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# Notes on Deviations from TBM

Paul Harrison

**Abstract**

Just some notes as an aide memoir, for the time being

## 1 Exact Basic Formulae

Unitarity may be used to write the observables of neutrino oscillations without using the elements of the first column of the MNS matrix. Neutrino oscillation disappearance probabilities are given by:

$$\begin{aligned}\frac{1}{4}(1 - P(\nu_\alpha \to \nu_\alpha)) &= |U_{\alpha 3}|^2(1 - |U_{\alpha 3}|^2)\sin^2\Delta_{31} + |U_{\alpha 2}|^2(1 - |U_{\alpha 2}|^2)\sin^2\Delta_{21} \\ &- 2|U_{\alpha 3}|^2|U_{\alpha 2}|^2 \sin\Delta_{31}\sin\Delta_{21}\cos\Delta_{32}. \qquad (1)\end{aligned}$$

where

$$\Delta_{ij} \equiv \frac{\Delta m^2_{ij} L}{4E}. \qquad (2)$$

Neutrino oscillation appearance probabilities are given by:

$$\begin{aligned}\frac{1}{4}P(\nu_\alpha \to \nu_\beta) &= |U_{\alpha 3}|^2|U_{\beta 3}|^2\sin^2\Delta_{31} + |U_{\alpha 2}|^2|U_{\beta 2}|^2\sin^2\Delta_{21} \\ &+ 2\,(K^{32}_{\alpha\beta}\cos\Delta_{32} \pm J\sin\Delta_{32})\sin\Delta_{31}\sin\Delta_{21}, \ \alpha \neq \beta \qquad (3)\end{aligned}$$

where $K^{32}_{\alpha\beta} = \mathrm{Re}(U_{\alpha 3}U^*_{\beta 3}U^*_{\alpha 2}U_{\beta 2})$.

## 2 Expansion of the MNS Matrix Around TBM

Let the PMNS matrix be given by:

$$U = P_0 O'_{23} O'_{12} P_{\delta'} O'_{13} P^{\dagger}_{\delta'} \tag{4}$$

where the $O'_{ij}$ are orthogonal rotations in the $ij$ plane, through, in general, different rotation angles (for the same physical mixing), than the standard parameterisation, owing to the non-commutativity of the rotation matrices, $P_{\delta'} = \mathrm{diag}(1, 1, \mathrm{e}^{i\delta'})$ and $P_0 = \mathrm{diag}(1, 1, -1)$. Then the tri-bimaximal mixing matrix is given by:

$$\sin\theta'_{12} = 1/\sqrt{3}, \quad \sin\theta'_{23} = 1/\sqrt{2}, \tag{5}$$

$$\sin\theta'_{13}\mathrm{e}^{-i\delta'} = 0 \tag{6}$$

where the two constraints of Eq. 5 are sufficient to ensure democracy, and a trimaximally mixed $\nu_2$ mass eigenstate, even in the more general case that $\theta'_{13}$ takes a more general value.

Given that the data are consistent with the TBM form, we will be interested in how best to establish deviations from it. We therefore expand in four small quantities around these particular values by instead letting:

$$\theta'_{12} = \sin^{-1}(1/\sqrt{3}) + \sqrt{2}\epsilon, \quad \theta'_{23} = \sin^{-1}(1/\sqrt{2}) + \mu, \tag{7}$$

$$\sin\theta'_{13}\mathrm{e}^{-i\delta'} = U_{e3}\,/\cos\theta'_{12}, \tag{8}$$

so that eg.

$$\begin{aligned}
\sin\theta'_{12} &= \frac{1}{\sqrt{3}}[\cos(\sqrt{2}\epsilon) + \sqrt{2}\sin(\sqrt{2}\epsilon)] && (9)\\
&\simeq \frac{1}{\sqrt{3}}[1 + 2\epsilon - \epsilon^2] && (10)\\
\sin\theta'_{23} &= \frac{1}{\sqrt{2}}[\cos\mu + \sin\mu] && (11)\\
&\simeq \frac{1}{\sqrt{2}}[1 + \mu - \frac{1}{2}\mu^2] && (12)
\end{aligned}$$

to second order in small quantities. Then (dropping primes)

$$U = \begin{pmatrix} c_{13}c_{12} & s_{12} & U_{e3} \\ -s_{12}c_{23}c_{13} - \frac{s_{23}}{c_{12}}U^*_{e3} & c_{12}c_{23} & c_{13}s_{23} - t_{12}c_{23}U_{e3} \\ -s_{12}s_{23}c_{13} + \frac{c_{23}}{c_{12}}U^*_{e3} & c_{12}s_{23} & -c_{13}c_{23} - t_{12}s_{23}U_{e3} \end{pmatrix} \tag{13}$$

