Lecture Notes PX263: Electromagnetic Theory and Optics

Y.A. Ramachers, University of Warwick
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Chapter 1

Introduction to Electromagnetic theory

This first part of the lecture should introduce you to the Electromagnetic Theory part of the lecture. As you will hopefully see and understand later, the Optics part follows from that theory.

1.1 A first look at the meaning of electromagnetic theory

Electromagnetic theory is the first field theory in physics that you will encounter and hence one of the most important and fundamental theories you'll meet during your entire course.

As you'll learn during your studies, all fundamental theories in physics (so far) are field theories. The concept of 'a field' is very important and electromagnetism gives you the chance to learn about that concept for the first time properly. Electromagnetism is excellent for studying fields for the first time since it shows all the fundamental features of any other field theory in a manageable format as well as all the problems, conceptual or technical. You can learn about those as well. Therefore for anyone with an interest in fundamental physics the material in this course can offer something of value.

Likewise, electromagnetism is extremely important for any practical applications of physics in real life. If you are in any way interested in applied physics, tangential or professional, then knowing your Maxwell equations is surprisingly important. They are not merely relevant for fundamental studies, they also offer a clear and short path right into proper, full applications of physics.

Here are a few select remarks on both main areas of electromagnetism, fundamental and practical:

- Studying the Maxwell equations would be a first full exploration of the fundamental concepts of a theory of charges (test particles) and fields and their intrinsically linked dependency. The Maxwell equations feature all the challenges this brings far more detailed than a test mass in a Newtonian gravitational potential. All the same problems have to be answered by any other field theory such as quantum field theories.

- The concept of a field as a physical entity and therefore radiation, i.e. time-dependent fields and their physics appears for the first time in electromagnetism and cost physicists almost 100 years to understand fully even after quantum theory was invented.

- Most applied engineering in the real world requries classical electromagnetism rather than any other physical theory. Most of optics is applied electromagnetism (quantum optics isn't as the name suggests). Clear engineering tasks like antenna design (for smart phones for instance) comes straight out of classical electromagnetism.

- Practically all applications of electromagnetism require a good understanding of the corresponding maths! For electromagnetism this would be mostly partial differential equations and vector calculus. Consider this as part of your toolbox to 'do' physics. Electromagnetism offers an excellent area where to practice these skills in an applied setting.

Finishing off these introductory remarks, electromagnetism also offers an excellent educational opportunity for you. Its non-trivial demands on maths skills combined with fundamental as well as very applied physics allows you to understand the practical importance of the role of maths in physics. Everything you have seen in the Mathematics for Scientists lecture or dedicated maths lectures really finds an application in physics, very directly in a lot of cases in electromagnetism. You didn't just learn all that maths for fun, it really means something in physics. Electromagnetism again offers a good platform from which to understand this link once and for all. Something that will help you enormously in later years.

1.2 Maxwell equations

The Maxwell equations in differential form are given by:

$$\begin{aligned} \operatorname{div}\mathbf{E} &= \frac{\rho}{\epsilon_0} && (1.1)\\ \operatorname{div}\mathbf{B} &= 0 && (1.2)\\ \operatorname{curl}\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && (1.3)\\ \operatorname{curl}\mathbf{B} &= \mu_0\left(\mathbf{j} + \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) && (1.4) \end{aligned}$$

and additionally there is the conservation law (somewhat redundant since it can be derived from the above, see exercises)

$$\operatorname{div}\mathbf{j} + \frac{\partial \rho}{\partial t} = 0 \qquad (1.5)$$

where ϵ_0 is the permittivity, μ_0 the permeability, ρ the charge density, $\mathbf{j}$ the current density and finally the equation of motion (Lorentz force law) is

$$\frac{d\mathbf{p}}{dt} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad (1.6)$$

The Maxwell equations are a set of partial differential equations describing the dynamics of the vector fields **E** and **B**. Their interpretation varies according to the situation but you can roughly call them electric and magnetic fields in that order.

The vector fields **E** and **B** are then the **solutions** of the differential equations. This makes the Maxwell equations field equations and classical electromagnetism a **field theory**. The solutions are defined locally, i.e. at every point, hence EM is a classical local field theory.

As for every set of differential equations, obtaining solutions requires you to specify boundary conditions. This is often the major challenge in solving the Maxwell equations for a specific problem setting. The integral form, see below, of writing the equations often represents a more practical approach to finding solutions since their form reveals boundary conditions explicitly. Apart from mixing boundary conditions and fundamental field equations the integral form of writing the Maxwell equations is mathematically equivalent to the above quoted field equations.

