Magic/S3 Group Mixing and its Derivatives

– Democracy in Neutrino Oscillations

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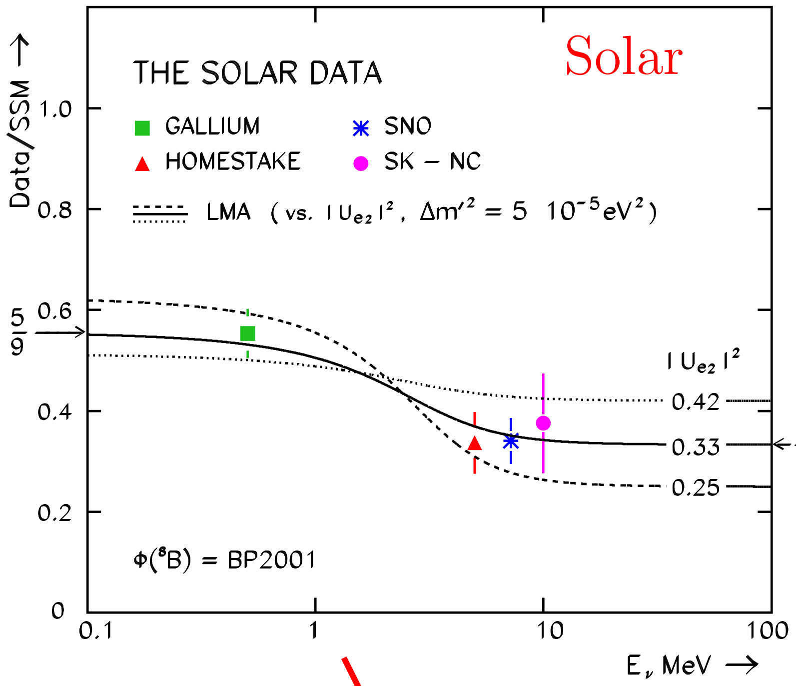
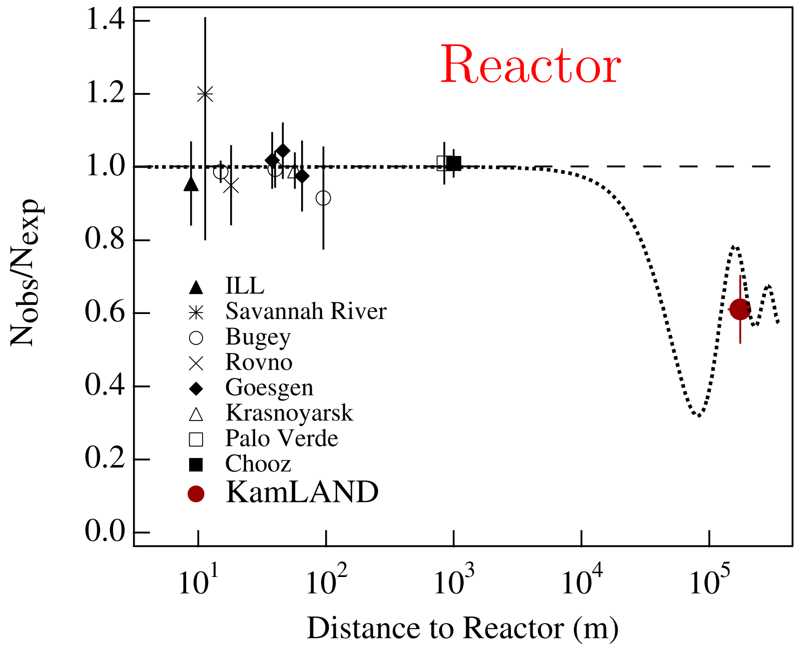
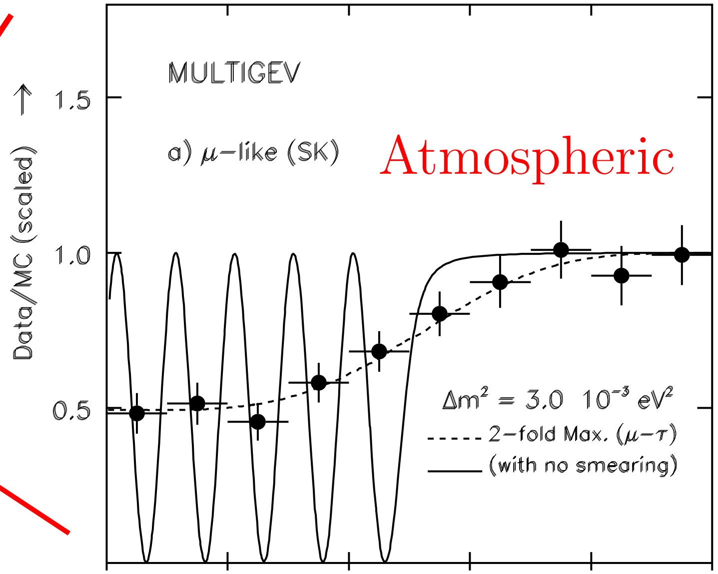
Outline of Talk

