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RQM Assessment: 1. Special Relativity & Lorentz Invariance

1) Show that length defined as $A^2 = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2$ is invariant under Lorentz transformation. [3]

For a Lorentz boost along the A^1 direction,

$$\begin{aligned}
A'^2 &= (A^{0\prime})^2 - (A^{1\prime})^2 - (A^{2\prime})^2 - (A^{3\prime})^2 \\
&= \gamma^2(A^0 - \beta A^1)^2 - \gamma^2(A^1 - \beta A^0)^2 - (A^2)^2 - (A^3)^2 \\
&= \gamma^2(A^0)^2 + \gamma^2\beta^2(A^1)^2 - \cancel{\gamma^2\beta^2 A^0A^1} - \cancel{\gamma^2\beta^2 A^1A^0} \\
&\quad - \gamma^2(A^1)^2 - \gamma^2\beta^2(A^0)^2 + \cancel{\gamma^2\beta^2 A^1A^0} + \cancel{\gamma^2\beta^2 A^0A^1} - (A^2)^2 - (A^3)^2 \\
&= \gamma^2(1-\beta^2)(A^0)^2 - \gamma^2(1-\beta^2)(A^1)^2 - (A^2)^2 - (A^3)^2 \\
&= (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2 \\
&= A^2
\end{aligned}$$

this shows just special case

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2) Show that $g_{\mu\nu}g^{\mu\nu} = 4$. [2]

$$g_{\mu\nu}g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{aligned} &(1)(1) + (0)(0) + (0)(0) + (0)(0) \\ &+ (0)(0) + (-1)(-1) + (0)(0) + (0)(0) \\ &+ (0)(0) + (0)(0) + (-1)(-1) + (0)(0) \\ &+ (0)(0) + (0)(0) + (0)(0) + (-1)(-1) \end{aligned}$$

$$= 1 + 1 + 1 + 1 = 4$$

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3) Consider π^+ and B^+ mesons with momentum $|\vec{p}| = 2$ GeV/c. What are their velocities? With which speed do protons travel in Tevatron ($E = 980$ GeV) and LHC ($E = 7$ TeV) colliders? [4]

$$E = \gamma m, \quad p = \beta\gamma m \quad \therefore \quad \beta = \frac{p}{E} = \frac{p}{\sqrt{p^2 + m^2}}$$

$$\beta_{\pi^+} = \frac{2\,\text{GeV/c}}{\sqrt{(2\,\text{GeV/c})^2 + (140\,\text{MeV/c}^2)^2}} = 0.998$$

$$\beta_{B^+} = \frac{2\,\text{GeV/c}}{\sqrt{(2\,\text{GeV/c})^2 + (5.280\,\text{GeV/c}^2)^2}} = 0.354$$

$$\beta_{Tev} = \frac{p}{E} = \frac{\sqrt{E^2 - m^2}}{E} = \frac{\sqrt{(980\,\text{GeV})^2 - (0.938\,\text{GeV})^2}}{(980\,\text{GeV})} = 0.99999954$$

$$\beta_{LHC} = \frac{\sqrt{(7000\,\text{GeV})^2 - (0.938\,\text{GeV})^2}}{7000\,\text{GeV}} = 0.999999991$$

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KG

4) In a few words explain the meaning of the principle of relativity to a layman. [1]

The laws of physics should be the same in all reference frames.

2: The Klein-Gordon Equation

1) Using the Schrödinger equation and definition of particle density $\rho = \Psi^*\Psi$ show that system satisfies continuity equation with current defined as

$$\vec{j} = \frac{\hbar}{2mi}\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right).$$ [4]

Multiply the Schrödinger equation by Ψ^*:

$$i\hbar\Psi^*\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Psi^*\nabla^2\Psi + V$$

Multiply the C.C. of the S.E. by Ψ:

$$-i\hbar\Psi\frac{\partial\Psi^*}{\partial t} = -\frac{\hbar^2}{2m}\Psi\nabla^2\Psi^* + V^*$$

Assuming $V=V^*$ and subtracting,

$$i\hbar\left(\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right) + \frac{\hbar^2}{2m}\left(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\right) = 0$$

$$\therefore = i\hbar\frac{\partial}{\partial t}\left(\Psi^*\Psi\right) + \frac{\hbar^2\vec{\nabla}}{2m}\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right) = 0$$

$$\therefore \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

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KG

2) Show that plane waves are solutions of the Klein-Gordon equations. Obtain energies of the solutions. [2]

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + \frac{m^2c^2}{\hbar^2}\right)Ne^{-i(\omega t-\vec{k}\cdot\vec{x})} = 0$$

$$= \left(-\frac{\omega^2}{c^2} + K^2 + \frac{m^2c^2}{\hbar^2}\right)Ne^{-i(\omega t-\vec{k}\cdot\vec{x})} = 0$$

$$\therefore \ -\hbar^2\omega^2 + \hbar^2c^2K^2 + m^2c^4 = 0$$

$$\therefore \ E = \hbar\omega = \pm\sqrt{\hbar^2c^2K^2 + m^2c^4}$$

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3) Derive continuity equation from Klein-Gordon equation. Check that for plane wave solutions current is given by

$$j^\mu_{KG} = \frac{2|N|^2}{c}p^\mu.$$ [4]

$$\left(\partial^\mu\partial_\mu + \frac{m^2c^2}{\hbar^2}\right)\phi(x^\mu) = 0$$

$$\left.\begin{aligned} \phi^*\partial^\mu\partial_\mu\phi + \frac{m^2c^2}{\hbar^2}\phi^*\phi &= 0 \\ \phi\partial^\mu\partial_\mu\phi^* + \frac{m^2c^2}{\hbar^2}\phi\phi^* &= 0 \end{aligned}\right\} -$$

$$\phi^*\partial^\mu\partial_\mu\phi - \phi\partial^\mu\partial_\mu\phi^* = 0$$

$$\partial_\mu\underbrace{(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)} = 0$$

$$\therefore \ \partial_\mu j^\mu_{KG} = 0$$

$$j^\mu_{KG} = |N|^2e^{i(\omega t-\vec{k}\cdot\vec{x})}(-i\omega - i\vec{k})e^{-i(\omega t-\vec{k}\cdot\vec{x})} - |N|^2e^{-i(\omega t-\vec{k}\cdot\vec{x})}(i\omega + i\vec{k})e^{i(\omega t-\vec{k}\cdot\vec{x})}$$

$$= 2|N|^2(-i\omega - i\vec{K})$$

derivative is only for single coordinate

?

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imaginary unit in current def. is convention for real current.