(4) Andrew Sturgess

(1) two important relations

$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

from clifford algebra, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

Use these together;

$[\gamma^\mu, \gamma^\nu] = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu = \gamma^\mu\gamma^\nu - (2g^{\mu\nu} - \gamma^\mu\gamma^\nu)$

$= 2\gamma^\mu\gamma^\nu - 2g^{\mu\nu}$

Hence $i\sigma^{\mu\nu} = -(\gamma^\mu\gamma^\nu - g^{\mu\nu})$

$= g^{\mu\nu} - \gamma^\mu\gamma^\nu$ [OR $\gamma^\nu\gamma^\mu - g^{\mu\nu}$]

Now calculate

$\bar{u}(p) i\sigma^{\mu\nu}(p_f - p_i) u(p)$ → this has a similar form to our final term

$= \bar{u}(p_f)[(\gamma^\nu\gamma^\mu - g^{\mu\nu})p_f - (g^{\mu\nu} - \gamma^\mu\gamma^\nu)p_i]u(p_i)$

$= \bar{u}(p_f)[\gamma^\nu p_{f\nu}\gamma^\mu - p_f^\mu - p_i^\mu + \gamma^\mu\gamma^\nu p_{i\nu}]u(p_i)$

$= \bar{u}(p_f)[\gamma p_f \gamma^\mu - (p_f + p_i)^\mu + \gamma^\mu \gamma p_i]u(p_i)$ → contraction

Now want to simplify using dirac equations for spinor and adjoint

$(\gamma \cdot p - m)u(p) = 0 \implies \gamma \cdot p u = Mu$

$\bar{u}(\gamma \cdot p' - m) = 0 \rightarrow \bar{u}\gamma \cdot p = \bar{u}M$

So the expression becomes

$= \bar{u}(p_f)[M\gamma^\mu - (p^f + p^i)^\mu + \gamma^\mu M]u(p_i)$

$= \bar{u}(p_f)[2M\gamma^\mu - (p_f + p^i)^\mu]u(p_i)$ ⊛

Want to now rearrange; note:

$\bar{u}(p_f) i\sigma^{\mu\nu}(p_f - p_i)_\nu u(p_i) = -\bar{u}(p_f)\left[\frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\right](p_f - p_i)_\nu u(p_i)$ (1)

⊛ $= 2M\bar{u}(p_f)\gamma^\mu u(p_i) - \bar{u}(p_f)(p_f + p_i)^\mu)\bar{u}(p_i) =$ (1)

Hence $\bar{u}(p_f)\gamma^\mu u(p_i) = \frac{1}{2M}\bar{u}(p_f)\left\{(p_f + p_i)^\mu - \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p_f - p_i)_\nu\right\}u(p_i)$

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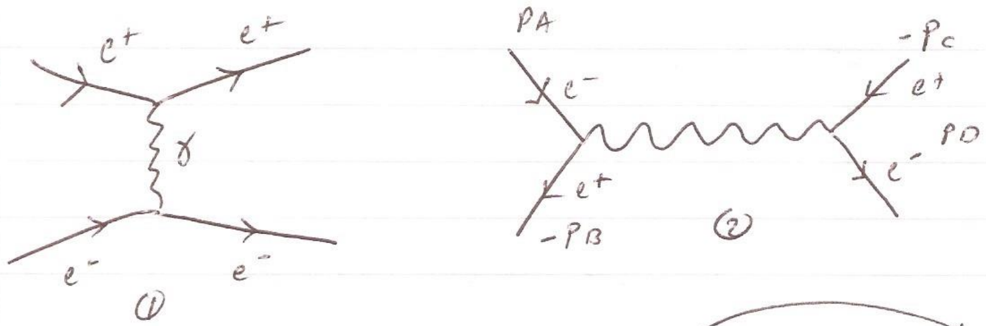
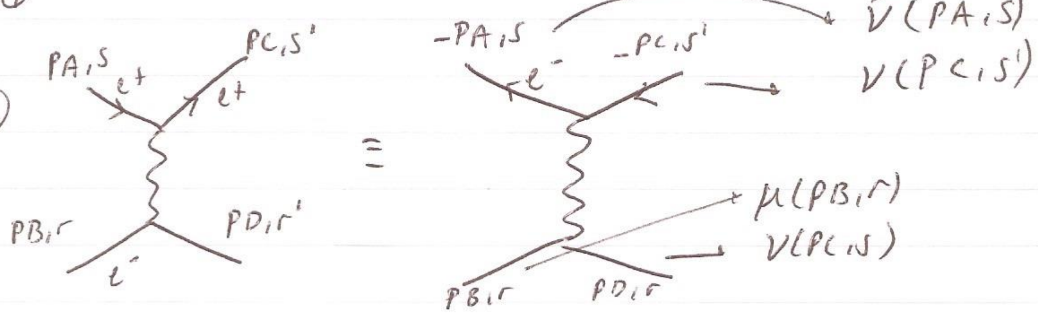
②