- Summary of ν Oscillation Data
- Magic/S3 Group Mixing
- Democracy in Neutrino Oscillations
- Some Special Cases of Magic/S3 Group Mixing
- The Friedman-Lee Model
- Discussion and Summary

Summary of Neutrino Oscillation Data







$$
(|U|^2) = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array}
\begin{array}{c}
\begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\
\begin{pmatrix}
\sim \frac{2}{3} & 0.31 \pm 0.04 & < 0.013 \\
\sim \frac{1}{6} & \sim \frac{1}{3} & 0.50 \pm 0.11 \\
\sim \frac{1}{6} & \sim \frac{1}{3} & \sim \frac{1}{2}
\end{pmatrix}
\end{array}
$$

NB. 68% CL!

Magic/S3 Group Mixing

This talk is about a mixing scheme in which ν_2 is trimaximally mixed:

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Unitarity $\Rightarrow$

$$U_{MNS} = \begin{pmatrix} \frac{2}{\sqrt{6}}C & \frac{1}{\sqrt{3}} & U_{e3} \\ -\frac{1}{\sqrt{6}}C - \frac{\sqrt{3}}{2}U^*_{e3} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}C - \frac{1}{2}U_{e3} \\ -\frac{1}{\sqrt{6}}C + \frac{\sqrt{3}}{2}U^*_{e3} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}C - \frac{1}{2}U_{e3} \end{pmatrix} \quad \text{“Magic/S3 Group Mixing”}$$

where $C = \sqrt{1 - \frac{3}{2}|U_{e3}|^2} \simeq 1$ (J.D.Bjorken, PFH and WGS, hep-ph/0511201).

2 constraints and 2 remaining mixing d.o.f.: $\mathrm{Re}(U_{e3})$ and $\mathrm{Im}(U_{e3})$.

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Ideas first developed during 2002-04 by PFH and WGS (hep-ph/0203209, 0302025, 0402006 and 0403278).

Lately by C.S.Lam (hep-ph/0606220), R.Friedberg and T.D.Lee (hep-ph/0606071).

Recent developments stimulate this review.

Corresponding $\boldsymbol{\nu}$ Mass Matrix

WLOG, choose to work with Hermitian-square of ν mass matrix (in flavour basis):

$$H_\nu = M_\nu M_\nu^\dagger.$$

(NB. $\frac{H_\nu}{2E}$ is ν-oscillation Hamiltonian).

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$$H_\nu = a_0\, I + \begin{pmatrix} x & y & z \\ y & z & x \\ z & x & y \end{pmatrix} + i\, d \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

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Has several interesting properties:

- Is an S3 group matrix (hep-ph/0302025) (hence the name "S3 Group Mixing")
- Is a (semi-)magic square - all its rows and columns add up to the same "magic" constant, $\lambda = a_0 + x + y + z$ (hep-ph/0402006) (hence the name "Magic Mixing"). Then $H_\nu \underline{\nu}_{\text{trimax}} = \lambda \underline{\nu}_{\text{trimax}}$, where $\underline{\nu}_{\text{trimax}} = \frac{1}{\sqrt{3}}(1, 1, 1,)^T$.

