Unitarity Triangles and Neutrino Phenomenology

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Outline of Talk

- Introduction to UTs - the general case
- Relevance of UTs to neutrino oscillations
- The trend of oscillation data: TBM mixing
- New approximate parameterisation of MNS matrix
- Simplified unitarity triangles for neutrinos

Work done in collaboration with Bill Scott and James Bjorken (hep-ph/0511201)

Introduction to UTs

Well-known that MNS matrix is unitary, ie.

$$UU^\dagger = I = U^\dagger U$$

1st equality $\implies \sum_i U_{\alpha i} U^*_{\beta i} = 0,\ \alpha \neq \beta$ (dot product of rows)

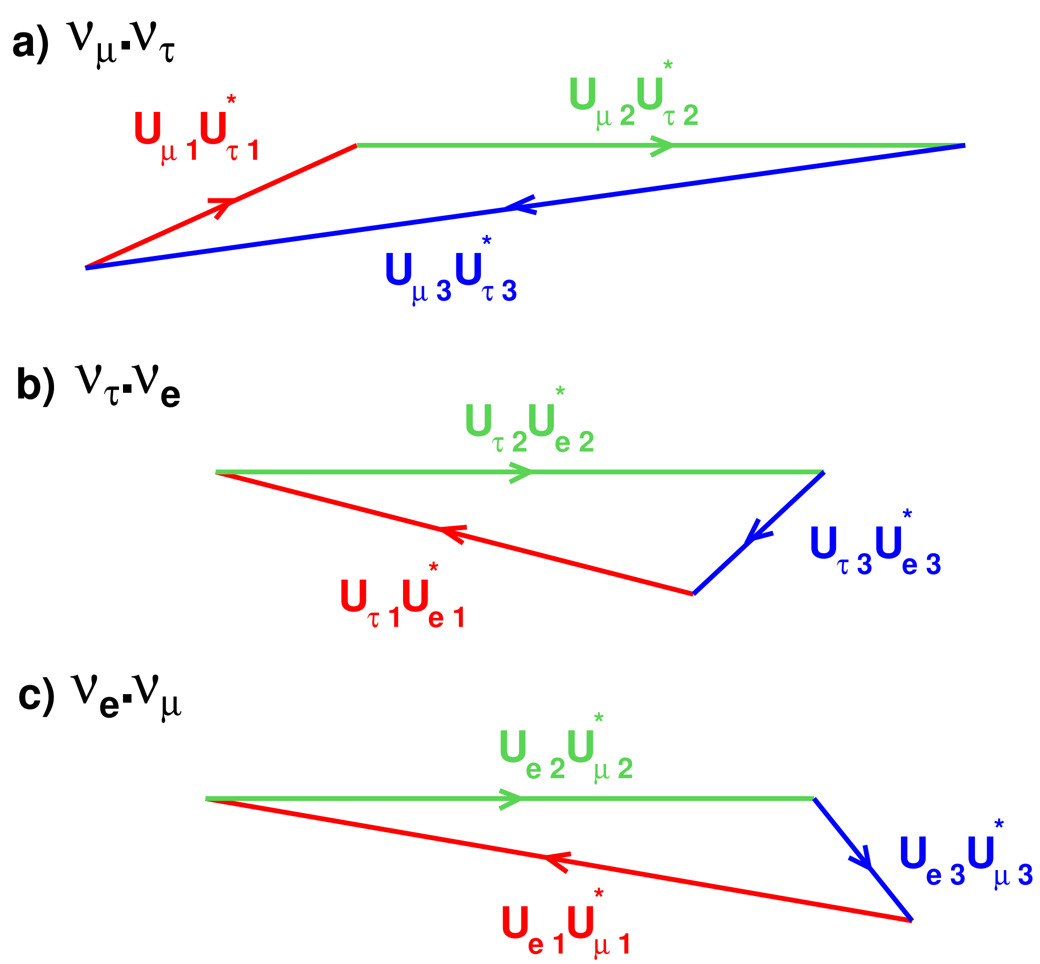
2nd equality $\implies \sum_\alpha U_{\alpha i} U^*_{\alpha j} = 0,\ i \neq j$ (dot product of columns)

Each implies a different triangle in the complex plane (6 altogether).