1.2.1 Vector fields

Consider classical fields as simply functions of coordinates, $T(x, y, z)$ for instance. You plug three numbers into T and get one number out - that would be called a **scalar function** or **scalar field**.

Time-dependent fields would simply be functions containing an explicit dependence on the time coordinate.

A **vector field** would for instance look like this:

$$\mathbf{F}(x,y,z,t) = F_x(x,y,z,t)\,\hat{\mathbf{i}} + F_y(x,y,z,t)\,\hat{\mathbf{j}} + F_z(x,y,z,t)\,\hat{\mathbf{k}}$$

with $F_x(x,y,z,t)$, $F_y(x,y,z,t)$ and $F_z(x,y,z,t)$ scalar functions in an Euclidean space with basis vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$.

Therefore, a vector field is a function into which you would plug some numbers, four coordinates in the example above, and get back a vector with three components $\mathbf{F}$. This vector is really just a vector with a direction and a magnitude and you know its components in $x-$ and $y-$ and $z-$direction.

An easy but non-trivial example is the electric field of a static charge Q

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{Q}{|\mathbf{r}-\mathbf{r}_0|^2}\widehat{(\mathbf{r}-\mathbf{r}_0)}$$

where $\mathbf{r}$ is short-hand notation for $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$. and it is assumed that the charge Q sits at position $\mathbf{r}_0$. The symbol $\widehat{\mathbf{r}-\mathbf{r}_0}$ denotes the unit vector (the hat says normalise to magnitude one) from the charge position $\mathbf{r}_0$ to the point of interest at $\mathbf{r}$.

Working with vector fields requires proficiency with vector calculus, a brief refresher will be given later. In essence, you need to know how to differentiate and integrate vector fields in addition to normal numbers.

1.2.2 Maxwell equations in integral form

The Maxwell equations in integral form are

$$\iint_{\partial V} \mathbf{E}\,d\mathbf{S} = \iiint_V \frac{\rho}{\epsilon_0} dV \tag{1.7}$$

$$\iint_{\partial V} \mathbf{B}\,d\mathbf{S} = 0 \tag{1.8}$$

$$\oint_{\partial S} \mathbf{E}\,d\mathbf{l} = -\iint_S \frac{\partial \mathbf{B}}{\partial t}\,d\mathbf{S} \tag{1.9}$$

$$\oint_{\partial S} \mathbf{B}\,d\mathbf{l} = \mu_0 \iint_S \mathbf{j}\,d\mathbf{S} + \mu_0\,\epsilon_0 \iint_S \frac{\partial \mathbf{E}}{\partial t}\,d\mathbf{S} \tag{1.10}$$

The integral form of the Maxwell equations hides the local nature of the field theory by doing something very useful for practical applications of the theory. They manifestly (visibly, more or less) combine the required boundary conditions for solving the differential equations with the differential equations themselves. That might be immediately obvious or not. Consider all those volumes, areas or curves on the integral signs. They represent one quite practical way to specify boundary conditions.

1.2.3 Maxwell equations in matter

When considering sources, i.e. charges and currents, in matter which after all is a little more realistic then something potentially confusing happens to the Maxwell equations:

$$\operatorname{div}\mathbf{D} = \rho_f \tag{1.11}$$

$$\operatorname{div}\mathbf{B} = 0 \tag{1.12}$$

$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.13}$$

$$\operatorname{curl}\mathbf{H} = \mathbf{j}_f + \frac{\partial \mathbf{D}}{\partial t} \tag{1.14}$$

These clearly look different but also almost the same. Some new labels, a little cleaner perhaps, i.e. Greek symbol constants disappeared.

First of all these are not the more fundamental or more general Maxwell equations as some textbooks have you to believe. One has to painstakingly derive(!) the above (the new symbols and their meaning, the 'D' and 'H') from the original Maxwell equations. That results in additional constitutive relations:

$$\mathbf{D} = \epsilon\mathbf{E} \tag{1.15}$$

$$\mathbf{B} = \mu\mathbf{H} \tag{1.16}$$

$$\epsilon = \epsilon_0\,(1+\chi_e) \tag{1.17}$$

$$\mu = \mu_0\,(1+\chi_m) \tag{1.18}$$

where χ_e and χ_m are susceptibilities of media, electric and magnetic, i.e. material properties. Worse, the constitutive relations above might or might not hold depending on the material properties.