What is an S3 Group Matrix?

The $S3$ group has 6 elements. May be written:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$P_{\mu\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_{\tau e} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_{e\mu} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

An $S3$GM is just a (complex) linear combination (in our case, Hermitian):

$$H_\nu = a_0 I + bP_+ + b^* P_- + xP_{\mu\tau} + yP_{\tau e} + zP_{e\mu}$$

(a_0, x, y, z real and b complex).

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(a_0, x, y, z real and b complex).
H_ν is invariant under:

$$a_0 \to a_0 + \mu, \quad b \to b + \mu, \quad x \to x - \mu, \quad y \to y - \mu, \quad z \to z - \mu$$

μ real. Can set $\mathrm{Re}(b) = 0 \implies$ 5 real parameters - corresp. to 3 ν masses and U_{e3}.

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S3 Group/Magic properties are cute, but do not fit classic "symmetry" paradigm.

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Yes! H_ν commutes with the Democracy operator: $[H_\nu, \mathcal{D}] = 0$, where

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The "usual" results of unitary symmetry follow:

- Mass eigenstates are also e-states of $\mathcal{D}$: $\mathcal{D}\nu_2 = \nu_2$; $\mathcal{D}\nu_1 = \mathcal{D}\nu_3 = 0$.
- Democracy is conserved in neutrino oscillations. Example is propagation of a pure ν_2 eigenstate from the surface of the Sun (prepared by its adiabatic evolution en route from the solar core). Outside Sun, it does not oscillate, and remains a pure ν_2. $\mathcal{D}$ remains unity.

Historical Aside on Democracy

"Democratic" mass matrix has long history (Harari, Haut, Weyers, PLB 78 (1978)).

In that ansatz, postulate that mass matrix itself has the democratic form, each of its elements being equal - 1 parameter only

Or each of equal magnitude (L. Lavoura. Phys. Lett. B 228 (1989)).

Fritzsch and Xing, employed small perturbations from this form (PLB 372 (1996), PLB 440 (1999), PRD61 (2000)).

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NB. $\mathcal{D}$ generates unitary transformations, under which H_ν is invariant:

Define $U_\mathcal{D} = \exp^{i\alpha\mathcal{D}}$ (α a real parameter). Then

$$H_\nu \rightarrow H'_\nu := U_\mathcal{D} H_\nu U_\mathcal{D}^\dagger = H_\nu .$$

I Emphasise:

Democracy Symmetry $\Longleftrightarrow$ S3 Group property $\Longleftrightarrow$ Magic property $\Longleftrightarrow$ At least one trimaximally mixed mass eigenstate: $\frac{1}{\sqrt{3}}(1, 1, 1)$

THE FOUR CONDITIONS ARE ALL EQUIVALENT!

The Unsquared $\boldsymbol{\nu}$ Mass Matrix, $\boldsymbol{M_\nu}$

In general, M_ν has no special form (cf. $H_\nu \equiv M_\nu M_\nu^\dagger$ is Hermitian).

C.S. Lam proposes (hep-ph/0606220) that M_ν itself be (semi-)magic:

$$M_\nu = \begin{pmatrix} a & b & c \\ e & d & a+b+c-d-e \\ b+c-e & a+c-d & d+e-c \end{pmatrix}$$

a, b, c, d, e complex.

In Dirac ν case, only 5 (of the 10) parameters are "physical" $\rightarrow$ 3 masses and U_{e3}.

In Majorana ν case, M_ν is symmetric, so that $e = b$. Remaining 8 parameters correspond to the 3 complex masses (one unphysical phase) and U_{e3}.

But $[M_\nu, \mathcal{D}] = 0 \implies [H_\nu, \mathcal{D}] = 0$, so that all such cases are covered by the already-defined Magic H_ν (unless one is really interested in the Majorana phases, which drop out of H_ν).