which is unitary, by construction. We note here that when $\epsilon = \mu = 0$, we recover "S3 group mixing", ie. democracy is respected, with $\mu$-$\tau$ reflection symmetry broken by $\mathrm{Re}(U_{e3}) \equiv \rho$ and $CP$ symmetry broken by $\mathrm{Im}(U_{e3}) \equiv \eta$ (where we have defined abbreviated names for these parameters - inspired by the quark case). Clearly, $\epsilon \neq 0$ and $\mu \neq 0$ each violate democracy, as expected. Additionally, $\mu \neq 0$ violates $\mu$-$\tau$ reflection symmetry (also as expected, given that $\sin\theta'_{12} = \frac{1}{\sqrt{3}}$ has already been recognised as a constraint of this symmetry), while $\epsilon$ respects it. Table 1 summarises which parameters respect which of the phenomenological symmetries which define tri-bimaximal mixing.

| Parameter | $CP$ | Democracy | $\mu$-$\tau$ Reflection |
|---|---|---|---|
| $\rho$ | Y | Y | N |
| $\eta$ | N | Y | Y |
| $\epsilon$ | Y | N | Y |
| $\mu$ | Y | N | N |

Table 1: Table indicating which of our four mixing parameters respect which of the three phenomenological symmetries which define tri-bimaximal mixing.

Expanding $U$ to second order:

$$
\begin{aligned}
U \simeq & \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{(zeroth order)} \\
+ & \begin{pmatrix} \cdot & \frac{1}{\sqrt{3}}\quad(2\epsilon) & U_{e3} \\ \cdot & \frac{1}{\sqrt{3}}(-\epsilon-\mu) & +\frac{1}{\sqrt{2}}(+\mu-\frac{U_{e3}}{\sqrt{2}}) \\ \cdot & \frac{1}{\sqrt{3}}(-\epsilon+\mu) & -\frac{1}{\sqrt{2}}(-\mu+\frac{U_{e3}}{\sqrt{2}}) \end{pmatrix} \quad \text{(first order)} \\
+ & \begin{pmatrix} \cdot & \frac{1}{\sqrt{3}}\quad(-\epsilon^2) & 0 \\ \cdot & \frac{1}{\sqrt{3}}(-\epsilon^2-\frac{1}{2}\mu^2+\epsilon\mu) & +\frac{1}{\sqrt{2}}(-\frac{3}{4}|U_{e3}|^2-\frac{1}{2}\mu^2-\frac{U_{e3}}{\sqrt{2}}(3\epsilon-\mu)) \\ \cdot & \frac{1}{\sqrt{3}}(-\epsilon^2-\frac{1}{2}\mu^2-\epsilon\mu) & -\frac{1}{\sqrt{2}}(-\frac{3}{4}|U_{e3}|^2-\frac{1}{2}\mu^2+\frac{U_{e3}}{\sqrt{2}}(3\epsilon+\mu)) \end{pmatrix} \text{(etc.)} \qquad (14)
\end{aligned}
$$

where the roles of the parameters in breaking the various symmetries of TBM are more explicit.

# 3 Useful Intermediate Results

We note from Eqs. 3 and 1, that we will need only six different $|U|^2$ values, three $K$-matrix entries and the value of $J$. Here, we give the expansions of these parameters to second order in small quantities:

$$|U_{e2}|^2 = \frac{1}{3}(1+4\epsilon+2\epsilon^2) \qquad (15)$$

$$|U_{\mu 2}|^2 = \frac{1}{3}[1-2(\epsilon+\mu)-\epsilon^2+4\epsilon\mu] \qquad (16)$$

$$
\begin{aligned}
|U_{\tau 2}|^2 &= \frac{1}{3}[1 - 2(\epsilon - \mu) - \epsilon^2 - 4\epsilon\mu] &(17)\\
|U_{e3}|^2 &= \rho^2 + \eta^2 &(18)\\
|U_{\mu 3}|^2 &= \frac{1}{2}[1 + 2\mu - \sqrt{2}\rho - \rho^2 - \eta^2 - 3\sqrt{2}\rho\epsilon] &(19)\\
|U_{\tau 3}|^2 &= \frac{1}{2}[1 - 2\mu + \sqrt{2}\rho - \rho^2 - \eta^2 + 3\sqrt{2}\rho\epsilon] &(20)\\
K^{32}_{\mu e} &= \frac{+\rho}{3\sqrt{2}}(1+\epsilon) - \frac{1}{6}(\rho^2 + \eta^2) &(21)\\
K^{32}_{\tau e} &= \frac{-\rho}{3\sqrt{2}}(1+\epsilon) - \frac{1}{6}(\rho^2 + \eta^2) &(22)\\
K^{32}_{\mu\tau} &= -\frac{1}{6}[1 - 2\epsilon - \epsilon^2 - 4\mu^2 + 2\sqrt{2}\mu\rho - 2(\rho^2 + \eta^2))] &(23)\\
J &= \frac{\eta}{3\sqrt{2}}(1+\epsilon) &(24)
\end{aligned}
$$