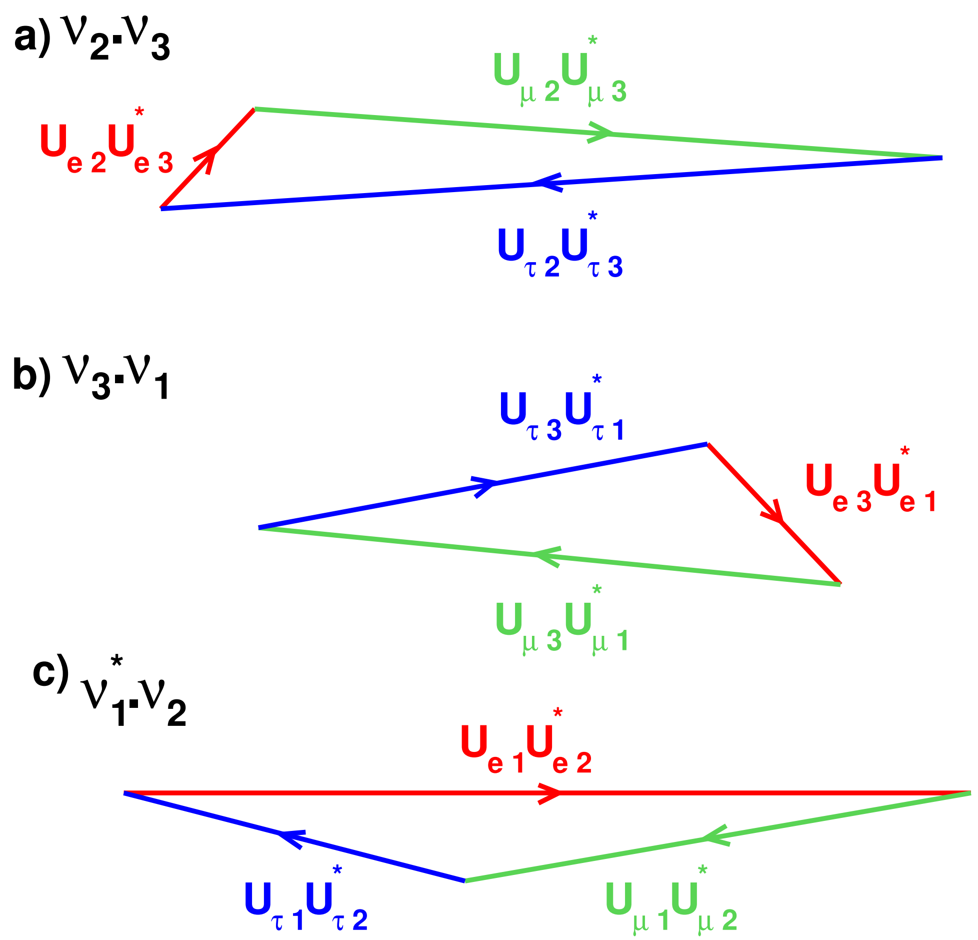
Analogous to quark case.

The Six UTs

ROW-BASED UNITARITY TRIANGLES



COLUMN-BASED UNITARITY TRIANGLES (x $\sqrt{3}$)



Tri-Angles

Each row-based triangle shares one of its angles with each column-based triangle and vice-versa, $\Longrightarrow$ $\exists$ only 9 distinct angles.

A simple and unique notation for the angles is to use two indices (like MNS matrix elements): $\phi_{\alpha i}$

Here, for row-based triangles, α denotes the triangle (ie. $\nu_\beta.\nu_\gamma$) while i denotes the side opposite the angle in question (ie. $U_{\beta i}U^*_{\gamma i}$)

For column-based triangles, interpretations of α and i are reversed.