3 Special Cases of S3 Group/Magic Mixing

- $\mathrm{Im}(U_{e3}) = 0$: Defined by 2 symmetries: Democracy and CP.

$$U_{MNS} = \begin{pmatrix} \sqrt{\frac{2}{3}}\cos\phi & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}\sin\phi \\ -\frac{\cos\phi}{\sqrt{6}} - \frac{\sin\phi}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{\cos\phi}{\sqrt{2}} - \frac{\sin\phi}{\sqrt{6}} \\ -\frac{\cos\phi}{\sqrt{6}} + \frac{\sin\phi}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{\cos\phi}{\sqrt{2}} - \frac{\sin\phi}{\sqrt{6}} \end{pmatrix}$$

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- $\mathrm{Re}(U_{e3}) = 0$: 2 symmetries: Democracy and Mutativity.

$$U_{MNS} = \begin{pmatrix} \sqrt{\frac{2}{3}}\cos\chi & \frac{1}{\sqrt{3}} & i\sqrt{\frac{2}{3}}\sin\chi \\ -\frac{\cos\chi}{\sqrt{6}} + i\frac{\sin\chi}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{\cos\chi}{\sqrt{2}} - i\frac{\sin\chi}{\sqrt{6}} \\ -\frac{\cos\chi}{\sqrt{6}} - i\frac{\sin\chi}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{\cos\chi}{\sqrt{2}} - i\frac{\sin\chi}{\sqrt{6}} \end{pmatrix}$$

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Magic plus $y = z$

"Tri-χ maximal" mixing

(PFH&WGS – hep-ph/0203209)

- $U_{e3} = 0$: All 3 symmetries: Democracy, Mutativity and CP.

$$U_{MNS} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Magic plus $y = z$, $d = 0$

"Tri-bimaximal" mixing (TBM)

(PFH, WGS and D.H.Perkins

– hep-ph/0202074)

Mutativity

The Mutativity transformation is defined (PFH and WGS, PLB547 (2002) 219) as simultaneous exchange of the μ and τ flavour labels and a CP transformation.

Eg. 1: Mass Matrix (symmetric under Mutativity):

$$\begin{pmatrix} a & b & b^* \\ d^* & a & c \\ d & c^* & a \end{pmatrix} \xrightarrow{\mu-\tau \text{ exchange}} \begin{pmatrix} a & b^* & b \\ d & a & c^* \\ d^* & c & a \end{pmatrix} \xrightarrow{CP} \begin{pmatrix} a & b & b^* \\ d^* & a & c \\ d & c^* & a \end{pmatrix}$$

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Eg. 2: Mixing Matrix (symmetric under Mutativity):

$$U_{MNS} = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ v_1^* & v_2^* & v_3^* \end{pmatrix} \xrightarrow{\mu-\tau \text{ exchange}} \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1^* & v_2^* & v_3^* \\ v_1 & v_2 & v_3 \end{pmatrix} \xrightarrow{CP} \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ v_1^* & v_2^* & v_3^* \end{pmatrix}$$

Key feature is $\nu_\mu = \nu_\tau^*$ (More generally, $|\nu_{\mu i}| = |\nu_{\tau i}| \ \forall \ i$).

The Simplest $\boldsymbol{\nu}$ Mass Matrix

The "Simplest" ν mass matrix is a special case of Tri-χ maximal mixing, defined by $y(=z)=0$:

$$H_\nu = a_0 I + x \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + id \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

3 parameters for 4 observables $\Longrightarrow$ 1 constraint between masses and mixing angles:

$$\mathrm{Im}(U_{e3}) = |U_{e3}| = \sin\theta_{13} = \sqrt{\frac{2\Delta m^2_{\mathrm{sol}}}{3\Delta m^2_{\mathrm{atm}}}} \simeq 0.13.$$

x and d are given in terms of oscillation observables by:

$$x = -\frac{\Delta m^2_{\mathrm{atm}}}{2}(1-3\sin^2\theta_{13}), \quad d = \frac{\Delta m^2_{\mathrm{atm}}}{\sqrt{2}}\sin\theta_{13}(1-\frac{3}{2}\sin^2\theta_{13})^{\frac{1}{2}}$$

The Friedberg-Lee $\boldsymbol{\nu}$ Mass Model

With different motivation, Friedberg and Lee (hep-ph/0606071) recently proposed a model in which M_ν has the form:

$$
\begin{aligned}
M_\nu^{FL} &= m_0\, I + \begin{pmatrix} b+c & -b & -c \\ -b & a+b & -a \\ -c & -a & c+a \end{pmatrix} \\
&= m_0'\, I - \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}
\end{aligned}
$$

with m_0, a, b and c real and $m_0' = m_0 + (a+b+c)$.

