin{array}{c c} & \begin{array}{ccc} \nu_e & \nu_\mu & \nu_\tau \end{array} \\ \begin{array}{c} \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} & \begin{pmatrix} s+t+u & u & u \\ u & s+u & t+u \\ u & t+u & s+u \end{pmatrix} \end{array}$$

where $s = (m_1^2 + m_3^2)/2$, $t = -\Delta m_{31}^2/2$ and $u = -\Delta m_{12}^2/3$.

Discussion of TBM

TBM form is redolent of symmetries:

- Taking ν_e, ν_μ, ν_τ to define the orientation of a cube, ν_2 lies along its body diagonal
- Same coefficients form the $M = 0$ subset of the $j \times j = 1 \times 1$ set of Clebsch-Gordan Coeffts.



Several authors have built TBM into models,

eg.

G. Altarelli and F. Feruglio, hep-ph/0512103

I. de Medeiros Varzielas, S. King and G. Ross, hep-ph/0512313

E. Ma PRD 73:057304, 2006 etc.

Symmetries of TBM: Democracy, $\boldsymbol{\mu - \tau}$ Reflection and $\boldsymbol{CP}$

Tri-bimaximal mixing is defined by three symmetries:

- CP Symmetry
- Democracy
- $\mu - \tau$ Reflection Symmetry (Mutativity)

First needs no introduction. We will illustrate the others.

WLOG, choose to work with Hermitian-square of ν mass matrix (in flavour basis):

$$H_\nu = M_\nu M_\nu^\dagger.$$

(NB. $\frac{H_\nu}{2E}$ is ν-oscillation Hamiltonian).

Democracy

Most striking feature of TBM is (arguably) the trimaximally mixed ν_2 eigenstate:

$$\nu_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Ensured by "Democracy", ie. H_ν commutes with the Democracy operator:

$$[H_\nu, \mathcal{D}] = 0, \text{ where } \mathcal{D} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(hep-ph/0402006, hep-ph/0403278).

The "usual" results of unitary symmetry follow:

- Mass eigenstates are also e-states of $\mathcal{D}$: $\mathcal{D}\nu_2 = \nu_2$; $\mathcal{D}\nu_1 = \mathcal{D}\nu_3 = 0$.
- Democracy Q. No. is conserved in neutrino oscillations.

NB. $\Longrightarrow H_\nu$ is a MAGIC SQUARE - rows and columns all add to same number.

Historical Aside on Democracy

"Democratic" mass matrix has long history (Harari, Haut, Weyers, PLB 78 (1978)). In that ansatz, postulate that mass matrix itself has the democratic form, each of its elements being equal - 1 parameter only

Or each of equal magnitude (L. Lavoura. Phys. Lett. B 228 (1989)).

Fritzsch and Xing, employed small perturbations from this form (PLB 372 (1996), PLB 440 (1999), PRD61 (2000)).

Our approach is distinct - require only that mass matrix commutes with $\mathcal{D}$

$\Longrightarrow$ different consequences (eg. more parameters - realistic phenomenology).

$\boxed{\text{I Emphasise:}}$

Democracy Symmetry $\Longleftrightarrow$ Magic property

$\Longleftrightarrow$ At least one trimaximally mixed mass eigenstate: $\frac{1}{\sqrt{3}}(1, 1, 1)$

THE THREE CONDITIONS ARE ALL EQUIVALENT!

$\boldsymbol{\mu-\tau}$ Reflection Symmetry (Mutativity)

The Mutativity transformation is defined (PFH and WGS, PLB547 (2002) 219) as simultaneous exchange of the μ and τ flavour labels and a CP transformation.

Eg. 1: H_ν (symmetric under Mutativity):

$$\begin{pmatrix} a & b & b^* \\ b^* & d & c \\ b & c^* & d \end{pmatrix} \xrightarrow{\mu-\tau \text{ exchange}} \begin{pmatrix} a & b^* & b \\ b & d & c^* \\ b^* & c & d \end{pmatrix} \xrightarrow{CP} \begin{pmatrix} a & b & b^* \\ b^* & d & c \\ b & c^* & d \end{pmatrix}$$

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Eg. 2: U_{MNS} (symmetric under Mutativity):

$$U_{MNS} = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ v_1^* & v_2^* & v_3^* \end{pmatrix} \xrightarrow{\mu - \tau \text{ exchange}} \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1^* & v_2^* & v_3^* \\ v_1 & v_2 & v_3 \end{pmatrix} \xrightarrow{CP} \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ v_1^* & v_2^* & v_3^* \end{pmatrix}$$

Key feature is $\nu_\mu = \nu_\tau^*$ (More generally, $|\nu_{\mu i}| = |\nu_{\tau i}| \ \forall \ i$).