Again, the breaking of $\mu$-$\tau$ symmetry by $\rho$ and $\mu$ is explicit. As cross-checks, it can be verified that the unitarity conditions, $|U_{e2}|^2 + |U_{\mu 2}|^2 + |U_{\tau 2}|^2 = 1$ and $|U_{e3}|^2 + |U_{\mu 3}|^2 + |U_{\tau 3}|^2 = 1$, are satisfied, as well as $K^{32}_{e\tau} + K^{32}_{\mu\tau} = -|U_{\tau 3}|^2 |U_{\tau 2}|^2$, etc.

We are going to expand the observable transition probabilities to second order in small quantities. While we have defined four small mixing parameters which we would like to determine, we will also define two other small quantities associated with the observables. For most current and next-generation (man-made) neutrino beams (other than the KAMLAND and similar very long baseline reactor experiments, for which a special case will need to be made), $\Delta_{21} \ll 1$ so that $\sin\Delta_{21} \simeq \Delta_{21} \ll 1$. We will handle this by writing

$$\Delta_{21} = h\,\Delta \tag{25}$$

where

$$\Delta \equiv < \Delta m^2_{31} > L/4E, \tag{26}$$

with $< \Delta m^2_{31} >$ the (known) World average value of $\Delta m^2_{31}$, and

$$h \equiv \frac{\Delta m^2_{21}}{< \Delta m^2_{31} >} \simeq 0.03 \tag{27}$$

describes the hierarchy of mass-squared differences, considered a small (known) parameter.

There is also the small uncertainty in the measured value of the atmospheric mass-squared difference, $\Delta m^2_{31} - < \Delta m^2_{31} >$, which we describe with the small dimensionless parameter:

$$\delta \equiv \frac{\Delta m^2_{31}}{< \Delta m^2_{31} >} - 1. \tag{28}$$

Hence

$$\Delta_{31} = (1+\delta)\,\Delta \tag{29}$$

and our parameters describing the $L/E$ dependence will be the (exactly known) $\Delta$ and the two small expansion parameters:

$$h\,\Delta,\ \ \delta\,\Delta << 1. \tag{30}$$

Then, to second order in small quantities, we have for the trigonometric functions of compound angles in Equations 1 and 3:

$$\begin{aligned} \sin\Delta_{31} &= S + \delta\,\Delta C - \frac{1}{2}(\delta\,\Delta)^2 S && (31)\\ \sin^2\Delta_{31} &= S^2 + 2\,\delta\,\Delta SC + (\delta\,\Delta)^2(1-2S^2) && (32) \end{aligned}$$

where $S = \sin\Delta$ and $C = \cos\Delta$, and noting that $\Delta_{32} = \Delta_{31} - \Delta_{21} = \Delta + (\delta - h)\Delta$, we have:

$$\begin{aligned} \sin\Delta_{32} &= S + (\delta-h)\Delta C - \frac{1}{2}(\delta-h)^2\Delta^2 S && (33)\\ \cos\Delta_{32} &= C - (\delta-h)\Delta S - \frac{1}{2}(\delta-h)^2\Delta^2 C. && (34) \end{aligned}$$

# 4 Expansion of Transition Probabilities to Second Order in $h\,\Delta$ and $\delta\,\Delta$

Inserting the expansions from Eqs. 31-34 in the formula, Eq. 1, for the general disappearance probability, and keeping terms to second order in $h\,\Delta$ and $\delta\,\Delta$ gives:

$$\begin{aligned} \frac{1}{4}(1 - P(\nu_\alpha \to \nu_\alpha)) &= S^2|U_{\alpha 3}|^2(1-|U_{\alpha 3}|^2)\\ &+ 2\Delta SC|U_{\alpha 3}|^2[\,\delta(1-|U_{\alpha 3}|^2) - h|U_{\alpha 2}|^2]\\ &+ \Delta^2\{\,\delta^2(1-2S^2)|U_{\alpha 3}|^2(1-|U_{\alpha 3}|^2) + h^2|U_{\alpha 2}|^2(1-|U_{\alpha 2}|^2)\\ &- \quad 2h|U_{\alpha 3}|^2|U_{\alpha 2}|^2[\delta - (2\delta - h)S^2]\,\} && (35) \end{aligned}$$