Is manifestly the real case of the Magic/S3 form ($\Longrightarrow$ Tri-ϕ maximal mixing).

Ie. has single trimaximally mixed ν mass eigenstate and real mixing matrix.

Friedberg-Lee – Next Step

Apart from Magic condition and CP conservation, FL model is otherwise general (in particular, ν masses and $\mathrm{Re}(U_{e3})$ are arbitrary).

As next step, they set $c = 0 \implies$ 3 parameters $(m_0,\, a,\, b)$ to accommodate 3 masses and $\mathrm{Re}(U_{e3}) \implies$ a non-trivial constraint

(cf. "Simplest" ansatz - in terms of the number of parameters, this is equally simple!).

Constraint is on M_ν, not H_ν, $\implies$ oscillation phenomenology is a little complex.

In terms of our (H_ν) parameters:

$$\begin{aligned} f(x,y,z,a_0) \quad := \quad & \frac{1}{4}(z-x)^2(z-y)^2 + z^2[(z-x)^2+(z-y)^2] \\ - \quad & z(z+a_0)(z-x)(z-y) \quad = 0. \end{aligned}$$

Gives essentially:

$$\mathrm{Re}(U_{e3}) = |U_{e3}| \simeq \frac{1}{2\sqrt{2}}\frac{\Delta m^2_{\mathrm{sol}}}{\Delta m^2_{\mathrm{atm}}} \simeq 0.01.$$

Friedberg-Lee with $\boldsymbol{CP}$ Violation

In an additional step, FL introduce CP (T) violation as follows:

$$
\begin{aligned}
M_\nu^{FL} &= m_0\, I + \begin{pmatrix} b+c & -b & -ce^{-i\eta} \\ -b & a+b & -a \\ -ce^{i\eta} & -a & c+a \end{pmatrix} \\
&= m_0'\, I - \begin{pmatrix} a & b & ce^{-i\eta} \\ b & c & a \\ ce^{i\eta} & a & b \end{pmatrix}
\end{aligned}
$$

This matrix is not Magic, and not an S3 GM. There is no trimaximally mixed eigenstate, although phenomenologically, it differs by only a small perturbation.

Moreover, this version of the model sacrifices its original motivation, as we will see.

Xing-Zhang-Zhou Ansatze

Seeks to generalise the FL model, by allowing the parameters of M_ν: m_0, a, b and c to be complex.

Corresponds in general to H_ν being a (Hermitian) S3 GM as outlined above.

XZZ define two special cases:

Scenario A - defined by m_0 and a real with $b = c^*$.

This gives exactly Tri-χ maximal mixing, defined by $y = z$, having the corresponding mixing matrix.

Scenario B - defined by m_0 complex and a b and c real.

This gives exactly Tri-ϕ maximal mixing, defined by $d = 0$ having the corresponding mixing matrix.

Our use of H_ν covers the *oscillation* observables equally well for both Majorana and Dirac cases.

We make no predictions for Majorana phases - those given by XZZ are new.

Setting $\boldsymbol{d = z = 0}$ in $\boldsymbol{H_\nu}$

Can be considered the equivalent of the additional constraints of FL, except making them on H_ν instead of M_ν.

$$H_\nu = a_0 I + x \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Because we deal directly with the oscillation Hamiltonian, correspondence with oscillation observables is more direct.

Special case of Tri-ϕ maximal mixing, with

$$\mathrm{Re}(U_{e3}) = |U_{e3}| \simeq \frac{1}{\sqrt{3}} \frac{\Delta m^2_{\mathrm{sol}}}{\Delta m^2_{\mathrm{atm}}} \simeq 0.02.$$

Could name it "Simplest Real" mass matrix.

