Leptonic Mixings from a Variational Principle

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Outline of Talk

- Flavour Physics - the Issues
- Common Origin of Masses and Mixings in the Standard Model
- Jarlskog Covariance
- Historical Attempts to Constrain Masses and Mixings
- Towards a Theory of Masses and Mixings
- Some Examples
- The Future

Draws heavily on, and develops earlier work by Cecilia Jarlskog.

Flavour Physics: the Issues

- Why 3 families?

  I can't answer that.

- What are the unmeasured parameters?

  We are working on neutrino and B experiments to try to answer that.

- Fermion mass spectra (fig.)

  Hope eventually to explain.

- Fermion mixing spectra (fig.)

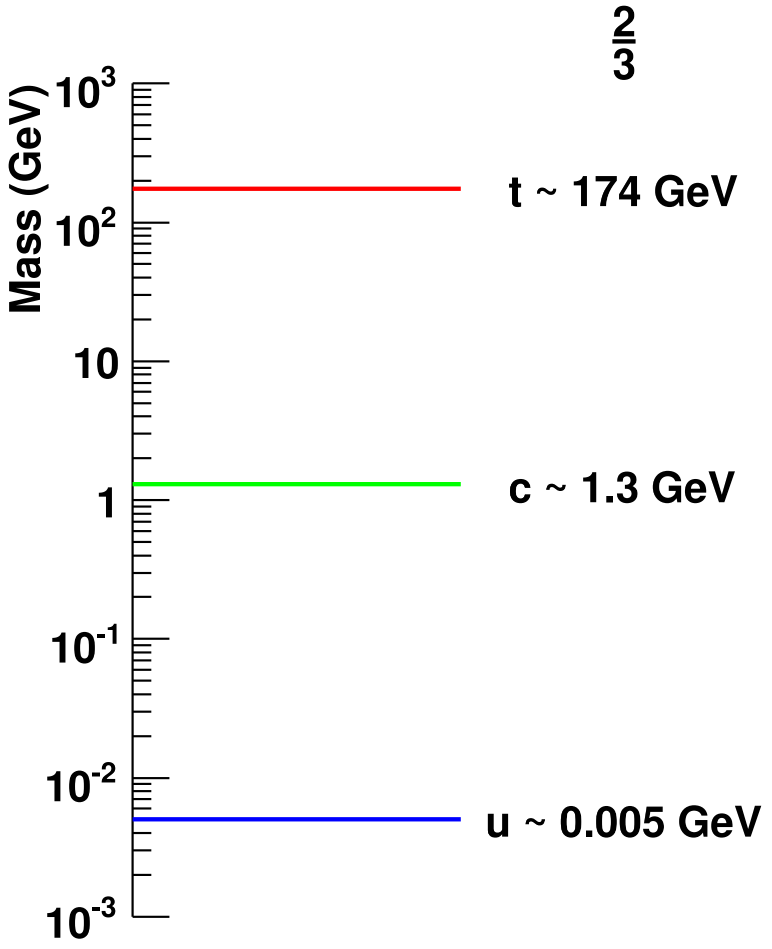
  Hope to explain

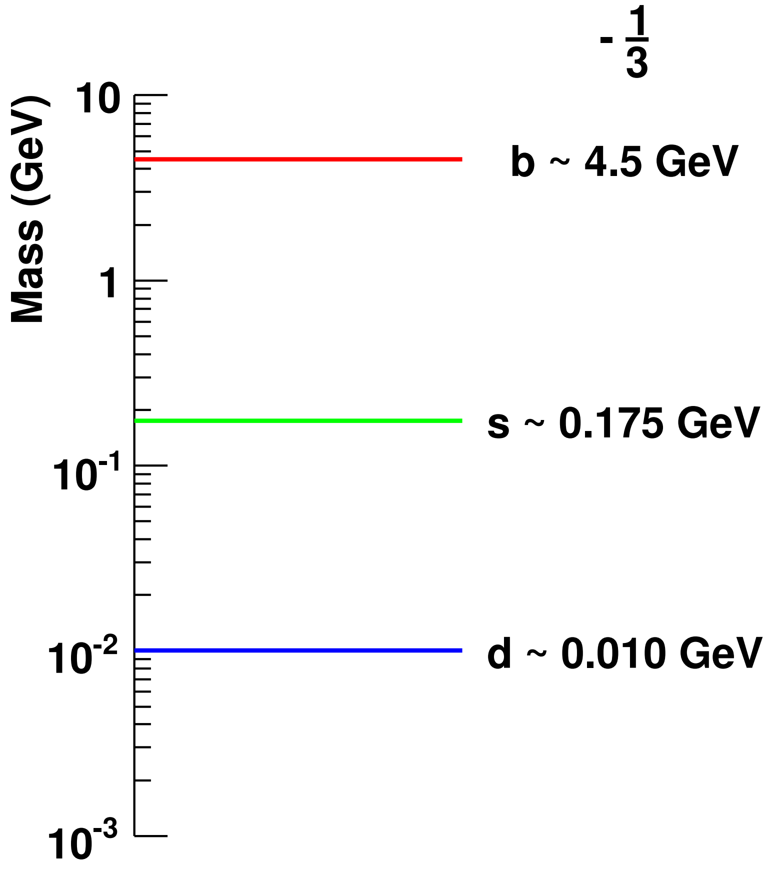
We suggest a simple approach to the last two issues.

Claim some success, with the lepton mixings, at least.

Why Such Peculiar Quark Mass Spectra?