Inserting the expansions from Eqs. 31-34 in the formula, Eq. 3, for the general appearance probability, and keeping terms to second order in $h\,\Delta$ and $\delta\,\Delta$ gives:

$$\begin{aligned} \frac{1}{4}P(\nu_\alpha \to \nu_\beta) &= S^2|U_{\alpha 3}|^2|U_{\beta 3}|^2\\ &+ 2\Delta S[\,\delta|U_{\alpha 3}|^2|U_{\beta 3}|^2 C + h(K^{32}_{\alpha\beta}C + JS)\,]\\ &+ \Delta^2\{\,\delta^2(1-2S^2)|U_{\alpha 3}|^2|U_{\beta 3}|^2 + h^2|U_{\alpha 2}|^2|U_{\beta 2}|^2\\ &+ \quad 2h[\delta K^{32}_{\alpha\beta} - (2\delta - h)S(K^{32}_{\alpha\beta}S - JC)]\,\} && (36) \end{aligned}$$

# 5 Expansion of Observables in Small Parameters

While a high intensity $\nu_\tau$ beam is a practical impossibility, beams of $\nu_e$ and $\nu_\mu$ and their anti-particles are possible, and it is possible, in principle, to detect all three flavoured final states. Hence, we are led to consider the six possible combinations for $P(\nu_\alpha \to \nu_\beta)$ with $\alpha \neq \tau$, and their anti-particle conjugates. Given that time-reversed and $CP$-conjugate probabilities are simply given by inverting the sign of $J$ (in the absence of matter effects), only 5 need to be considered: $P(\nu_e \to \nu_e)$, $P(\nu_e \to \nu_\mu)$, $P(\nu_e \to \nu_\tau)$, $P(\nu_\mu \to \nu_\mu)$, and $P(\nu_\mu \to \nu_\tau)$.

We will of course parameterise in terms of our four small mixing parameters: $\epsilon$, $\mu$, $\rho$ and $\eta$, and also $h\,\Delta$ and $\delta\,\Delta$. Deciding how to present the formulae presents us with a choice. The formulae in Eqs. **??** and **??** above have been organised in terms of their $L/E$ dependence. However, on multiplying out the dependence on mixing parameters, these terms will be of mixed order and there is the option below to present them in terms of strictly increasing powers in small quantities. I have taken the latter option. While this leads to a somewhat more cluttered appearance to the formulae, it makes the extraction of what is actually measurable, perhaps, a little clearer.

## 5.1 $P(\nu_e \to \nu_e)$

In this case, we have retained sufficient information to determine this quantity to third order. We start by giving the factors dependent on the MNS matrix elements, to the order required in each case (which is determined by a comparison of powers of small quantities between the MNS and the L/E-dependent factors):

$$|U_{e3}|^2(1-|U_{e3}|^2) = \rho^2+\eta^2 \quad \text{(3rd order)} \tag{37}$$

$$|U_{e2}|^2|U_{e3}|^2 = \frac{1}{3}(\rho^2+\eta^2) \quad \text{(2nd order)} \tag{38}$$

$$|U_{e2}|^2(1-|U_{e2}|^2) = \frac{2}{9}(1+2\epsilon) \quad \text{(1st order)} \tag{39}$$

Substituting in Eq. 35 gives:

$$\begin{aligned}\frac{1}{4}(1-P(\nu_e\to\nu_e)) &= S^2(\rho^2+\eta^2)+\frac{2}{9}\Delta^2h^2 && \text{(2nd order)}\\ &+ 2\Delta SC(\rho^2+\eta^2)(\,\delta-\frac{1}{3}h)+\frac{4}{9}\Delta^2h^2\epsilon && \text{(3rd order)}\end{aligned} \tag{40}$$

### 5.2 $P(\nu_\mu \to \nu_\mu)$

$$
\begin{aligned}
|U_{\mu 3}|^2(1-|U_{\mu 3}|^2) &= \frac{1}{4}[1-(2\mu-\sqrt{2}\rho)^2+2(2\mu-\sqrt{2}\rho)(\rho^2+\eta^2+3\sqrt{2}\rho\epsilon)] \text{ (3rd)} && (41)\\
|U_{\mu 2}|^2|U_{\mu 3}|^2 &= \frac{1}{6}[1-2\epsilon-\sqrt{2}\rho-\epsilon^2-4\mu^2-\rho^2-\eta^2+\sqrt{2}\rho(2\mu-\epsilon)] \text{ (2nd)} && (42)\\
|U_{\mu 2}|^2(1-|U_{\mu 2}|^2) &= \frac{2}{9}(1-\epsilon-\mu) \quad \text{(1st order)} && (43)
\end{aligned}
$$