Take ①



Hence from this, have

$$-iM_{①} = v(p_C, s')(ie\gamma^{\mu})\bar{v}(p_A, s)\left(\frac{-ig^{\mu\nu}}{q^2}\right)\bar{\mu}(p_D, r')(ie\gamma^{\mu})\mu(p_B, r)$$

Also have a contribution ② → use crossing symetry!

$$-iM_{②} = v(p_B, s')(ie\gamma^{\mu})\bar{v}(p_A, s)\left(\frac{-ig^{\mu\nu}}{q^2}\right)\bar{\mu}(p_D, r')(ie\gamma^{\mu})\mu(p_C, r)$$

$$\cancel{|M|^2 = \frac{e^4}{q^4}\left[v(p_C, s')\gamma^{\sigma}\bar{v}(p_A, s)\bar{\mu}(p_D, r')\gamma_{\sigma}\mu(p_B, r)\right]}$$

$$|M|^2 = |M_{①} + M_{②}|^2 = M_1^2 + M_2^2 + 2M_1 \cdot M_2$$
$$= M_1^* M_1 + M_2^* M_2 + 2M_1 \cdot M_2$$

so
$$M_1^* M_1 = \left[v(p_C, s')\gamma^{\sigma}\bar{v}(p_A, s)\bar{\mu}(p_D, r')\gamma_{\sigma}\mu(p_B, r)\right]$$
$$\left[v(p_C, s')\gamma^{\rho}\bar{v}(p_A, s)\bar{\mu}(p_D, r')\gamma_{\rho}\mu(p_B, r)\right]^*$$

Exploit the conjugate as in the lecture notes

$$(\bar{a}\gamma^{\mu}b)^{+} = b^{+}\gamma^{\mu +}\gamma^{0}a^{+} = \bar{b}\gamma^{\mu}a$$

$$M_1 M_1^* = \left.\begin{array}{l}\left[v(p_C, s')\gamma^{\sigma}\bar{v}(p_A, s)\bar{\mu}(p_D, r')\gamma_{\sigma}\mu(p_B, r)\right] \times \\ \left[\bar{\mu}(p_B, r)\gamma^{\rho}\mu(p_D, r)\bar{v}(p_A, s)\gamma_{\rho}v(p_C, s)\right]\end{array}\right\} L^{\sigma\rho}_{(e^+)}$$

$$M_2 M_2^* = \left.\begin{array}{l}\left[v(p_B, s')\gamma^{\sigma}\bar{v}(p_A, s)\bar{\mu}(p_D, r')\gamma_{\sigma}\mu(p_C, r)\right] \times \\ \left[\bar{\mu}(p_C, r)\gamma^{\rho}\mu(p_D, r')\bar{v}(p_B, s)\gamma_{\rho}v(p_A, s')\right]\end{array}\right\} L^{\sigma\rho}_{(e^-)}$$

$$\overbrace{2M_1 \cdot M_2 \;\sum\left[v(p_C,s')\gamma^\sigma \bar{v}(p_A,s)\bar{\mu}(p_D,r')\gamma_\sigma \mu(p_B,r)\right] + \left[v(p_B,s')\gamma^\rho \bar{v}(p_A,s)\bar{\mu}(p_D,r')\gamma_\rho \mu(p_C,r)\right]}^{2L^{\sigma\rho}_{(e^+e^-)}}$$

$$\text{Hence } |M|^2 = \frac{e^4}{q^4}\left[L^{\sigma\rho}_{(e^+)} + L^{\sigma\rho}_{(e^-)} + 2L^{\sigma\rho}_{(e^+e^-)}\right]$$

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(10) Simplest way to solve, start for one process, do crossing symmetry, then determine crossterms

$$|M|^2 = \underset{\substack{\text{one}\\ \text{term}}}{M_1^2} + \underset{\substack{\text{obtain}\\ \text{by}\\ \text{sym}}}{M_2^2} + \underset{\text{dot product}}{2M_1 M_2}$$