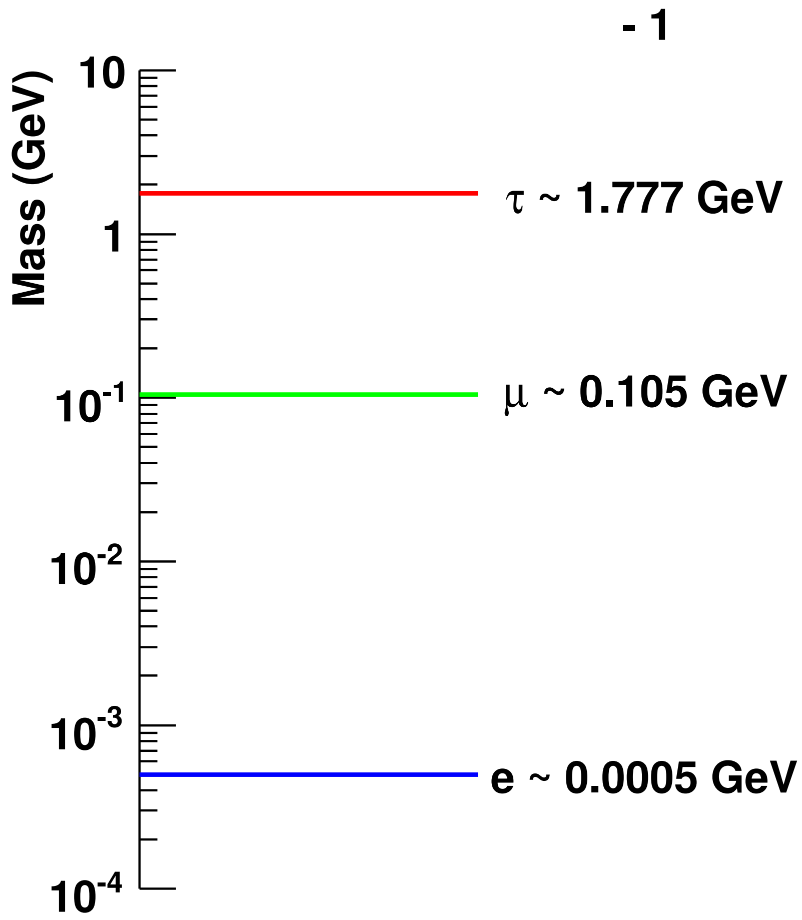
Charge $= +\frac{2}{3}$ Charge $= -\frac{1}{3}$



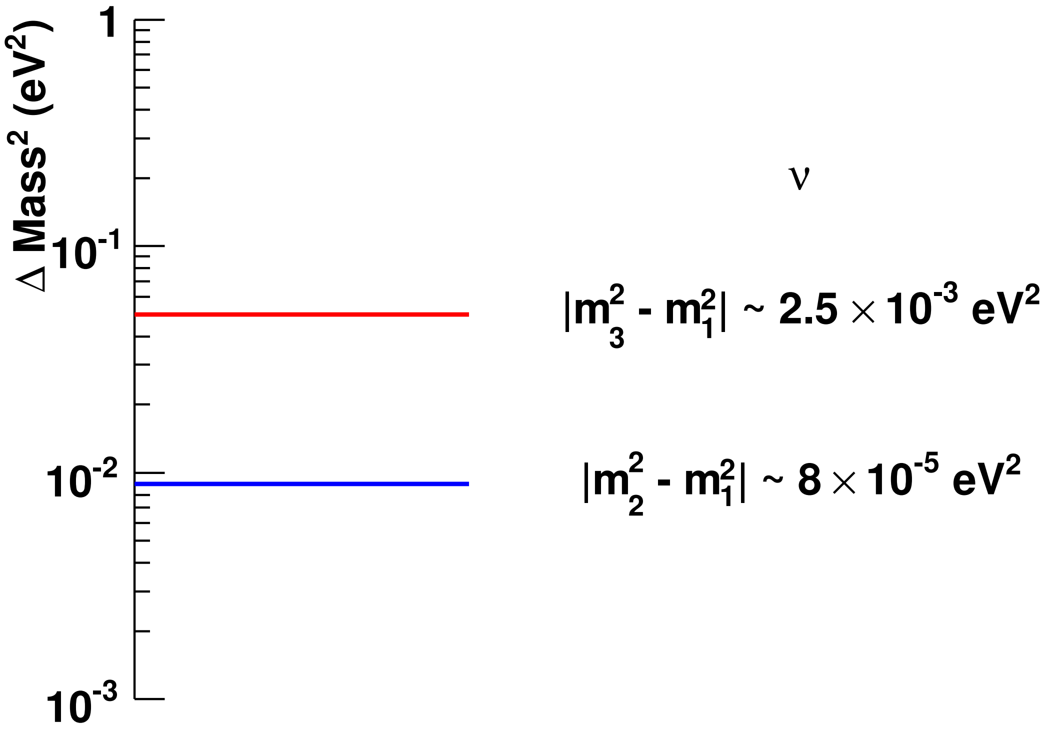


Why Such Peculiar Lepton Mass Spectra?

Charge $= -1$



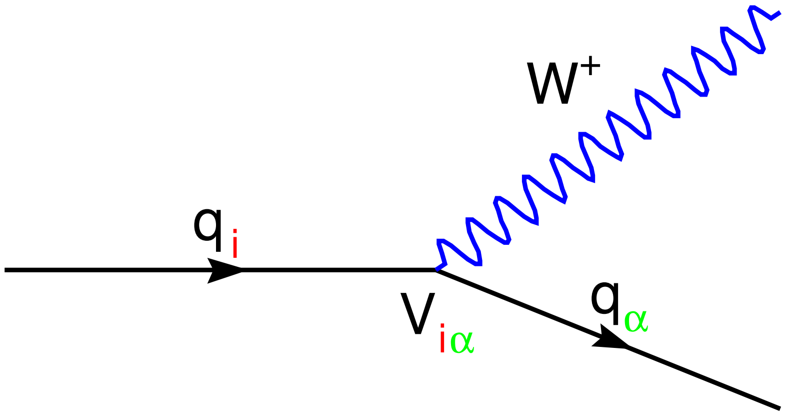
Charge $= 0$



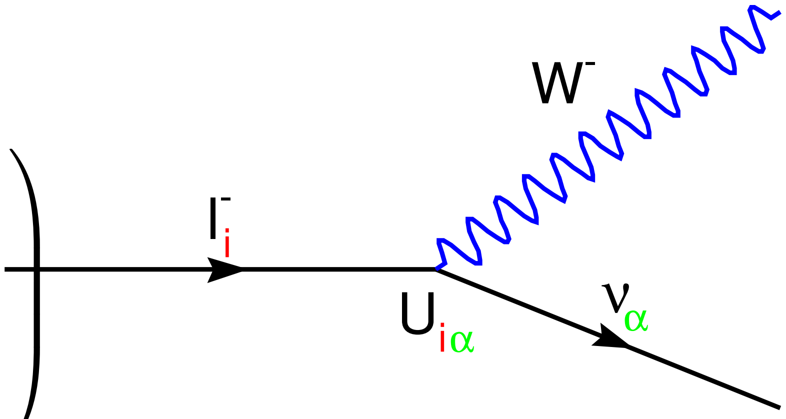
Why Such Peculiar Fermion Mixing Matrices?