Even though we have calculated the mixing factors to 3rd order, since we calculated the $L/E$ dependence only to 2nd order (and since moreover, the mixing factors have terms of zeroth order here), we have only enough information to determine $P(\nu_\mu \to \nu_\mu)$ overall to second order. This should not be much of a problem, given that experiment is unlikely to probe third-order for a very long time (still only sensitive at zeroth order today). Extending to third order is anyway straightforward, if required. Substituting in Eq. 35 gives:

$$
\begin{aligned}
\frac{1}{4}(1-P(\nu_\mu \to \nu_\mu)) &= \frac{1}{4}S^2+\Delta SC(\frac{\delta}{2}-\frac{h}{3}) && \text{(0th \& 1st order)}\\
&- S^2(\mu-\frac{\rho}{\sqrt{2}})^2+\frac{2}{3}\Delta SCh(\epsilon+\frac{\rho}{\sqrt{2}})\\
&+ \Delta^2[(1-2S^2)\delta(\frac{\delta}{4}-\frac{h}{3})+(2-3S^2)\frac{h^2}{9}] && \text{(2nd order)} \quad (44)
\end{aligned}
$$

Remembering that $S$ (actually, only $< \Delta m^2_{31} >$) is known exactly and that $h$ is known to a reasonable approximation allows us to see that measuring $P(\nu_\mu \to \nu_\mu)$ and subtracting known quantities measures $\delta$ at lowest order. This is not particularly surprising, but was certainly not obvious a priori (at least to me). The point here is that other unknowns $(\mu, \rho, \epsilon)$ first enter at higher (second) order, allowing $\delta$ to be extracted relatiely cleanly in precision measurements. Another feature is that $\eta$ is completely absent.

### 5.3 $P(\nu_e \to \nu_\mu)$

We note that this is the same as its time-reversed conjugate, apart from the sign of $\eta$.

$$
\begin{aligned}
|U_{e3}|^2|U_{\mu 3}|^2 &= \frac{1}{2}(\rho^2+\eta^2)(1+2\mu-\sqrt{2}\rho) && \text{(3rd)} \quad (45)\\
K^{32}_{e\mu} &= \frac{+\rho}{3\sqrt{2}}(1+\epsilon) - \frac{1}{6}(\rho^2+\eta^2) && \text{(2nd)} \quad (46)\\
J &= \frac{\eta}{3\sqrt{2}}(1+\epsilon) && \text{(2nd)} \quad (47)\\
|U_{e2}|^2|U_{\mu 2}|^2 &= \frac{1}{9}[1+2(\epsilon-\mu)] && \text{(1st)} \quad (48)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{4}P(\nu_e \to \nu_\mu) &\simeq \frac{1}{2}S^2(\rho^2+\eta^2)(1+2\mu-\sqrt{2}\rho)\\
&+ \Delta S[(\delta-\frac{h}{3})(\rho^2+\eta^2)C + \frac{\sqrt{2}h}{3}(1+\epsilon)(\rho C+\eta S)]\\
&+ \Delta^2\frac{h}{3}\{\frac{h}{3}[1+2(\epsilon-\mu)] + \sqrt{2}[\delta\rho-(2\delta-h)S(\rho S-\eta C)]\} \quad (49)\\
&= \frac{1}{2}S^2(\rho^2+\eta^2) + \Delta S\frac{\sqrt{2}h}{3}(\rho C+\eta S) + \Delta^2\frac{h^2}{9} \quad \text{(2nd order)}\\
&+ S^2(\rho^2+\eta^2)(\mu-\frac{\rho}{\sqrt{2}})\\
&+ \Delta S[(\delta-\frac{h}{3})(\rho^2+\eta^2)C + \frac{\sqrt{2}h\epsilon}{3}(\rho C+\eta S)]\\
&+ \Delta^2\frac{\sqrt{2}h}{3}[\frac{\sqrt{2}h}{3}(\epsilon-\mu)+\delta\rho-(2\delta-h)S(\rho S-\eta C)] \quad \text{(3rd order)} \quad (50)
\end{aligned}
$$