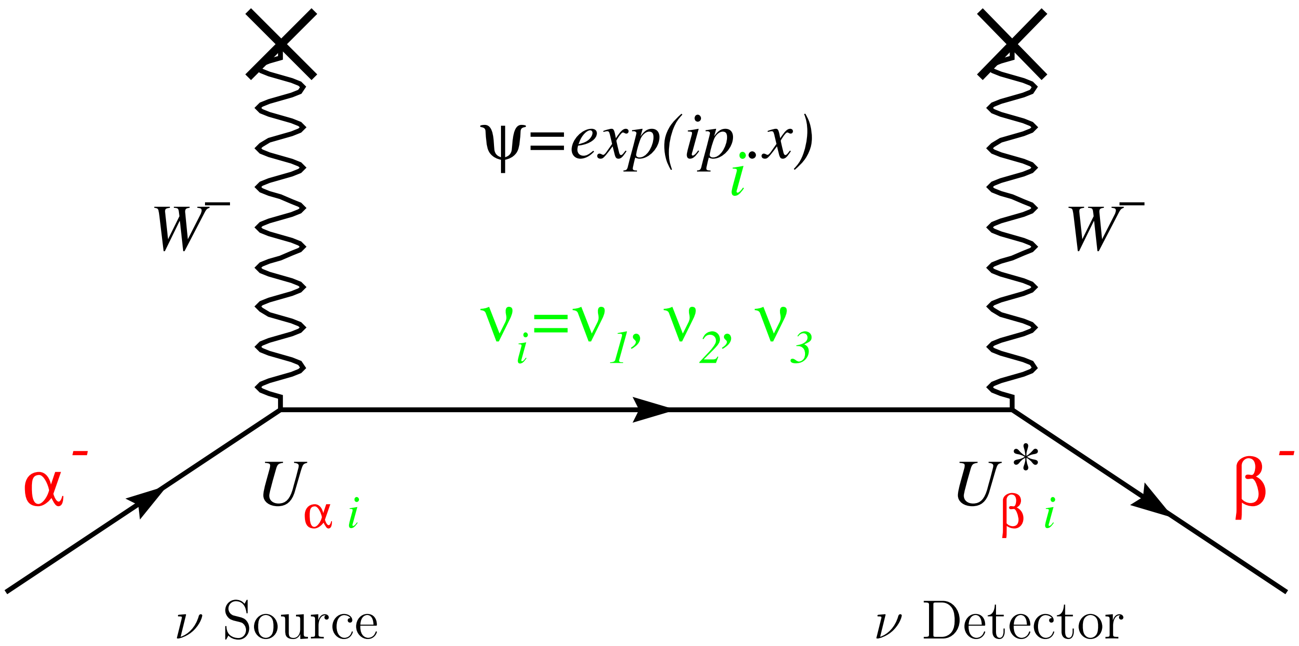
The $\phi_{\alpha i}$ form a matrix which happens to be a magic square (each row and column sums to π), $\Longrightarrow$ $\exists$ exactly 4 independent angles.

Contains equivalent information to the MNS matrix.

Triangles are "Observables" of $\boldsymbol{\nu}$ Oscillations

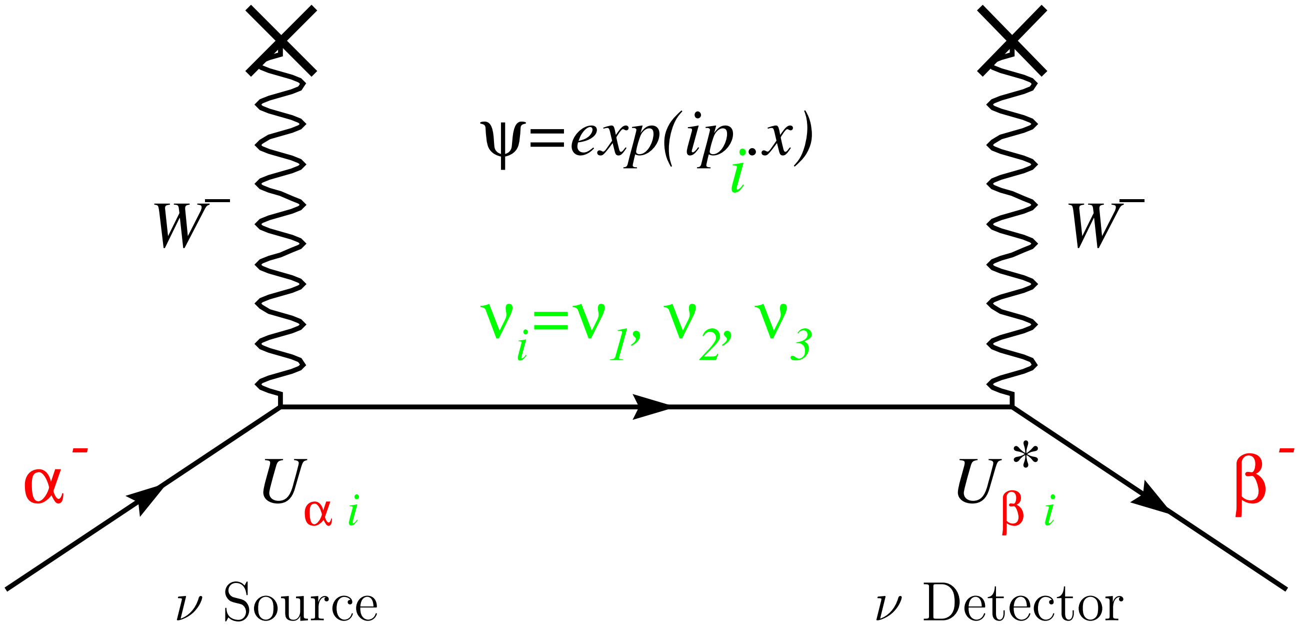
- Phases of mass and flavour eigenstates are not fixed $\Longrightarrow$ rows and columns of MNS matrix may be rephased at will.
- Corresponds to rotating triangles $\Longrightarrow$ orientation of triangles is unphysical.
- BUT side lengths, opening angles and areas correspond to observables.