MNS (U_{MNS}) and CKM (V_{CKM}) matrices parameterise couplings of mass eigenstates to W:

$$|V_{CKM}| \sim \begin{array}{c c} & \begin{array}{ccc} d & s & b \end{array} \\ \begin{array}{c} u \\ c \\ t \end{array} & \begin{pmatrix} 0.974 & 0.224 & 0.004 \\ 0.224 & 0.974 & 0.04 \\ 0.01 & 0.04 & 0.999 \end{pmatrix} \end{array}$$



$$|U_{MNS}| \sim \begin{array}{c c} & \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{array}{c} e \\ \mu \\ \tau \end{array} & \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & < 0.17 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \end{array}$$



Common Origin of Masses and Mixing in the SM

- In SM, they have common origin in fermion-Higgs "Yukawa couplings" (multiplied by Higgs Vevs).
- Will focus on leptons – consider 3 families of leptonic $SU(2)$ doublets:

$$\psi_a = \begin{pmatrix} \nu_a \\ \ell_a^- \end{pmatrix}, \quad \psi_b = \begin{pmatrix} \nu_b \\ \ell_b^- \end{pmatrix}, \quad \psi_c = \begin{pmatrix} \nu_c \\ \ell_c^- \end{pmatrix}$$

  Each transforms among itself under $SU(2)$ weak transformations in usual way.
- Define also two three-component vectors in family-space:

$$\underline{\nu} = \left(\nu_a, \nu_b, \nu_c\right), \quad \underline{\ell} = \left(\ell_a^-, \ell_b^-, \ell_c^-\right),$$

Origin of Masses and Mixing (contd.)

- Accommodate ν masses in SM with simplest extension: Dirac masses only $\rightarrow$ νSM (extension to Majarana neutrinos is on the "to do" list).

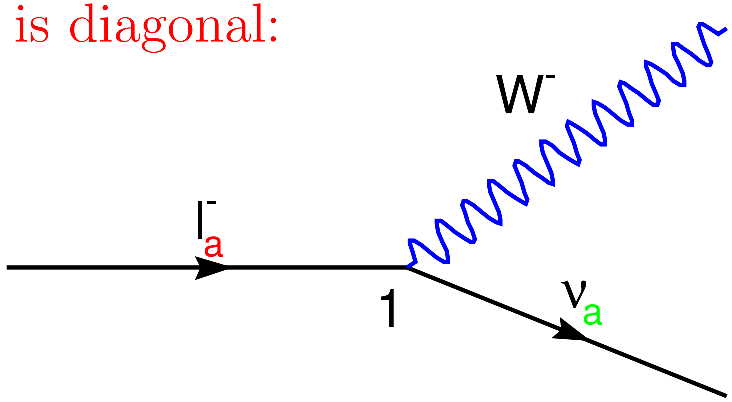
- After SSB, mass terms appear in Lagrangian (Dirac structure assumed):

$$\mathcal{L}_{mass} = \bar{\underline{\ell}}_L \, L \, \underline{\ell}_R + \bar{\underline{\nu}}_L \, N \, \underline{\nu}_R + H.C.$$

- L and N, are arbitrary, 3×3 mass matrices.

- Charged-current weak interaction (same basis) is diagonal:

$$\mathcal{L}_{cc} = g(\bar{\underline{\ell}}_L \, . \, \underline{\nu}_L)W^+ + H.C.$$



Origin of Masses and Mixing (contd.)

- Diagonalise L and N, $\rightarrow$ "physical" states.

$$\underline{\ell}_L \rightarrow \underline{\ell}^m_L = U_\ell \, \underline{\ell}_L \,, \qquad \underline{\nu}_L \rightarrow \underline{\nu}^m_L = U_\nu \, \underline{\nu}_L \quad \text{(similarly for RH components.)}$$

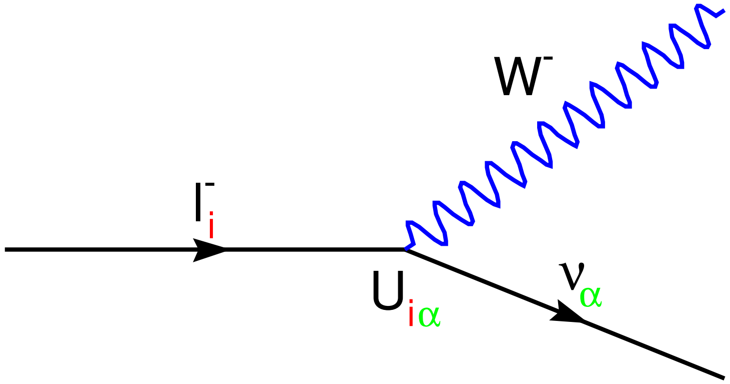
- Then

$$\mathcal{L}_{mass} \quad = \quad \underline{\bar{\ell}}^m_L D_\ell \, \underline{\ell}^m_R + \underline{\bar{\nu}}^m_L D_\nu \, \underline{\nu}^m_R + H.C.$$

where $D_\ell \equiv U_\ell L U_\ell^\dagger$ and $D_\nu \equiv U_\nu N U_\nu^\dagger$ are diagonal mass matrices.

- Also

$$\mathcal{L}_{cc} \quad = \quad g \, (\underline{\bar{\ell}}^m_L \, U_{MNS} \, \underline{\nu}^m_L) W^+ + H.C.$$



where $U_{MNS} \equiv U_\ell U_\nu^\dagger$ is the lepton mixing matrix. NB. Analogous for quarks.

Jarlskog (ie. Weak Basis) Covariance

- Instead of diagonalising, could have made arbitrary *simultaneous* transf. (dropping "$_L$", "$_R$" subscripts):

$$\underline{\ell} \to \underline{\ell}' = U\underline{\ell}\,, \qquad \underline{\nu} \to \underline{\nu}' = U\underline{\nu} \qquad \text{(ie. same } U\text{)}$$

- Then

$$\mathcal{L}_{mass} \;=\; \underline{\bar{\ell}}'L'\,\underline{\ell}' + \underline{\bar{\nu}}'N'\,\underline{\nu}'$$

where $L' \equiv ULU^\dagger$ and $N' \equiv UNU^\dagger$ are new arbitrary Hermitian mass matrices.