### 5.4 $P(\nu_e \to \nu_\tau)$

This can be written down immediately from the formula for $P(\nu_e \to \nu_\mu)$, making the substitutions (see eg. Eqs. 15-24) $\mu \to -\mu$, $\rho \to -\rho$ and $\eta \to -\eta$ ($\mu$-$\tau$ reflection includes a $CP$ transformation):

$$
\begin{aligned}
\frac{1}{4}P(\nu_e \to \nu_\tau) &\simeq \frac{1}{2}S^2(\rho^2+\eta^2)(1-2\mu+\sqrt{2}\rho) \\
&+ \Delta S[(\delta - \frac{h}{3})(\rho^2+\eta^2)C - \frac{\sqrt{2}h}{3}(1+\epsilon)(\rho C+\eta S)] \\
&+ \Delta^2\frac{h}{3}\{\frac{h}{3}[1+2(\epsilon+\mu)] - \sqrt{2}[\delta\rho-(2\delta-h)S(\rho S-\eta C)]\} \qquad (51) \\
&= \frac{1}{2}S^2(\rho^2+\eta^2) - \Delta S\frac{\sqrt{2}h}{3}(\rho C+\eta S) + \Delta^2\frac{h^2}{9} \qquad \text{(2nd order)} \\
&- S^2(\rho^2+\eta^2)(\mu - \frac{\rho}{\sqrt{2}}) \\
&+ \Delta S[(\delta-\frac{h}{3})(\rho^2+\eta^2)C - \frac{\sqrt{2}h\epsilon}{3}(\rho C+\eta S)] \\
&+ \Delta^2\frac{\sqrt{2}h}{3}[\frac{\sqrt{2}h}{3}(\epsilon+\mu) - \delta\rho + (2\delta-h)S(\rho S-\eta C)] \quad \text{(3rd order)} \qquad (52)
\end{aligned}
$$

A significant sanity check is available at this point. The sum $P(\nu_e \to \nu_e) + P(\nu_e \to \nu_\mu) + P(\nu_e \to \nu_\tau) = 1$ should hold, as indeed, it does when substituting from Eqs. 40, 50 and 52.

### 5.5 $P(\nu_\mu \to \nu_\tau)$

This can be written down immediately from the formulae for $P(\nu_\mu \to \nu_\mu)$, and $P(\nu_\mu \to \nu_e)$, using unitarity $(P(\nu_\mu \to \nu_e) + P(\nu_\mu \to \nu_\mu) + P(\nu_\mu \to \nu_\tau) = 1)$, where in turn, $P(\nu_\mu \to \nu_e)$ can be obtained from $P(\nu_e \to \nu_\mu)$ on making the substitution $\eta \to -\eta$.

$$\begin{aligned}
\frac{1}{4}(P(\nu_\mu \to \nu_\tau)) &= \frac{1}{4}S^2 + \Delta SC(\frac{\delta}{2} - \frac{h}{3}) && \text{(0th \& 1st order)} \\
&- S^2[(\mu - \frac{\rho}{\sqrt{2}})^2 + \frac{1}{2}(\rho^2 + \eta^2)] + \frac{2}{3}\Delta Sh(\epsilon C + \frac{\eta}{\sqrt{2}}S) \\
&+ \Delta^2[(1-2S^2)\delta(\frac{\delta}{4} - \frac{h}{3}) + (1-3S^2)\frac{h^2}{9}] && \text{(2nd order)} \qquad (53)
\end{aligned}$$