$\boldsymbol{\nu}$-Oscillations



$$p_i = \sqrt{E^2 - m_i^2} \simeq E - \frac{m_i^2}{2E} \qquad\qquad \alpha, \beta = e, \mu, \tau \qquad i = 1, 2, 3$$

ν-Oscillations



$$p_i = \sqrt{E^2 - m_i^2} \simeq E - \frac{m_i^2}{2E} \qquad\qquad \alpha, \beta = e, \mu, \tau \qquad i = 1, 2, 3$$

Amplitude is:

$$A_{\alpha\beta}(x) = \sum_{i=1,3} U_{\alpha i} U^*_{\beta i} \exp\{i(E - \frac{m_i^2}{2E})x\} \qquad \text{Define } \Delta m^2_{ij} = m_i^2 - m_j^2$$

$$\sim \quad U_{\alpha 1} U^*_{\beta 1} + U_{\alpha 2} U^*_{\beta 2} \exp\{-i\frac{\Delta m^2_{21} x}{2E}\} + U_{\alpha 3} U^*_{\beta 3} \exp\{-i\frac{\Delta m^2_{31} x}{2E}\}$$

Coefficients, $U_{\alpha i} U^*_{\beta i}$ $(\alpha \neq \beta)$, are exactly the sides of the (row-based) UTs.

Appearance Probabilities

Conventionally:

$$P(\nu_\alpha \to \nu_\beta) = -4 \sum_{i,j\ cycl} \mathrm{Re}(U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}) \sin^2 \Delta_{ij} - 8\, J \sin\Delta_{12} \sin\Delta_{23} \sin\Delta_{31}$$

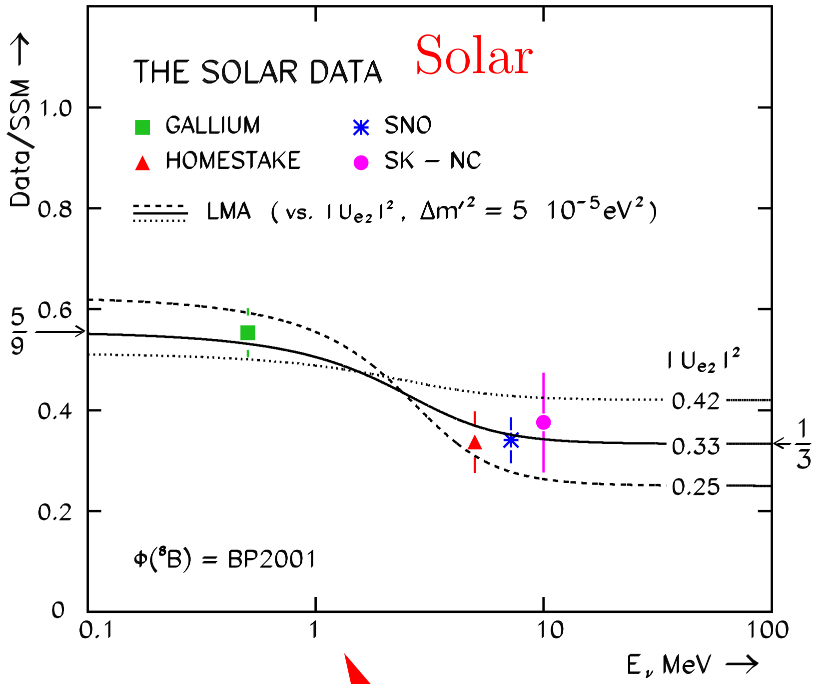
where $\Delta_{ij} \equiv \Delta m^2_{ij} L/4E$, and $J = \mathrm{Im}(U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j})$ is twice the area of each unitarity triangle.