In summary, one target of this lecture will be to work out the meaning of the fields in the Maxwell equations which should reveal that the fields causing forces on charged particles and fields due to sources are distinct in the theory of classical electromagnetism.

1.2.4 More variations of the Maxwell equations

The set of equations 1.1-1.4 can be (and has been) considered from all sorts of perspectives, from purely formal variations to incorporating additional physics ingredients, some more motivated than others. As a consequence what started out to aid clarity of understanding could look a bit like a mess. It isn't a mess but merely a signature of the vast amount of physics electromagnetic theory describes.

Most of the formal variations have to do with attempts to put more emphasis on a somewhat hidden, major property of the Maxwell equations: they are **Lorentz transformation invariant**. In fact Lorentz derived his set of transformations using the Maxwell equations! This very fundamental physics revelation just isn't obvious at all from eqns 1.1-1.4. Casting them in a different mathematical format however, can make that property obvious (not given here since you lack the maths and there isn't time to get that far on this module). Likewise, attempts to put Maxwells theory into a wider context can result in different manifestations of the equations. Again, none of that is of much help to get going on electromagnetic theory itself.

One formal variation, however, might be useful to quote to point out a general characteristic of all field theories, something that is considered to reflect beauty in physics. See whether that works for you; the statement is believed to come from Einstein about his own field equations. Casting the Maxwell equations into the following format [3]:

$$\operatorname{div}\mathbf{B} = 0 \tag{1.19}$$

$$\operatorname{curl}\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \tag{1.20}$$

$$\operatorname{div}\mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1.21}$$

$$\operatorname{curl}\mathbf{B} - \mu_0\,\epsilon_0\frac{\partial\mathbf{E}}{\partial t} = \mu_0\,\mathbf{j} \tag{1.22}$$

puts all field-related terms on the left and source terms on the right. This reflects the essence of the field theory: fields tell sources what to do and sources tell fields what to do, they are intrinsically inseparable! Einstein comes into play since for his field equations of general relativity he chose exactly this format (space-time terms on the left, matter on the right) on purpose to make that very same point.

Chapter 2

Vector calculus refresher

In principle this chapter shouldn't really be necessary. You all should feel comfortable with vector calculus at this stage. The Maths Methods I lecture last term dealt with this. However, Christmas holidays and a lack of revision for that maths lecture can do wonders for cleansing the mind from unwanted mathematical symbols and practice. In order to give you at least a fighting chance to follow this lecture at all, a refresher on vector calculus seems like a good idea. Please note the word 'refresher'. If you need any kind of detail beyond the meagre notes from this lecture then remember that you have the most extensive notes from last term on exactly this topic. Use them!

2.1 Taking derivatives of functions

You all know of course how to take the derivative of a function but I am not so sure whether all of you have seen the geometry of that operation. A visualisation of the process of taking the derivative can help a bit when soon you'll learn about taking derivatives of functions of several variables.

Taking the derivative of the function $f(x) = x^2$ gives you the slope of every tangent at every point along the parabola curve. This collection of slope values is itself a function, $f'(x) = 2\,x$ as you well know. The 'tangent' statement is the geometry interpretation of the process of taking derivatives and it works on all functions, no matter how many variables there are.

Consider the parabola as a curve or path you can walk along. At any point you can stop walking, mark a point on the path, say at x_1, take another step and put down a second marker, x_2. For both markers, you know the y-axis values, y_1 and y_2 since it is a parabola. Now take a ruler or a stick and connect the two markers. The geometry will be easier to imagine if you think of the stick to be really long, longer than the distance between the two markers. You can get the slope of that stick by building

$$\text{slope} = \frac{\triangle y}{\triangle x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Taking the derivative means taking the limit of the fraction from above at a specific, fixed point. Say that fixed point shall be marker one at x_1, i.e. $x_2 = x_1 + \epsilon$.

$$\text{slope} = \lim_{\epsilon \to 0} \frac{\triangle y}{\triangle x}$$

You do know the y-values as $f(x)$ and you know the x-values so what you really do is

$$\text{slope} = \lim_{\epsilon \to 0} \frac{f(x_1 + \epsilon) - f(x_1)}{x_1 + \epsilon - x_1}$$

Geometrically, you move the second marker at $x_2 = x_1 + \epsilon$ closer and closer to the first marker as you let ϵ tend to zero. As a result, your stick will look like the tangent at a single point, x_1 of the parabola. The slope of that stick will be $f'(x) = 2\,x$. That's what it means to take the derivative of a function.