Friedberg-Lee Motivation

Cited motivation for FL model is that it is invariant under the transformation

$$\nu_e \longrightarrow \nu_e + z, \quad \nu_\mu \longrightarrow \nu_\mu + z, \quad \nu_\tau \longrightarrow \nu_\tau + z$$

which we may write: $\underline{\nu} \rightarrow \underline{\nu} + \sqrt{3} z \, \underline{\nu}_{\text{trimax}}$.

Effective mass term of Lagrangian is $\overline{\underline{\nu}} \, M_\nu \, \underline{\nu}$.

Sufficient condition for FL invariance is that $M_\nu \underline{\nu}_{\text{trimax}} = 0$ and $\overline{\underline{\nu}}_{\text{trimax}} M_\nu = 0$.

Necessary condition for this is $[M_\nu, \mathcal{D}] = 0$, ie. M_ν is Magic, is S3 GM and has Democracy symmetry.

Hence, FL invariance is intimately related to Democracy

Moreover, FL CP violation term breaks FL symmetry/Democracy, whereas our explicit Magic/S3 Group mixing formulation accommodates CP violation, while preserving this fascinating property.

Conclusions

- TBM ansatz gaining in popularity. A number of models proposed which "explain" why it could be so (see eg. E. Ma talk).
- Defined by 3 symmetries, it allows no flexibility in the mixing, in particular, no CP violation. We should consider its generalisations.
- Can abandon any or all of the symmetries - already in literature.
- Here, we focussed on consequences of abandoning the two with the weakest evidence (CP and mutativity), keeping the one with the most (democracy).
- Resulting mixing matrix has one trimaximally mixed mass eigensate and two remaining free parameters (U_{e3}):

$$U_{MNS} = \begin{pmatrix} \frac{2}{\sqrt{6}}C & \frac{1}{\sqrt{3}} & U_{e3} \\ -\frac{1}{\sqrt{6}}C - \frac{\sqrt{3}}{2}U_{e3}^* & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}C - \frac{1}{2}U_{e3} \\ -\frac{1}{\sqrt{6}}C + \frac{\sqrt{3}}{2}U_{e3}^* & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}C - \frac{1}{2}U_{e3} \end{pmatrix} \quad \text{with } C \simeq 1.$$

Conclusions (contd.)

- Resulting mass(-squared) matrix in the flavour basis:
  - Is an S3 group matrix
  - Is Magic
  - Respects Friedberg-Lee Symmetry

  All the above properties are equivalent.

- We believe that they "motivate" this ansatz as a better "approximate" form for the lepton mixing matrix than TBM.

Summary

Table 1: Special cases of the Magic/S3 Group mixing ansatz.

H_ν Constraint(s)	Extra Symms.	Given Name	U_{MNS} Constraint(s)
$y = z$	$\mu - \tau^*$	Tri-χ maximal, XZZ(A)	$\mathrm{Re}(U_{e3}) = 0$
$d = 0$	CP	Tri-ϕ maximal, XZZ(B)	$\mathrm{Im}(U_{e3}) = 0$
$y = z$ and $d = 0$	$\mu - \tau^*$ and CP	Tri-bimaximal mixing	$U_{e3} = 0$
$y = z = 0$	$\mu - \tau^*$	Simplest mass matrix	$\mathrm{Re}(U_{e3}) = 0$ and $\mathrm{Im}(U_{e3}) = \sqrt{\frac{2}{3}}\sqrt{\frac{\Delta m^2_{\mathrm{sol}}}{\Delta m^2_{\mathrm{atm}}}}$
$d = 0$ and $f(x, y, z, a_0) = 0$	CP	Friedberg-Lee model	$\mathrm{Im}(U_{e3}) = 0$ and $\mathrm{Re}(U_{e3}) \simeq \frac{1}{2\sqrt{2}}\frac{\Delta m^2_{\mathrm{sol}}}{\Delta m^2_{\mathrm{atm}}}$
$d = z = 0$	CP	Simplest Real mass matrix	$\mathrm{Im}(U_{e3}) = 0$ and $\mathrm{Re}(U_{e3}) \simeq \frac{1}{\sqrt{3}}\frac{\Delta m^2_{\mathrm{sol}}}{\Delta m^2_{\mathrm{atm}}}$