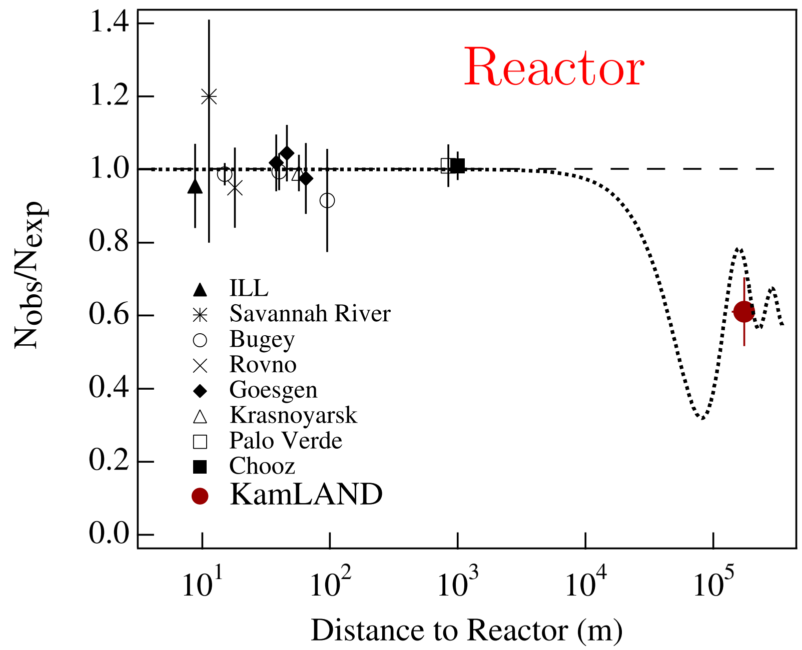
Less conventionally:

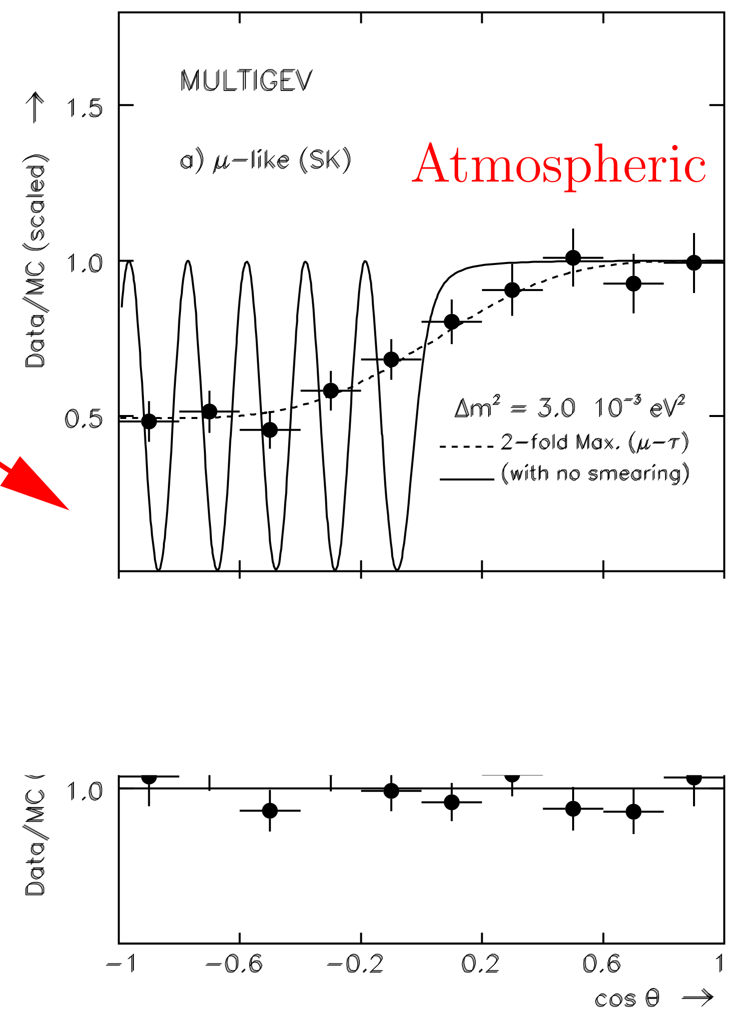
$$\frac{1}{4} P(\nu_\alpha \to \nu_\beta) = |U_{\alpha 2} U^*_{\beta 2}|^2 \sin^2\Delta_{12} + |U_{\alpha 3} U^*_{\beta 3}|^2 \sin^2\Delta_{31} + 2|U_{\alpha 2} U^*_{\beta 2}||U_{\alpha 3} U^*_{\beta 3}| \sin\Delta_{12} \sin\Delta_{31} \cos(\Delta_{23} + \phi_{\gamma 1})$$

where $\phi_{\gamma 1} \equiv \mathrm{Arg}(U_{\alpha 2} U^*_{\beta 2} U^*_{\alpha 3} U_{\beta 3})$ is an opening angle of the $\nu_\alpha.\nu_\beta$ UT.