- In this basis, charged current is:

$$\begin{aligned} \mathcal{L}_{cc} &= g(\underline{\bar{\ell}}'\,UU^\dagger\,\underline{\nu}')W^+ + H.C. \\ &= g(\underline{\bar{\ell}}'.\underline{\nu}')W^+ + H.C. \end{aligned}$$

ie. still diagonal, but mass matrices have changed ($\equiv$ "Jarlskog-covariant")

- New basis looks like just as good a starting point...

...and Observables are Invariant:

- Now

$$\underline{\ell}^{\,m} = U_\ell U^\dagger \underline{\ell}^{\,\prime}, \qquad \underline{\nu}^m = U_\nu U^\dagger \underline{\nu}'$$

and

$$\mathcal{L}_{mass} \;=\; \bar{\underline{\ell}}^{\,m} D'_\ell \underline{\ell}^{\,m} + \bar{\underline{\nu}}^{\,m} D'_\nu \underline{\nu}^m$$

where $D'_\ell \equiv U_\ell U^\dagger L' U U_\ell^\dagger = D_\ell$ and $D'_\nu \equiv U_\nu U^\dagger N' U U_\nu^\dagger = D_\nu$ are unchanged.

- Also

$$\mathcal{L}_{cc} \;=\; g(\bar{\underline{\ell}}^{\,m} U'_{MNS} \underline{\nu}^m) W^+ + H.C.$$

where $U'_{MNS} \equiv U_\ell U^\dagger U U_\nu^\dagger = U_{MNS}$ is also unchanged.

- ie. Masses and mixing angles are "Jarlskog-invariant".
- NB. Is not a "symmetry" (that would be if $L' = L$), but an invariance of νSM

Historical Attempts to Constrain Fermion Masses and Mixings

- Inspired by phenomenological relationships in quark mixing, eg.

$$\tan\theta_C \simeq 0.23 \simeq \sqrt{\frac{m_d}{m_s}}; \qquad \tan\theta_{cb} \simeq 0.04 \simeq \sqrt{\frac{m_d}{m_b}}$$

  and common origin of masses and mixings. Many attempts made to relate them.

- Best known is by H. Fritzsch from 1977 and 1978: mass matrices:

$$M^{\frac{2}{3}} = \begin{pmatrix} 0 & a & 0 \\ a^* & 0 & b \\ 0 & b^* & c \end{pmatrix}; \qquad M^{-\frac{1}{3}} = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha^* & 0 & \beta \\ 0 & \beta^* & \gamma \end{pmatrix}$$

- Predicts (after approximations) above θ_C relation, but also

$$\tan\theta_{cb} > \sqrt{\frac{m_s}{m_b}} - \sqrt{\frac{m_c}{m_t}} \simeq 0.09, \qquad (\text{now excluded}).$$

How does Fritzsch *Ansatz* work?

- Mass matrices encode 10 pieces of information (6 masses and 4 mixing parameters) using only 6 magnitudes and one phase, $\implies$ 3 constraints.
- Spawned many refinements, some phenomenologically successful.
- Modern models appeal to "flavour symmetries" to enforce zeroes or other "textures".
- Many build-in supersymmetry and/or complicated Higgs structures etc.
- However, generally assume that $\exists$ a "special" basis, in which the mass matrices take some "particular" form (which limits number of parameters to < 10).
- "Specialness" of this basis is never explained.
- Since 1905, we have been alert to excessive reliance on a given frame or basis.

Towards a Theory of Masses and Mixings (ToMM)

Look to νSM for some principles on which to base a theory:

- νSM as it stands has no preferred basis in which to write the lepton (and quark) mass matrices. A ToMM should respect this. If not, it should at least explain existence of a preferred basis. (Who could argue with that?).

And yet few authors even pay lip-service to the idea of weak basis-invariance in their models.

- νSM as it now stands treats all families on an equal footing. A ToMM should (ideally?) respect this, ie. mass/mixing constraints should be flavour-symmetric.

Historically, concepts of basis/frame/label-independence have been a great guide to building successful theories.

Decided to adopt both weak-basis invariance AND flavour-symmetry as guiding principles.

Flavour-Symmetric Jarlskog Invariants

- "Usual" observables are not flavour-symmetric. eg. the electron mass, m_e, (electron label) and U_{e3} (flavour and mass labels).

- Jarlskog showed how to write flavour-symmetric invariant observables.

- Examples are (all matrices in an arbitrary weak-basis, primes dropped):

$$\mathrm{Tr}(L) = m_e + m_\mu + m_\tau; \quad \mathrm{Tr}(L^2) = m_e^2 + m_\mu^2 + m_\tau^2; \quad \mathrm{Tr}(L^3) = m_e^3 + m_\mu^3 + m_\tau^3$$

  which form a complete set.

- All traces of polynomials of L and N are invariant (and FS), eg.:

$$\mathrm{Tr}(L'N') = \mathrm{Tr}(ULU^\dagger UNU^\dagger) = \mathrm{Tr}(ULNU^\dagger) = \mathrm{Tr}(LNU^\dagger U) = \mathrm{Tr}(LN)$$

  etc.

For Mixing

- From earlier: $U_{MNS} \equiv U_\ell U_\nu^\dagger$.
- Clearly, $U_\nu = U_\ell \implies U_{MNS} = I \implies$ no mixing.
- So, experiment $\implies U_\nu \neq U_\ell$: L and N are diagonalised by different U's.
- Hence,

$$[L, N] \equiv iC \neq 0$$

- Now, under a change of weak basis,

$$C \to C' = UCU^\dagger$$

and likewise for any polynomial of C. But, as before, traces are invariant.

- NB: $\mathrm{Tr}C = 0$ for any commutator.

$\implies \exists$ two indepenent commutator invariants for C: $\mathrm{Tr}C^2$ and $\mathrm{Tr}C^3$.