## 6 Summary of Results

$$
\begin{aligned}
\frac{1}{4}(1-P(\nu_e\to\nu_e)) &= S^2(\rho^2+\eta^2)+\frac{2}{9}\Delta^2h^2 && \text{(2nd order)}\\
&+ 2\Delta SC(\rho^2+\eta^2)(\delta-\frac{1}{3}h)+\frac{4}{9}\Delta^2h^2\epsilon && \text{(3rd order)}\\
\frac{1}{4}(1-P(\nu_\mu\to\nu_\mu)) &= \frac{1}{4}S^2+\Delta SC(\frac{\delta}{2}-\frac{h}{3}) && \text{(0th \& 1st order)}\\
&- S^2(\mu-\frac{\rho}{\sqrt{2}})^2+\frac{2}{3}\Delta SCh(\epsilon+\frac{\rho}{\sqrt{2}})\\
&+ \Delta^2[(1-2S^2)\delta(\frac{\delta}{4}-\frac{h}{3})+(2-3S^2)\frac{h^2}{9}] && \text{(2nd order)}\\
\frac{1}{4}P(\nu_e\to\nu_\mu) &= \frac{1}{2}S^2(\rho^2+\eta^2)+\Delta S\frac{\sqrt{2}h}{3}(\rho C+\eta S)+\Delta^2\frac{h^2}{9} && \text{(2nd order)}\\
&+ S^2(\rho^2+\eta^2)(\mu-\frac{\rho}{\sqrt{2}})\\
&+ \Delta S[(\delta-\frac{h}{3})(\rho^2+\eta^2)C+\frac{\sqrt{2}h\epsilon}{3}(\rho C+\eta S)]\\
&+ \Delta^2\frac{\sqrt{2}h}{3}[\frac{\sqrt{2}h}{3}(\epsilon-\mu)+\delta\rho-(2\delta-h)S(\rho S-\eta C)] && \text{(3rd order)}\\
\frac{1}{4}P(\nu_e\to\nu_\tau) &= \frac{1}{2}S^2(\rho^2+\eta^2)-\Delta S\frac{\sqrt{2}h}{3}(\rho C+\eta S)+\Delta^2\frac{h^2}{9} && \text{(2nd order)}\\
&- S^2(\rho^2+\eta^2)(\mu-\frac{\rho}{\sqrt{2}})\\
&+ \Delta S[(\delta-\frac{h}{3})(\rho^2+\eta^2)C-\frac{\sqrt{2}h\epsilon}{3}(\rho C+\eta S)]\\
&+ \Delta^2\frac{\sqrt{2}h}{3}[\frac{\sqrt{2}h}{3}(\epsilon+\mu)-\delta\rho+(2\delta-h)S(\rho S-\eta C)] && \text{(3rd order)}\\
\frac{1}{4}(P(\nu_\mu\to\nu_\tau)) &= \frac{1}{4}S^2+\Delta SC(\frac{\delta}{2}-\frac{h}{3}) && \text{(0th \& 1st order)}\\
&- S^2[(\mu-\frac{\rho}{\sqrt{2}})^2+\frac{1}{2}(\rho^2+\eta^2)]+\frac{2}{3}\Delta Sh(\epsilon C+\frac{\eta}{\sqrt{2}}S)\\
&+ \Delta^2[(1-2S^2)\delta(\frac{\delta}{4}-\frac{h}{3})+(1-3S^2)\frac{h^2}{9}] && \text{(2nd order)}
\end{aligned}
$$

# 7 Discussion

Need first to discuss $\rho$ and $\eta$.

As already discussed, one of the first things to come-out of the study is that there is no better way to determine $\delta$, other than precision measurements of $P(\nu_\mu \to \nu_\mu)$. It is therefore good that MINOS should be able to do this quite well. It would be useful if eg. T2K could do even better. While on the sublect of $\delta$, we see that it is important in the leading corrections to the extraction of $\rho^2 + \eta^2\,(\equiv |U_{e3}|^2)$ from the measurements of $P(\nu_e \to \nu_e)$ and $P(\nu_\mu \to \nu_e)$, although the different $L/E$-dependence of this correction ($\propto \Delta SC$, compared with $S^2$) would help here, and this makes a strong case for the measurement of the energy-dependence in long baseline neutrino oscillation experiments at eg. T2K and NOVA. Finally, $\delta$ almost always occurs in conjunction with $h$, with some fractional relative coefficient.

Another feature to emerge is the difficulty of measuring $\epsilon$. It never appears in leading order (this is, essentially by construction) and when it does enter, it is always weighted by at least one power of the hierarchy parameter, $h \simeq 0.03$.

Another feature is the difficulty of measuring $\mu$. It only appears in probabilities where either initial or final state is $\mu$ or $\tau$. It never appears in leading order (this is, essentially by construction) and when it does enter, it is nearly always in the combination $(\mu - \rho/\sqrt{2})$. In the cases where this is not true, it is weighted by two powers of the hierarchy parameter, $\propto h^2 \simeq 0.001$. The fact that it usually enters as a linear combination with $\rho$ should not be too surprising, as both are are known to violate $\mu$-$\tau$ symmetry. The best way to measure $\mu$ would, apparently, be to compare $P(\nu_e \to \nu_\mu)$ with $P(\nu_e \to \nu_\tau)$ at, say, a beta beam facility and subtract the (hopefully known by then) value of $\rho/\sqrt{2}$. An alternative possibility might involve comparing the rates of cosmogenic $\mu$ and $\tau$ neutrinos, but that is beyond the scope of this note.

# 8 Other Observables

The above expansion for $\Delta_{21} \ll 1$ is not relevant for the KAMLAND experiment, which operates at significantly larger values of $L/E$. It also does not take into account matter effects, which can safely be ignored for most reactor experiments, and for T2K and MINOS. However, for NOVA and a neutrino factory, these need to be taken into account. The LMA MSW effect in the sun is also not accounted for, and this is a significant drawback. Finally, some experments, such as atmospheric neutrinos measure combinations of the above observables, and are, moreover sensitive to matter effects, at least to some extent.

We start with a brief discussion of the LMA MSW effect.