Summary of Neutrino Oscillation Data







$$
(|U|^2) \sim \begin{array}{c} \\ e \\ \mu \\ \tau \end{array}
\begin{array}{c}
\begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\
\begin{pmatrix} . & 0.31 \pm 0.04 & < 0.012 \\ . & . & 0.50 \pm 0.11 \\ . & . & . \end{pmatrix}
\end{array}
$$

NB. 68% CL!

TriBimaximal Mixing

So, in leading approximation:

$$(|U_{MNS}|^2) \simeq \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \!\!\! \begin{array}{c} \begin{array}{ccc} \quad\nu_1 & \quad\nu_2 & \quad\nu_3 \end{array} \\ \left(\begin{array}{ccc} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{array} \right) \end{array} .$$

Most striking feature is arguably the trimaximally mixed ν_2 mass eigenstate.

Ensured by "Democracy", ie. symmetry of ν mass matrix under transformation

$$M \rightarrow U_D M U_D^\dagger \text{ where } U_D = \mathrm{e}^{i\alpha D} \text{ and } D = \left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right)$$

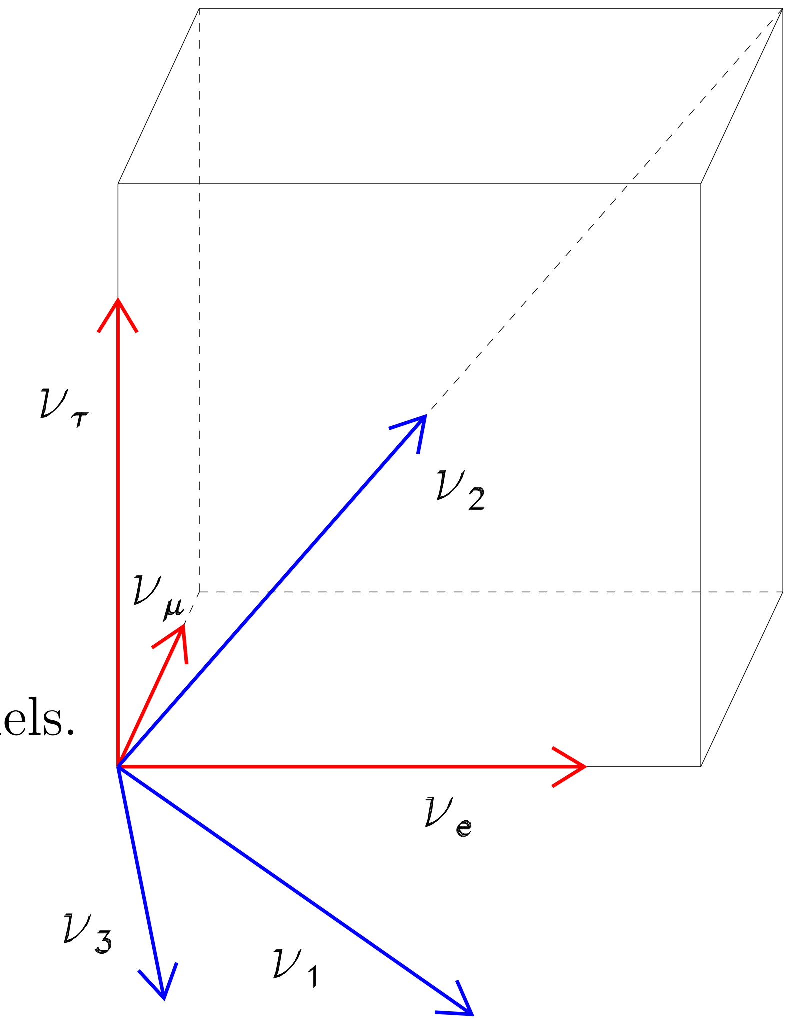
ie. $[M, D] = 0$. In addition, TBM is defined by two other symmetries: CP and $\mu - \tau$ symmetry (hep-ph/0403278).

Discussion of TBM

Redolent of symmetries in other respects:

- Taking ν_e, ν_μ, ν_τ to define the orientation of a cube, ν_2 lies along its body diagonal
- Same coefficients form the $M = 0$ subset of the $j \times j = 1 \times 1$ set of Clebsch-Gordan Coeffts.

Several authors have built TBM into models.