For Example

- Now,

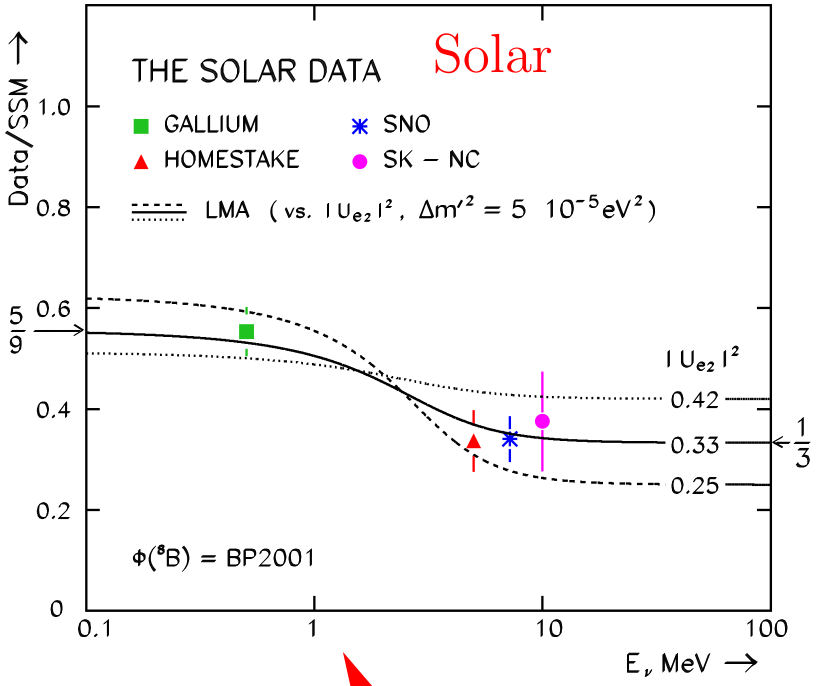
$$\begin{aligned}\mathrm{Tr}\, C^3/3 &= \mathrm{Det}\, C \\ &= -2(m_\mu - m_\tau)(m_\tau - m_e)(m_e - m_\mu)(m_2 - m_3)(m_3 - m_1)(m_1 - m_2)J\end{aligned}$$

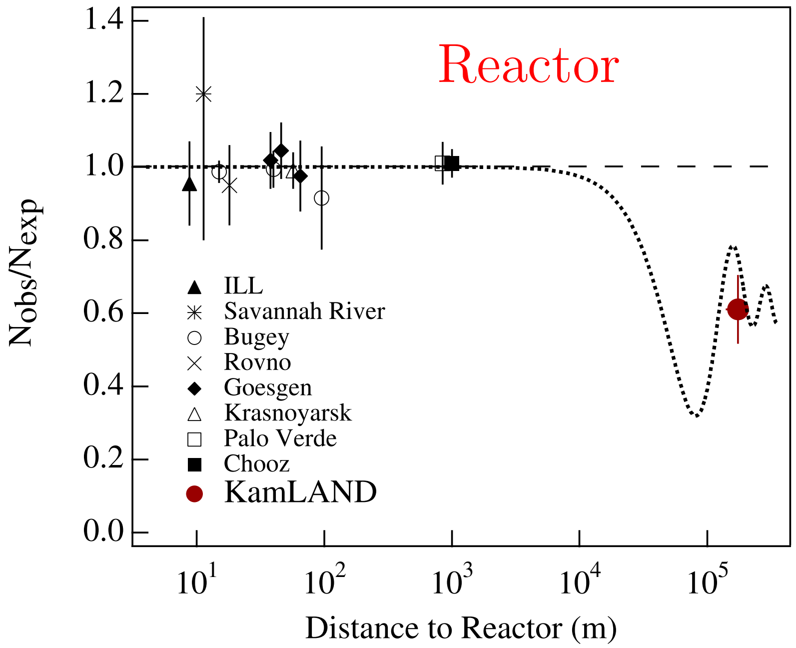
is a pure-CP-violating observable. (J is twice the area of the unitarity triangle) – manifestly symmetric among all flavour and mass indices.

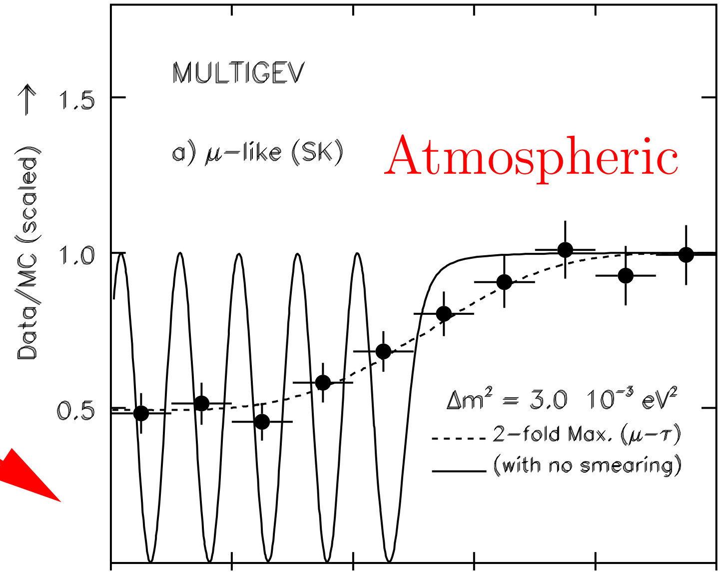
- $\mathrm{Tr}C^2$ is a pure-CP-conserving function of lepton masses and mixings (complicated to write out). Also manifestly symmetric among flavour and mass labels.

- Represent independent flavour-symmetric *measures* of mixing (they $\rightarrow 0$ as mixing $\rightarrow 0$).

Summary of Neutrino Oscillation Data







$$
|U| \sim \begin{array}{c} \\ e \\ \mu \\ \tau \end{array}
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1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\
1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2}
\end{pmatrix}
\end{array}
$$

An Approach to Jarlskog-Covariant Mixing Constraints?

- Motivated by quasi-maximal nature of leptonic mixing, attempt to "explain" it by maximising (extremising) a flavour-symmetric Jarlskog invariant:

$$\mathcal{A} = \mathrm{Tr}[f(L, N)].$$

ie. treat mass matrix entries (ie. Yukawas $\times$ Higgs Vevs) as dynamical variables.

- $\Longrightarrow$ find values for which derivatives are zero.
- Q: Derivative wrt what? (mixing angles, masses?).
- A: Derivatives wrt matrix elements themselves:

$$\partial(\mathcal{A})/\partial L \equiv \partial_L(\mathcal{A}) = 0; \qquad \partial(\mathcal{A})/\partial N \equiv \partial_N(\mathcal{A}) = 0.$$

- Treats all indices symmetrically, and always $\rightarrow$ covariant matrix constraints!
- But which invariant? (We tried several).