The nice part about this visualisation is that it works for any function, only limited by your geometrical imagination. Say having a function of two variables, $f(x, y)$. The drawing could look like a curved plane in 3D space, a mountain range for instance or a simple flat plane. The derivative at some fixed point on that function would look like a tangent plane, touching the original function drawing at that single fixed point. It can very well intersect the original function somewhere else but at the point of interest it would touch the function tangentially and nothing more.

However, which tangent? There are an infinite amount of them, all embedded in that tangent-plane! That's where the partial differentiation operation comes in. Given a coordinate system, say for this example, x,y and f(x,y) axes, you have two special directions, x and y, along which you can take derivatives and tangent lines along these directions fix the tangent plane uniquely (unless one is zero which means this isn't a function of two variables). The point to take home is that the derivative at a fixed point is a simple geometric object of the same order as the function you differentiate, i.e. a tangent line for the 1D function, a tangent-plane for a 2D function a tangent hyperplane for a N-D function, always giving the collection of slopes or changes at the chosen, fixed point. That means differentiation is a **local** operation (at a point), quantifying how something changes at that fixed point.

2.2 Differentiation

Given a function $\phi(x, y, z)$, a scalar field, see above, then we can take partial derivatives with respect to each variable. However, there is a way to do this which reveals what we are actually doing with this partial derivative creation.

Combine all partial derivatives into a vector and what you get is the direction of change in all coordinate directions combined, a proper vector of total change with direction and magnitude. That is what is called the **gradient** of a function. This operation of taking partial derivatives and putting them all into one vector can be cast into a most useful and intuitive form when defining a new operator: the **nabla or del** operator defined as

$$\nabla = \frac{\partial}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial}{\partial z}\,\hat{\mathbf{k}} \tag{2.1}$$

in Cartesian coordinates with unit basis vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ or written as column vector

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} \tag{2.2}$$

Useful, since it now permits to take derivatives of more than just scalar fields! Using the nabla operator one can define derivatives of vector fields very transparently, i.e. taking the **divergence** and the **curl** of a vector field:

$$\mathrm{div}\,\mathbf{E} = \nabla \cdot \mathbf{E} = \frac{\partial}{\partial x}\,E_x + \frac{\partial}{\partial y}\,E_y + \frac{\partial}{\partial z}\,E_z\ .$$

Like every self-respecting dot product this operation delivers a number (more generally, a number machine, i.e. a function) while the cross product operation

$$\mathrm{curl}\,\mathbf{E} = \nabla \times \mathbf{E} = \begin{pmatrix} \frac{\partial}{\partial y}\,E_z - \frac{\partial}{\partial z}\,E_y \\ \frac{\partial}{\partial z}\,E_x - \frac{\partial}{\partial x}\,E_z \\ \frac{\partial}{\partial x}\,E_y - \frac{\partial}{\partial y}\,E_x \end{pmatrix}$$

delivers a vector (vector function) as expected.

2.3 Integration

The main intellectual challenge when integrating in electromagnetism deals with geometry more than with solving integrals as such. Typically, the determinate integrals are rather simple were it not for the boundaries, i.e. the curves and areas and volumes over which to take integrals of something. Fair enough, sometimes the integrand is non-trivial and it cost effort to find the integral itself but mostly you need to work on the geometry to integrate over.

Last year, all those integrals were taken over simple geometries like spheres and cylinders and that will mostly continue this year. As a reminder though this year you will have to be familiar and comfortable with the various integral theorems. There are several but starting with the lowest dimensional integrals, we can go from lines to areas and back and have **Stokes theorem**:

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS \ , \tag{2.3}$$

where the contour line C describes the closed perimeter of the area S on the right hand side which is characterised by the surface normal $\hat{\mathbf{n}}$. There is a sign convention built into this geometry description and unfortunately, it is often rather important to be careful with this sign. The rule to remember is the **right-hand rule**, i.e. your curled fingers of the right hand point in the direction of travel along the curve C then the surface normal points in the same direction as your thumb.

Secondly, there is the **Gauss divergence theorem**:

$$\oint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV \ , \tag{2.4}$$

where the closed area S is connected to the enclosed volume V on the right hand side. Both theorems can be derived from the much more general Green theorem, after George Green.