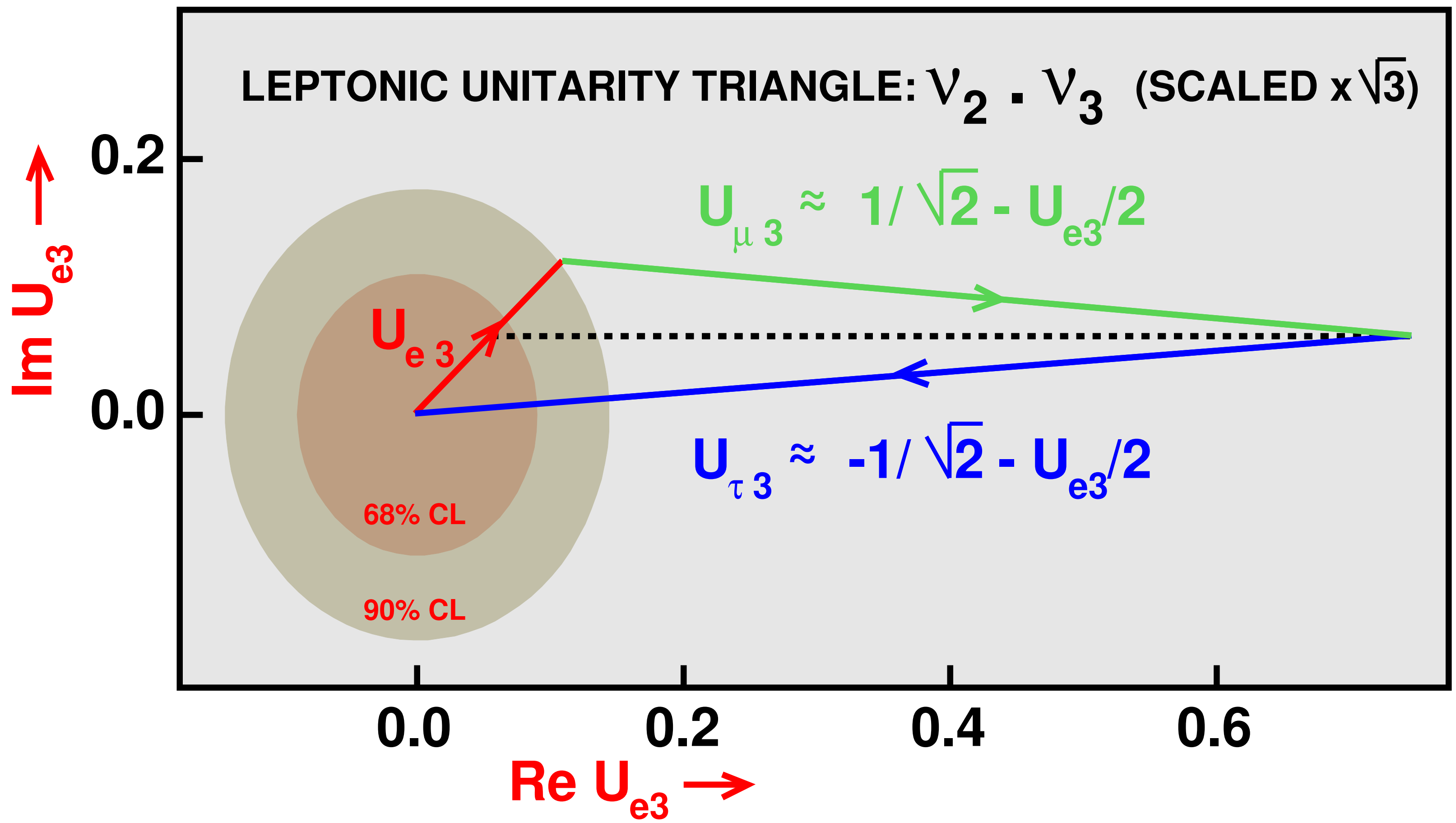
Generalisation to Non-zero U_{e3}

Want to find a useful generalisation of TBM to non-zero U_{e3} while maintaining trimaximally mixed ν_2, ie.

$$
\begin{aligned}
U_{MNS} &\simeq \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} C & 0 & \sqrt{\frac{3}{2}}U_{e3} \\ 0 & 1 & 0 \\ -\sqrt{\frac{3}{2}}U_{e3}^* & 0 & C \end{pmatrix} \\
&= \begin{pmatrix} \frac{2}{\sqrt{6}}C & \frac{1}{\sqrt{3}} & U_{e3} \\ -\frac{1}{\sqrt{6}}C - \frac{\sqrt{3}}{2}U_{e3}^* & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}C - \frac{1}{2}U_{e3} \\ -\frac{1}{\sqrt{6}}C + \frac{\sqrt{3}}{2}U_{e3}^* & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}C - \frac{1}{2}U_{e3} \end{pmatrix} \\
&\simeq \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & U_{e3} \\ -\frac{1}{\sqrt{6}} + \frac{\sqrt{3}}{2}U_{e3}^* & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{1}{2}U_{e3} \\ -\frac{1}{\sqrt{6}} - \frac{\sqrt{3}}{2}U_{e3}^* & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} - \frac{1}{2}U_{e3} \end{pmatrix}.
\end{aligned}
$$

where $C = \sqrt{1 - \frac{3}{2}|U_{e3}|^2} \simeq 1$. Now triangles become especially simple.