Use result of $\underline{\underline{e^-\mu^- \to e^-\mu^-}}$; NOTE change of relation due to having $\bar{v}$ and v

from lecture notes, use Trace theorems

$$|M|^2 = \frac{e^4}{q^2}\underbrace{Tr\left[(\not{p}_C - m)\gamma^\sigma(\not{p}_A - m)\gamma^\rho\right]}_{①} Tr\left[(\not{p}_D + m)\gamma_\sigma(\not{p}_B - m)\gamma_\rho\right]$$

$$① = \not{p}_C\gamma^\sigma\not{p}_A\gamma^\rho - \underbrace{\not{p}_C\gamma^\sigma m\gamma^\rho - m\gamma^\sigma\not{p}_A\gamma^\rho}_{=0 \text{ due to odd numbers!}} + m_e^2\gamma^\sigma\gamma^\rho$$

$$\text{Hence } ① = Tr[\not{p}_C\gamma^\sigma\not{p}_A\gamma^\rho] + Tr[m_e^2\gamma^\sigma\gamma^\rho]$$

$$Tr[\not{p}_C\gamma^\sigma\not{p}_A\gamma^\rho] = Tr[\gamma^\mu p_{C\mu}\gamma^\sigma\gamma^\nu p_{A\nu}\gamma^\rho] = p_{C\mu}p_{A\nu}Tr[\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho]$$

$$= m_e^2\frac{1}{2}Tr[\gamma^\sigma\gamma^\rho + \gamma^\rho\gamma^\sigma] = m_e^2\frac{1}{2}(2g^{\mu\nu})Tr[\hat{\mathbb{I}}] = 4m_e^2 g^{\sigma\rho}$$

Use anticommrelations + symmetries

$$① = 4\left[p_A^\sigma p_C^\rho - g^{\sigma\rho}(p_A \cdot p_C) + p_A^\sigma p_C^\rho\right]$$

$$\text{Hence } |M|^2 = M_1^2$$

$$= \underset{\substack{\text{spin}\\ \text{states}}}{\frac{e^4}{4q^4}} \cdot \left[16\left\{p_A^\sigma p_C^\rho + p_A^\rho p_C^\sigma - [(p_A \cdot p_C) - m_e^2]g^{\sigma\rho}\right\} \times \left\{p_B^\sigma p_D^\rho + p_B^\rho p_D^\sigma - [(p_B \cdot p_D) - m_e^2]g^{\sigma\rho}\right\}\right]$$

many terms cancel!

Using $g^{\mu\nu}g_{\mu\nu} = 4$

$$= \frac{4e^4}{q^4}\Big[(p_A^\sigma p_C^\rho)(p_B^\sigma p_D^\rho) + (p_A^\rho p_C^\sigma p_B^\sigma p_D) + (p_B^\sigma \cdot p_D^\rho)(p_A^\sigma p_C^\rho) + (p_B^\rho \cdot p_D^\rho)(p_A^\sigma p_C^\sigma) - m_e^2(p_B \cdot p_D) - m_e^2(p_A \cdot p_C)\Big] - 2m_e^4$$

$$= \frac{8e^4}{q^4}\Big[(p_A \cdot p_B)(p_C \cdot p_D) + (p_A \cdot p_D)(p_B \cdot p_C) - m_e^2(p_B \cdot p_D) - m_e^2(p_A \cdot p_C) - 2m_e^4\Big]$$

High E limit ; $p >> m_e$

$$= \frac{8e^4}{q^2}\Big[(p_A \cdot p_B)(p_C \cdot p_B) + (p_A \cdot p_D)(p_B \cdot p_C)\Big]$$

Using crossing symmetry $\rightarrow e^+e^- \rightarrow e^+e^-$

$$\left.\begin{aligned} s &= 2p_A \cdot p_B \\ t &= -2p_A \cdot p_C \\ u &= -2p_A \cdot p_D \end{aligned}\right\} \quad \left.\begin{aligned} p_B &\rightarrow -p_C \\ p_C &\rightarrow p_B \end{aligned}\right\} \quad \begin{aligned} s &\rightarrow t \\ t &\rightarrow s \end{aligned}$$

$$|M_1|^2 = 2e^4\left[\frac{t^2+u^2}{s^2}\right]$$

BUT ALSO! $|M_2|^2 = e^-\mu^- \rightarrow e^-\mu^-$ case

$$|M_2^2| = 2e^4\left[\frac{s^2+u^2}{t^2}\right]$$

$$2|M_1 \cdot M_2| =$$

$$S = 2e^4\left[\frac{t^2+u^2}{s^2} + \frac{s^2+u^2}{t^2} + 2\right]$$

NOT SURE?

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you could use Tr theorems on product of γ-matrices