3 Generalisations of TriBimaximal Mixing

- 2 symmetries: Democracy and CP.

$$U_{MNS} = \begin{pmatrix} \sqrt{\frac{2}{3}}\cos\phi & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}\sin\phi \\ -\frac{\cos\phi}{\sqrt{6}} - \frac{\sin\phi}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{\cos\phi}{\sqrt{2}} - \frac{\sin\phi}{\sqrt{6}} \\ -\frac{\cos\phi}{\sqrt{6}} + \frac{\sin\phi}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{\cos\phi}{\sqrt{2}} - \frac{\sin\phi}{\sqrt{6}} \end{pmatrix}$$

"Tri-ϕ maximal" mixing

(PFH&WGS – hep-ph/0203209)

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- 1 symmetry: just Democracy

$$U_{MNS} = \begin{pmatrix} \frac{2}{\sqrt{6}}C & \frac{1}{\sqrt{3}} & U_{e3} \\ -\frac{1}{\sqrt{6}}C - \frac{\sqrt{3}}{2}U_{e3}^* & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}C - \frac{1}{2}U_{e3} \\ -\frac{1}{\sqrt{6}}C + \frac{\sqrt{3}}{2}U_{e3}^* & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}C - \frac{1}{2}U_{e3} \end{pmatrix}$$

"Magic/S3 Group Mixing"

where $C = \sqrt{1 - \frac{3}{2}|U_{e3}|^2} \simeq 1$ (J.D.Bjorken, PFH and WGS, hep-ph/0511201).

Flavour-Symmetric Mixing Observables

The Jarlskog CP-violating observable has several interesting and useful properties.

As well as parameterisaing CP violation, it is flavour- and mass-label symmetric.

Is also phase-convention independent.

It may be written in terms of the elements of a single plaquette:

$$J = Im(U_{ai}U^*_{aj}U^*_{bi}U_{bj}), \text{ for ANY plaquette, eg. : } \begin{pmatrix} U_{ai} & . & U_{aj} \\ U_{bi} & . & U_{bj} \\ . & . & . \end{pmatrix}$$

It may also be written in terms of the charged lepton (L) and neutrino (N) mass matrices:

$$\begin{aligned} Det([L,N]) &= 2i(m_e - m_\mu)(m_\mu - m_\tau)(m_\tau - m_e)(m_1 - m_2)(m_2 - m_3)(m_3 - m_1)J \\ &= 2iL_\Delta N_\Delta J \end{aligned}$$

J is odd under interchange of charged lepton flavour labels or ν mass labels.

Another Flavour-odd Invariant

First we define the neutrino mixing probability matrix:

$$P \equiv \begin{pmatrix} |U_{e1}|^2 & |U_{e2}|^2 & |U_{e3}|^2 \\ |U_{\mu 1}|^2 & |U_{\mu 2}|^2 & |U_{\mu 3}|^2 \\ |U_{\tau 1}|^2 & |U_{\tau 2}|^2 & |U_{\tau 3}|^2 \end{pmatrix}.$$

In terms of P, can define another flavour-odd invariant which shares some of the interesting properties of J:

$$\mathcal{F} \equiv Det(P) = -(P_{\alpha i} + P_{\beta j} - P_{\beta i} - P_{\alpha j}) + 3(P_{\alpha i}P_{\beta j} - P_{\beta i}P_{\alpha j})$$

(have used unitarity to write it in terms of the elements of a single plaquette.)

NB. Same formula for any plaquette!

$\mathcal{F}$, like J, is observable, odd under flavour- and mass-label interchange, quartic in U_{MNS} and parameterises the deviation from certain symmetries.

Writing $\mathcal{F}$ in Terms of Mass Matrices

$\mathcal{F}$, like J, can also be written in terms of the lepton mass matrices in a weak-basis invariant way.