## 8.1 Solar Neutrinos and the LMA MSW Effect

The large mixing angle (LMA) MSW Effect has become the conventional explanation of the solar neutrino deficit. In this interpretation of the data, the density of electrons near the core of the sun distorts the neutrino Hamiltonian locally such that the electron neutrino is essentially the $\nu_2$ neutrino mass eigenstate (more precisely, it is the neutrino with the larger mass-squared of the closely spaced pair in mass-squared). Moreover, the Sun's density varies slowly enough on the neutrino's journey out of its core to the vacuum outside, that the Hamiltonian changes slowly enough for the neutrino wavefunction to remain close to the $\nu_2$ mass eigenstate, until it exits the sun as a (vacuum) $\nu_2$ mass eigenstate. What is measured in the charged current interaction, is of course, an electron flavour eigenstate, so that the measurement projects out the electron-flavoured component from the mass eigenstate. So, in this idealised picture, the measured suppression, $P(\nu_\mu \to \nu_\mu)$, is exactly $|U_{e3}|^2$. Of course, this is only an approximation, and there are corrections, which lie outside the scope of this note, but are probably at only the few percent level. We can see from Eq. 14 that, apart from the approximations inherent in the LMA interpretation, the suppression is

$$|U_{e3}|^2 = \frac{1}{3}(1 + 4\epsilon) \tag{54}$$

to first order. This shows that solar neutrino experiments are perhaps the best method of measureing $\epsilon$ (which is even amplified in this approach, by a factor of 4).

The suppression obtained in the previous paragraph allows us to get the current bound on $\epsilon$. From the SNO experiment, we have that:

$$|U_{e3}|^2 = 0.31 \pm 0.07 = 0.31(1 \pm 0.23) \quad (90\% \text{ CL}) \tag{55}$$

from which we obtain $-0.070 \lesssim \epsilon \lesssim 0.035$ at 90% CL.

### 8.2 Atmospheric Neutrinos

The two observables which can be straightforwardly measured using atmospheric neutrinos are the rates of muon neutrinos and of electron neutrinos normalised to their unoscillated rates,averaged over both signs of flavour. These may be written

$$''P''_{\mu\mu} \simeq \frac{1}{2}[P(\nu_\mu \to \nu_\mu) + \frac{1}{2}P(\nu_e \to \nu_\mu) + P(\overline{\nu}_\mu \to \overline{\nu}_\mu) + \frac{1}{2}P(\overline{\nu}_e \to \overline{\nu}_\mu)] \quad (56)$$

$$\begin{aligned} &= 1 - S^2 - 2\Delta SC(\delta - \frac{2h}{3}) \qquad \text{(0th \& 1st order)} \\ &+ 4S^2(\mu - \frac{\rho}{\sqrt{2}})^2 - \frac{8}{3}\Delta SCh(\epsilon + \frac{\rho}{\sqrt{2}}) \\ &- \Delta^2[(1-2S^2)\delta(\delta - \frac{4h}{3}) + (2-3S^2)\frac{4h^2}{9}] \quad \text{(2nd order)} \end{aligned} \quad (57)$$

### 8.3 KAMLAND and Very Long Baseline Reactor Experiments

# 9 Original Stuff

Using Eq. 3, to next-to-leading (third) order in small quantities, the electron neutrino appearance probability in a long baseline muon neutrino beam is (ignoring matter effects) is:

$$
\begin{aligned}
P(\nu_\mu \to \nu_e) &\simeq 2|U_{e3}|^2(1+2\mu-\sqrt{2}\rho)\sin^2\Delta_{31} + \frac{4}{9}(1+2\epsilon-2\mu)h^2\Delta_{31}^2 \\
&+ \frac{8h}{3\sqrt{2}}\Delta_{31}\sin\Delta_{31}[(1+\epsilon)(\rho\cos\Delta_{31}+\eta\sin\Delta_{31}) \\
&+ h\Delta_{31}(\rho\sin\Delta_{31}-\eta\cos\Delta_{31}) - \frac{1}{\sqrt{2}}(\rho^2+\eta^2)\cos\Delta_{31}]. \quad (58)
\end{aligned}
$$

We note, in particular, that $\epsilon$ and $\mu$ do not enter at leading order, vindicating the use of the democratic approximation for this observable.

The electron anti-neutrino disappearance probability at a (not too distant) reactor experiment is given, again to leading (second) order in small quantities, from Eq. **??**, by:

$$1 - P(\overline{\nu}_e \to \overline{\nu}_e) \simeq 4|U_{e3}|^2\sin^2\Delta_{31} + \frac{8}{9}h^2\Delta_{31}^2. \quad (59)$$

# References