Eg. Consider $\mathbf{Tr}\boldsymbol{C^3}$

- Extremisation conditions are:

$$\begin{aligned} \partial_L \,\mathrm{Tr}\, C^3/3 &= -i[N, C^2]^T = 0, \\ \partial_N \,\mathrm{Tr}\, C^3/3 &= +i[L, C^2]^T = 0. \end{aligned}$$

- Manifestly covariant constraints: both sides of equation are evidently form-invariant under an arbitrary transformation $L \to U L U^\dagger$, $N \to U N U^\dagger$.
- Would get same constraints doing the extremisation in any basis.
- Equations treat all flavour and mass labels symmetrically $\implies$ satisfy our two principal requirements.
- Above properties do not depend on initially chosen invariant - general properties of our extremisation procedure.

Make a Modest Start

- Decide to extremise at *fixed* lepton masses.

  (Allowing them to vary is the ultimate goal)

- Extremisation with constraints $\Longrightarrow$ Lagrange multipliers.

- Outcome is to modify RHS of equations:

$$\begin{aligned} (\partial_L \,\mathrm{Tr}\, C^3/3)^T = -i[N, C^2] &= \lambda_{L0} + \lambda_{L1}\, L + \lambda_{L2}\, L^2 \\ (\partial_N \,\mathrm{Tr}\, C^3/3)^T = +i[L, C^2] &= \lambda_{N0} + \lambda_{N1}\, N + \lambda_{N2}\, N^2 \end{aligned}$$

  where the arbitrary constants λ are the LMs. (Eqs. still Jarlskog-covariant).

- $\Longrightarrow$ 18 constraint eqns. but not all independent.

- They exactly constrain the mixings, while all masses are fixed to their exptl. values by LMs.

Solution Method

- Use covariance to choose any basis in which to solve. Choose flavour basis (L is diagonal, N is not) - off diagonal elements of ∂_L Eqn. $= 0$.

- Further choose phase convention in which imaginary parts of N are all of equal modulus:

$$N = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} e & \mu & \tau \end{array} \\ \begin{pmatrix} a & z+id & y-id \\ z-id & b & x+id \\ y+id & x-id & c \end{pmatrix} \end{array}$$

Variables a, b, c, x, y, z and d determine the three neutrino masses and the four mixing parameters

Solution for $\mathbf{Tr}\boldsymbol{C}^{\mathbf{3}}$

- Solution to constraint equations corresponds to trimaximal lepton mixing (an old friend):

$$U_{MNS} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^* \\ 1 & \omega^* & \omega \end{pmatrix}$$

where $\omega = e^{2i\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, a cube-root of unity.

- $\Longrightarrow$ maximal CP violation.

- Suggestive, but excluded by experiment.

Try Instead $\mathbf{Tr}\boldsymbol{C^2}$

- Constraints are now:

$$(-\partial_L \mathrm{Tr}\, C^2/2)^T = +[N, C] \;=\; \lambda_{L0} + \lambda_{L1}\, L + \lambda_{L2}\, L^2$$

$$(-\partial_N \mathrm{Tr}\, C^2/2)^T = -[L, C] \;=\; \lambda_{N0} + \lambda_{N1}\, N + \lambda_{N2}\, N^2.$$

- Get CP-conserving solution:

$$x = \pm\sqrt{(a-b)(c-a)E} \qquad E = \frac{(m_e - m_\mu)(m_\tau - m_e)}{(m_e + m_\mu - 2m_\tau)(m_\tau + m_e - 2m_\mu)}$$

$$y = \pm\sqrt{(b-c)(a-b)M} \qquad M = \frac{(m_\mu - m_\tau)(m_e - m_\mu)}{(m_\mu + m_\tau - 2m_e)(m_e + m_\mu - 2m_\tau)}$$

$$z = \pm\sqrt{(c-a)(b-c)T} \qquad T = \frac{(m_\tau - m_e)(m_\mu - m_\tau)}{(m_\tau + m_e - 2m_\mu)(m_\mu + m_\tau - 2m_e)}.$$

with a, b, c fixed by the ν masses.

Solution for $\mathbf{Tr}\boldsymbol{C}^{\mathbf{2}}$

- Interestingly, the observed ν mass-squared difference hierarchy is correctly obtained setting $\frac{b-a}{a-c} = 1$ ($\implies \frac{\Delta m^2_{21}}{\Delta m^2_{31}} = 0.03$).
- The mixing matrix $\rightarrow$

$$(|U_{MNS}|^2) = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} .33333 & .33333 & .33333 \\ .17079 & .16257 & .66663 \\ .49587 & .50409 & .00003 \end{pmatrix} \end{array} \simeq \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/6 & 1/6 & 2/3 \\ 1/2 & 1/2 & 0 \end{pmatrix} \end{array} .$$

– also inconsistent with experiment.

- BUT it is a simple permutation of the (viable) tribimaximal mixing matrix!

What Next?

- We tried the obvious next step, ie. a linear combination:

$$\mathcal{A} = i\mathrm{Tr}C^3 + r\,\mathrm{Tr}C^2.$$

where r is a JI constant of dimensions m^2.

- Solution gives a trajectory of mixing matrices, as a function of r (and the unknown offset of the neutrino masses).
- Has some interesting properties including CP violation and viable mixing matrices.
- Have failed to find a solution which simultaneously fits all observables.
- But "trend" is good, ie. experiment-consistent mixing is obtained with masses if correct hierarchy. The required hierarchy is too extreme.
- Tried alternative strategies - eg. using squared mass matrices, $MM^\dagger$ etc. but to no avail.

What Does Work

- Try a linear combination:

$$\mathcal{A} = \mathrm{Tr}C^2 + q\,\mathrm{Tr}(C.[L^2, N]),$$

where q is a JI constant of dimensions m^{-1}.

- Only CP-conserving solutions.
- $\rightarrow$ a trajectory of mixing matrices, as a function of q (and unknown m_{ν_1}).
- (Arbitrarily) choose $m_1 = 0$ (for simplicity).
- Look for solution with one trimaximally mixed mass eigenstate ie. $|U_{\alpha i}|^2 = 1/3, \forall\, \alpha$ and for some i (to fit phenomenology).
- Everything is now determined in terms of fixed ν mass-squared differences (we first need to find q to fit the known spectrum).