Eg. $\nu_2.\nu_3$ Triangle



Symmetries in this parameterisation

With

$$U_{MNS} \quad = \quad \begin{pmatrix} \frac{2}{\sqrt{6}}C & \frac{1}{\sqrt{3}} & U_{e3} \\ -\frac{1}{\sqrt{6}}C - \frac{\sqrt{3}}{2}U_{e3}^* & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}C - \frac{1}{2}U_{e3} \\ -\frac{1}{\sqrt{6}}C + \frac{\sqrt{3}}{2}U_{e3}^* & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}C - \frac{1}{2}U_{e3} \end{pmatrix}$$

Democracy is respected.

Obviously, $\mathrm{Im}(U_{e3})$ parameterises CP violation:

$$|J_{CP}| = \frac{C\,\mathrm{Im}(U_{e3})}{3\sqrt{2}}.$$

while $\mathrm{Re}(U_{e3})$ parameterises violation of $\mu - \tau$ reflection symmetry (simultaneous $\mu - \tau$ exchange and CP transformation).

Relationship to the Standard Parameterisation

Here, $|U_{e3}|$ corresponds exactly to standard one:

$$|U_{e3}| = \sin\theta_{13}$$

while our phase convention differs: $\delta' \equiv -\mathrm{Arg}(U_{e3})$ given by

$$\begin{aligned} \sin\delta' &= \sin 2\theta_{23} \sin\delta \\ &\simeq \sin\delta \end{aligned}$$

Get equality in 2 special cases of exact $\mu - \tau$ symmetry, and exact CP conservation.

Remaing two degrees of freedom, θ_{12} and θ_{23}, fixed in terms of θ_{13} and δ:

$$\begin{aligned} \sin\theta_{12} &= \frac{1}{\sqrt{3}\cos\theta_{13}} \\ \cot 2\theta_{23} = \tan 2\,(\frac{\pi}{4} - \theta_{23}) &= \frac{\sqrt{2 - 3\sin^2\theta_{13}}\sin\theta_{13}\cos\delta}{\cos 2\theta_{13}} \\ \Rightarrow \frac{\pi}{4} - \theta_{23} &\simeq \frac{1}{\sqrt{2}}\mathrm{Re}(U_{e3}) \end{aligned}$$

where we have used the small U_{e3} approximation in Eq. 1. (Fact that $\mathrm{Re}(U_{e3}) = 0$ corresponds to $\mu - \tau$ symmetry is explicit here: $\theta_{23} = \frac{\pi}{4}$ in this case.

$\boldsymbol{\nu}$ Oscillation Probabilities in this Approximation

ν_e appearance probability at leading (second) order:

$$P(\nu_\mu \to \nu_e) \simeq 2|U_{e3}|^2 \sin^2\Delta_{31} + \frac{4}{9}\alpha^2 {\Delta_{31}}^2 + \frac{8\alpha|U_{e3}|}{3\sqrt{2}}\Delta_{31} \sin\Delta_{31} \cos\left(\Delta_{31} + \delta'\right)$$

where $\delta' \simeq \delta \simeq \phi_{\tau 1}$ at this order.

ν_e disappearance probability at a (not too distant) reactor (to leading (second) order):

$$1 - P(\overline{\nu}_e \to \overline{\nu}_e) \simeq 4|U_{e3}|^2 \sin^2\Delta_{31} + \frac{8}{9}\alpha^2 {\Delta_{31}}^2.$$

Summary

- Data is described satisfactorily, at present, by the TBM scheme.
- Next generation of experiments is sensitive to U_{e3}
- Have generalised TBM to accommodate non-zero U_{e3}, while retaining as much of the original symmetry as possible.
- Unitarity triangles then take a very simple form, as do ν apperance and survival probabilities.
- This approximate parameterisation should serve useful, as long as we are sensitive only to the next order of approximation.
- The sensitivity at ν factories will (hopefully) require to go even beyond this level of approximation.