2.4 Conservative fields

Looking back at gradients you might feel it would be so much easier to work with scalar fields rather than vector fields. Sometimes that is indeed possible, and yes, it is indeed quite a bit easier to do so in many circumstances. In electromagnetism these functions come along under the name of potentials.

The topic of conservative fields tells you under which condition a vector field can be written as a gradient of some scalar field. This definition of a conservative field can be cast into a proper theorem but here it suffices to simply list the four possibilities to check in order to decide whether a field is conservative or not. Doesn't mean any of these conditions actually delivers the scalar field, merely that it exists. Getting it explicitly is a different matter. A vector field $\mathbf{F}$ is conservative if any of the following four conditions is fulfilled:

- for any oriented simple closed curve C, have $\oint_C \mathbf{F} \cdot d\mathbf{l} = 0$.
- For any two oriented simple curves C_1 and C_2 that have the same endpoints, get
$$\int_{C_1} \mathbf{F} \cdot d\mathbf{l} = \int_{C_2} \mathbf{F} \cdot d\mathbf{l}$$
which expresses integration–path independence.
- $\mathbf{F}$ is the gradient of some function f; that is $\mathbf{F} = \nabla f$.
- $\nabla \times \mathbf{F} = 0$

If one of the above is true then all are true. The most familiar is probably the last condition, stating every conservative field is curl-free. That is a good one to remember. Mathematicians would call this conditions merely necessary not sufficient but for practically all applications in EM is it perfectly fine to consider it even sufficient. For instance, if you look at the Maxwell equations, eqn. 1.3, you see immediately that for static magnetic fields, the electric field can be written as a scalar field since its curl vanishes. This leads to the domain of electrostatics which was covered in your first year module.

Chapter 3

Potentials

When the fields are time-dependent, $\mathbf{E}$ can no longer be found simply as the gradient of a scalar potential since $\nabla \times \mathbf{E} \neq 0$. In all cases, however, $\nabla \cdot \mathbf{B} = 0$, so that we can still express $\mathbf{B}$ as $\mathbf{B} = \nabla \times \mathbf{A}$, with $\mathbf{A}$ a vector potential. This arises formally from the vector identity (eqn. A.5, appendix) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$. The interpretation as magnetic vector potential requires some more work.

3.1 The magnetic vector potential

It would be convenient if $\mathbf{B}$ could generally be derived from a scalar potential. Unfortunately, since the curl of $\mathbf{B}$ is not generally zero, $\mathbf{B}$ cannot be expressed as the gradient of a scalar function. Under special circumstances when the current density vanishes and all fields are static, the magnetic field can be expressed by a scalar potential. It can be helpful for solving magnetostatic problems but it does not carry any fundamental meaning. That function is preserved for the magnetic vector potential.

With the definition $\mathbf{B} = \nabla \times \mathbf{A}$ using a simple vector identity, an interpretation is required. Ampere's law serves as a useful starting point. Recall the Biot-Savart law which gives you a recipe to calculate the magnetic field given a current distribution. The general expression to note for the Biot-Savart law is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d^3r' \tag{3.1}$$

which represents the equivalent role for the magnetic field that the Coulomb law plays for the electric field.

Starting with eqn. 3.1, a useful exercise in algebra will yield an equivalent expression from which to read-off the interpretation immediately, worth the effort. This is how it starts: a frequently used relation (note - useful to know)

$$\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

suggests a substitution in eqn. 3.1 as

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\,\pi} \int_V \mathbf{j}(\mathbf{r}') \times \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) \, d^3r'$$

Almost there, just missing to apply a single vector identity (eqn.A.8) for the curl on a scalar times vector function:

$$\nabla \times (f\,\mathbf{j}) = \nabla f \times \mathbf{j} + f\,\nabla \times \mathbf{j}$$

re-write for this particular case, i.e. plug in what we have for the scalar function to get

$$-\mathbf{j} \times \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) = \nabla \times \left(\frac{\mathbf{j}}{|\mathbf{r}-\mathbf{r}'|} \right) - \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) \nabla \times \mathbf{j}\,.$$

Observe that the current by definition only depends on the primed coordinates while the differential operator only acts on un-primed coordinates to see that $\nabla \times \mathbf{j} = 0$ and hence the term containing the curl of the current vanishes identically. Finally, the integral is over the primed coordinates, the remaining curl can be taken outside the integral to yield

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\,\pi} \nabla \times \int_V \left(\frac{\mathbf{j}}{|\mathbf{r}-\mathbf{r}'|} \right) \, d^3r' \tag{3.2}$$

The vector potential definition, see above, now yields immediately

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\,\pi} \int_V \left(\frac{\mathbf{j}}{|\mathbf{r}-\mathbf{r}'|} \right) \, d^3r' \tag{3.3}$$

an expression for the magnetic potential involving the current density as any self-respecting magnetic entity should.