Solution

- First allow masses to float to examine properties of trajectory as fn. of q.
- Has 3 poles, each corresponding to a permutation of the tribimaximal form, and (in limit as we approach each pole) has a single ν mass degeneracy.
- As we approach "correct" pole, Δm^2_{21} tends to zero (not bad, cf. data). $\Longrightarrow$ smallness of $\sin\theta_{13}$ related to smallness of $\Delta m^2_{21}/\Delta m^2_{31}$.
- Of course, $\Delta m^2_{21}/\Delta m^2_{31}$ is known (roughly) which fixes q and the mixings:

$$(|U_{l\nu}|^2) = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} 0.662 & 0.333 & 0.005 \\ 0.219 & 0.333 & 0.448 \\ 0.120 & 0.333 & 0.547 \end{pmatrix} \end{array} \simeq \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{pmatrix} \end{array}.$$

– fully compatible with experiment.

Where has Flavour Symmetry Gone?

It is spontaneously broken by the solution. There are three solutions, one corresponding to the trimaximally mixed column being in each of the three possible positions.

This is also a generic feature of our approach.

What Have we Actually Done?

- Our approach is not an *extension* of the νStandard Model – there are no new particles, no new symmetries and no new interactions (boring!?)
- It works entirely *within* the existing νSM:
  - it relies on existing transformation properties of the νSM, merely requiring that new BSM constraints respect them.
  - it utilises existing degrees-of-freedom (the Yukawa coupling "constants"), merely treating them instead like dynamical variables (determined by a conventional variational principle).
- If anything, it is a *contraction* of the νSM – we reduce the number of free parameters in the lepton sector from $6 + 4$ to $6 + 1$.
- Though original in the context of flavour physics, our procedure builds on tried and trusted traditions of our field - looks like the principle of least action.

The Future

- We do not consider the successful invariant

$$\mathcal{A} = \mathrm{Tr}C^2 + q\,\mathrm{Tr}(C.[L^2, N])$$

  to be the last word.

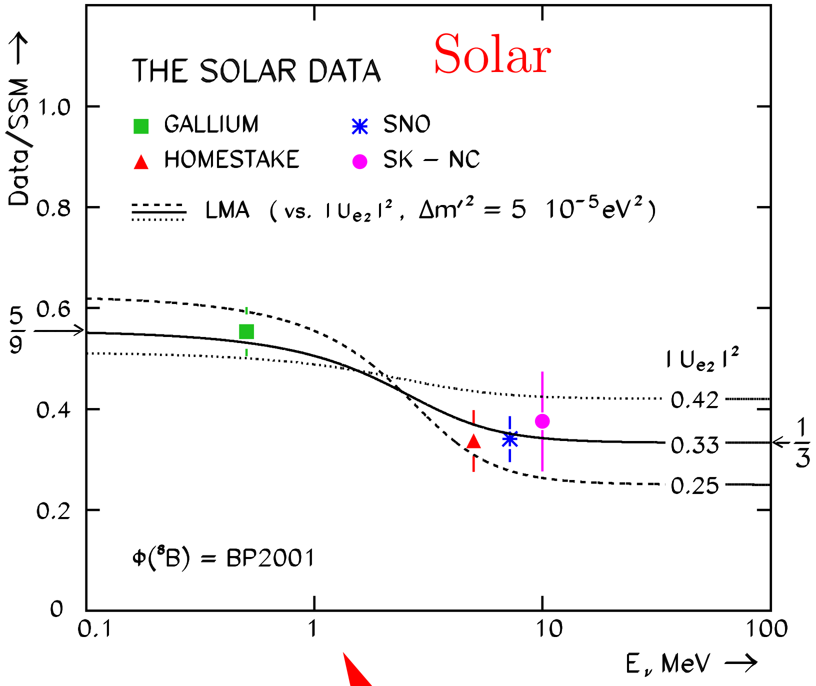
  – does not accommodate CP violation

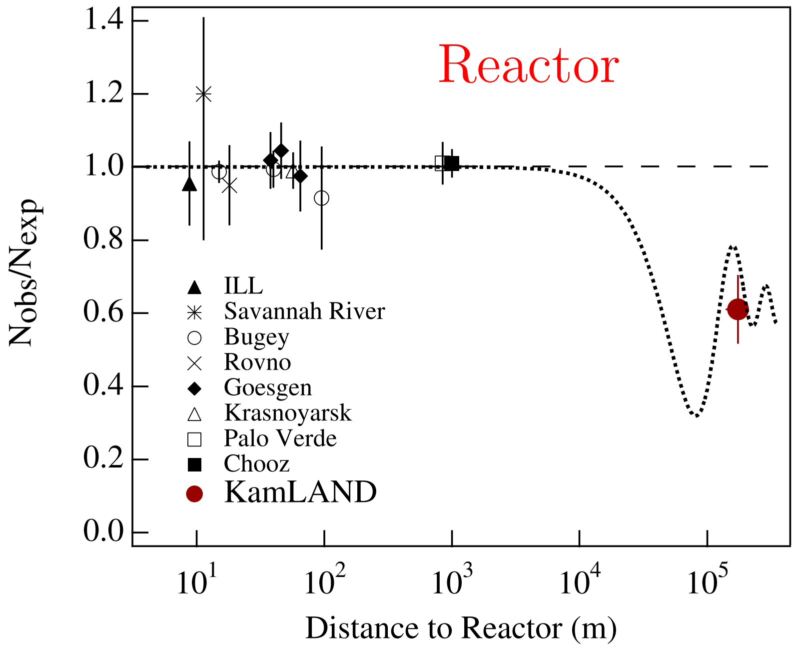
  – extremisation is performed at fixed masses.