First define the double mass-moment transform of P (PFH, WGS and Tom Weiler - hep-ph/0607335):

$$T_{mn} \equiv \mathrm{Tr}(L^m N^n) = (\underline{m}_l^m)^T \cdot P \cdot \underline{m}_\nu^n$$

$$\text{where} \qquad (\underline{m}_l^m)^T = (m_e^m, m_\mu^m, m_\tau^m) \;\; \text{and} \;\; (\underline{m}_\nu^n)^T = (m_1^n, m_2^n, m_3^n).$$

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Then

$$Det(T) = L_\Delta N_\Delta \, \mathcal{F}$$

cf.

$$Det([L, N]) = 2i L_\Delta N_\Delta \, J$$

NB. $-1 \leq \mathcal{F} \leq 1$

Democracy and $\mathcal{F}$

Democracy implies that all three elements in a given row (or column) of P are equal to each other (and moreover, $= \frac{1}{3}$) - leaves 2 free parameters:

$$P \equiv |U_{MNS}|^2 = \begin{pmatrix} \frac{2}{3} - |U_{e3}|^2 & \frac{1}{3} & |U_{e3}|^2 \\ \frac{1}{6} + \frac{1}{2}|U_{e3}|^2 - \mu & \frac{1}{3} & \frac{1}{2}(1 - |U_{e3}|^2) + \mu \\ \frac{1}{6} + \frac{1}{2}|U_{e3}|^2 + \mu & \frac{1}{3} & \frac{1}{2}(1 - |U_{e3}|^2) - \mu \end{pmatrix}$$

"Magic/S3 GroupMixing"

$$\begin{aligned} \Longrightarrow \mathcal{F} &= -(P_{\alpha i} + P_{\beta j} - P_{\beta i} - P_{\alpha j}) + 3(P_{\alpha i}P_{\beta j} - P_{\beta i}P_{\alpha j}) \\ &= 0 \quad \forall \text{ plaquettes} \end{aligned}$$

and for any row or any column being trimaximally mixed.

Mutativity and $\mathcal{F}$

$\mu - \tau$ Reflection Symmetry implies that the μ and τ rows of P are equal to each other (leaves 2 free parameters):

$$P \equiv |U_{MNS}|^2 = \begin{pmatrix} \frac{2}{3} - \epsilon - |U_{e3}|^2 & \frac{1}{3} + \epsilon & |U_{e3}|^2 \\ \frac{1}{6} + \frac{1}{2}(\epsilon + |U_{e3}|^2) & \frac{1}{3} - \frac{1}{2}\epsilon & \frac{1}{2}(1 - |U_{e3}|^2) \\ \frac{1}{6} + \frac{1}{2}(\epsilon + |U_{e3}|^2) & \frac{1}{3} - \frac{1}{2}\epsilon & \frac{1}{2}(1 - |U_{e3}|^2) \end{pmatrix}$$

$$\Longrightarrow \mathcal{F} \equiv Det(P) = 0$$

True, whichever pair of rows (or columns) is equal.

So, both Democracy and $\mu - \tau$ reflection symmetry lead to $\mathcal{F} = 0$.

Violation of either symmetry leads to $\mathcal{F} \neq 0$.

We find for the data $|F| \lesssim 0.15$ at 90% CL.

Another (Fun) Way to Write $\mathcal{F}$ in Terms of Mass Matrices

By analogy with the Jarlskog commutator, define generalised mass matrix commutator: $C_{mn} = [L^m, N^n]$.

Jarlskog Determinant may be written $Det(C_{11}) = \frac{1}{3}Tr(C_{11}^3)$.

Define generalised cubic commutator trace: $Tr(C_{ij}C_{kl}C_{mn})$ (hep-ph/0607335).

Those with repeated C_{ij} may be labelled (only) by their powers in L and N.

Can form a 3x3 matrix for powers ranging from 3-5.

$$M \equiv \begin{pmatrix} Tr(C_{11}^3) & Tr(C_{11}^2 C_{12}) & Tr(C_{11} C_{12}^2) \\ Tr(C_{11}^2 C_{21}) & Tr(C_{11}^2 C_{22}) & . \\ . & . & \text{etc.} \end{pmatrix}$$

Then

$$\begin{aligned} Det(M) &= -8iL_\Delta^4 N_\Delta^4\, J^3\, \mathcal{F} \\ \Longrightarrow\ L_\Delta N_\Delta\, \mathcal{F} &= \frac{Det(M)}{Det^3(C_{11})} \end{aligned}$$

Conclusions

- TBM ansatz gaining in popularity. A number of models proposed which "explain" why it could be so (see eg. E. Ma talk).
- Defined by 3 symmetries, it allows no flexibility in the mixing, in particular, no CP violation. We should consider its generalisations.
- Can abandon any of the symmetries singly or in pairs to give viable phenomenology.
- Here, we focussed on mixings with just Democracy or Mutativity.
- Have defined a new flavour- and mass-index-odd, observable quantity, $\mathcal{F}$, with some properties in common with J.
- Each of Democracy and Mutativity separately $\implies$ $\mathcal{F} = 0$.