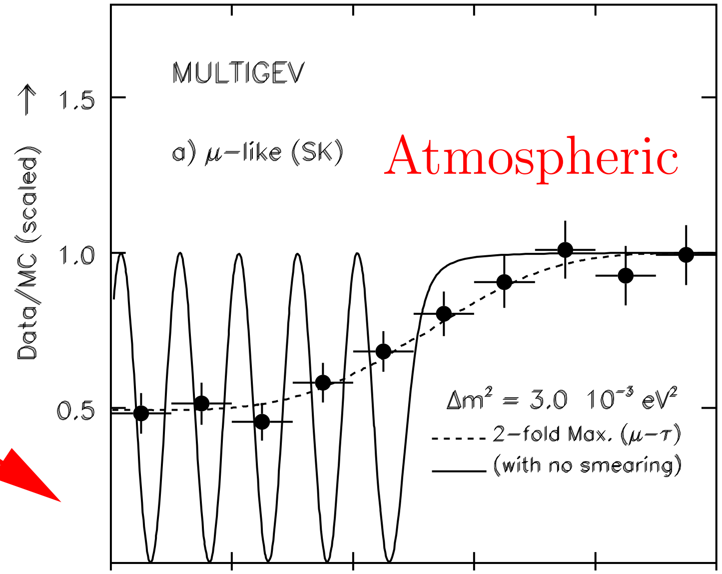
- Effort currently underway to incorporate CP violation.
- Would like to find an invariant which can be extremised without using LMs $\rightarrow$ predict masses also.
- Need to consider Majorana neutrinos and interaction with Seesaw mechanism
- Need to find successful application of our procedure to the quarks.

Back-up Slides

Summary of Neutrino Oscillation Data







$$
|U| \sim \begin{array}{c} \\ e \\ \mu \\ \tau \end{array}
\begin{array}{c}
\begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\
\begin{pmatrix}
0.82 & 0.58 & < 0.17 \\
0.41 & 0.58 & 0.71 \\
0.41 & 0.58 & 0.71
\end{pmatrix}
\end{array}
$$

Nomenclature for Particular Bases

- "Mass basis" - (specified by the "m" superscript) is unique, up to a bunch of possible re-phasings (which I may come back to).
- "Weak (or gauge) basis" - Any basis in which the weak interaction is diagonal (eg. those specified by the un-superscripted fields, and/or by the "primed" fields). WB is not unique - different ones are all related by the U transformation. Any WB can be used to show gauge invariance, etc.
- "Flavour basis" - a particular choice of WB in which the charged lepton mass matrix, L', is diagonal (while N' is not). Corresponds to a special choice of U, and is unique (up to the definition of the phases of the charged lepton fields). It is the basis in which all 6 leptons carry one of the flavour indices e, μ, τ. It is the *conventional* basis.
- "Neutrino Mass basis" - the particular WB for which N' is diagonal and L' is not. Is not often used.

Solution Method

- Covariance allows to choose any basis in which to solve $\Longrightarrow$ take flavour basis
- L is diagonal (N not) $\Longrightarrow$ off-diagonal elements of ∂_L Eqn. $= 0$.
- Further, choose phases such that imaginary parts of N all have equal moduli:

$$N = \begin{pmatrix} a & z & y \\ z & b & x \\ y & x & c \end{pmatrix} + id \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

- Variables a, b, c, x, y, z and d encode the three neutrino masses and the four mixing parameters

Try Instead $\mathbf{Tr}\boldsymbol{C}^{\mathbf{2}}$

Constraints are now:

$$
\begin{aligned}
(-\partial_L \mathrm{Tr}\, C^2/2)^T = +[N, C] &= \lambda_{L0} + \lambda_{L1}\, L + \lambda_{L2}\, L^2 \\
(-\partial_N \mathrm{Tr}\, C^2/2)^T = -[L, C] &= \lambda_{N0} + \lambda_{N1}\, N + \lambda_{N2}\, N^2.
\end{aligned}
$$

Get CP-conserving solution:

$$
x = \pm\sqrt{(a-b)(c-a)E} \qquad E = \frac{(m_e - m_\mu)(m_\tau - m_e)}{(m_e + m_\mu - 2m_\tau)(m_\tau + m_e - 2m_\mu)}
$$

$$
y = \pm\sqrt{(b-c)(a-b)M} \qquad M = \frac{(m_\mu - m_\tau)(m_e - m_\mu)}{(m_\mu + m_\tau - 2m_e)(m_e + m_\mu - 2m_\tau)}
$$

$$
z = \pm\sqrt{(c-a)(b-c)T} \qquad T = \frac{(m_\tau - m_e)(m_\mu - m_\tau)}{(m_\tau + m_e - 2m_\mu)(m_\mu + m_\tau - 2m_e)}.
$$

with a, b, c fixed by the ν masses.

Solution for $\mathbf{Tr}\boldsymbol{C}^{\mathbf{2}}$

Interestingly, the observed ν mass-squared difference hierarchy is correctly obtained setting $\frac{b-a}{a-c} = 1$ ($\Longrightarrow \frac{\Delta m^2_{21}}{\Delta m^2_{31}} = 0.03$).

The mixing matrix $\rightarrow$

$$
(|U_{MNS}|^2) \; = \; \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} .33333 & .33333 & .33333 \\ .17079 & .16257 & .66663 \\ .49587 & .50409 & .00003 \end{pmatrix} \end{array} \simeq \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/6 & 1/6 & 2/3 \\ 1/2 & 1/2 & 0 \end{pmatrix} \end{array} .
$$

which is inconsistent with experiment.

BUT it is a simple permutation of the (viable) tribimaximal mixing matrix!

What Next?

We tried the obvious next step, ie. a linear combination:

$$\mathcal{A} = i\mathrm{Tr}C^3 + r\,\mathrm{Tr}C^2.$$

where r is a JI constant of dimensions m^2.

The solution gives a trajectory of mixing matrices, as a function of r (and the unknown offset of the neutrino masses).

Has some very interesting properties (see below), including viable mixing matrices and CP violation.

However, so far, have failed to find a solution which simultaneously fits observed masses and mixings.

Complete characterisation of solution space is work in progress.

Complete Set of Flavour-Symmetric, Invariant Measures of Mixing

$\exists$ four observables of mixing $\implies$ must be possible to find four independent flavour-symmetric invariant measures of mixing. We extended the set as follows:

Commutators of powers of L and N are independent measures of mixing:

$$C_{ij} \equiv [L^i, N^j].$$

Complete set is obtained for $i, j \leq 2$ (higher powers of L and N may be reduced to functions of lower powers and their traces, using the characteristic equation).

While these are not invariants, it is easy to construct (CP-conserving) invariants by analogy with $\mathrm{Tr}C_{11}^2$ considered already:

$$Q_{mn} \equiv \mathrm{Tr}C_{ij}.C_{kl}$$

where $m = i + k - 1$ and $n = j + l - 1$. There are exactly 9 of these, any four of which are independent.