3.2 Electric field in terms of potentials

From Faraday's law we have

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial\,t} = -\frac{\partial}{\partial\,t} (\nabla \times \mathbf{A}) = -\nabla \times \left(\frac{\partial \mathbf{A}}{\partial\,t} \right)$$

or

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \, .$$

Therefore one can conclude that *not* $\mathbf{E}$, but instead $\mathbf{E} + \partial \mathbf{A}/\partial t$, is expressible as the gradient of a scalar potential. We have then $\mathbf{E} + \partial \mathbf{A}/\partial t = -\nabla \phi$, or

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \, . \tag{3.4}$$

Taking the divergence of eqn. 3.4, we obtain the equation

$$\nabla \cdot \mathbf{E} = \nabla \cdot \left(-\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \right)$$

or, equating the right hand side to charge density from Gauss law

$$-\nabla^2 \phi - \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = \frac{\rho}{\epsilon_0} \tag{3.5}$$

Similarly, putting $\mathbf{B} = \nabla \times \mathbf{A}$ into Ampere's law we obtain the analogous equation for the vector potential $\mathbf{A}$:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla \, (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{j} - \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\nabla \phi + \frac{\partial \mathbf{A}}{\partial t} \right)$$

or

$$\left(\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) - \nabla \left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} \right) = -\mu_0 \mathbf{j} \tag{3.6}$$

What we have then in eqns. 3.5, 3.6 are the Maxwell equations again but without the identities, Faraday's law and the divergence-free magnetic field, involving only source terms, the charge and the current, and the potentials to solve for. Admittedly, these equations look a lot more complicated to solve than the previous version of the Maxwell equations but they have their use. In particular, when considering quantum mechanics, forces carry no meaning anymore and hence potentials rather than electric and magnetic fields are the important terms to understand physics. Likewise, considering electromagnetism as a fully relativistic theory, potentials also become more fundamental than the classic force fields.

3.3 Lorentz force and canonical momentum (not examinable)

In order to give you a chance to appreciate in later lectures how electromagnetism is introduced in other theories one can look at the Lorentz force and derive something called a canonical momentum, i.e. what is being used in other theories, like quantum theories, to insert electromagnetism. This subsection should be understood as purely educational and not examinable on this module.

Write the Lorentz force as the rate of change of momentum of a charge in an electromagnetic field,

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) = q\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A})\right] .$$

The last term may be expanded using eqn. A.9 from the appendix:

$$\nabla(\mathbf{v} \cdot \mathbf{A}) = (\mathbf{v} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{v}) .$$

If $\mathbf{v}$ is not a function of position, we may eliminate spatial derivatives of $\mathbf{v}$ from the expansion above to obtain

$$\mathbf{v} \times (\nabla \times \mathbf{A}) = \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A} .$$

Replacing the last term in the momentum expression from above by this equation for the triple cross product, we find

$$\frac{d\mathbf{p}}{dt} = q\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A}\right]$$

The pair of terms

$$\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{A} \equiv \frac{d\mathbf{A}}{dt} .$$

is called the **convective derivative** of $\mathbf{A}$. As a charge moves through space, the change in $\mathbf{A}$ it experiences arises not only from the temporal change in $\mathbf{A}$ but also from the fact that it samples $\mathbf{A}$ in different locations. Substituting $d\mathbf{A}/dt$ for the pair of terms, the momentum relation can be written as

$$\frac{d\mathbf{p}}{dt} = q\left(-\nabla(\phi - \mathbf{v} \cdot \mathbf{A}) - \frac{d\mathbf{A}}{dt}\right)$$

Grouping like terms gives the desired outcome

$$\frac{d}{dt}\left(\mathbf{p} + q\,\mathbf{A}\right) = -\nabla\left(q\,\phi - q\,\mathbf{v} \cdot \mathbf{A}\right) \tag{3.7}$$

The argument of the gradient on the right hand side is the potential that enters Lagrange's equation as the potential energy of a charged particle in an electromagnetic field and the term on the left is the momentum conjugate to the coordinates, the canonical momentum.

3.4 Gauge transformations

The magnetic vector potential **A** is not unique since one can add any vector field whose curl vanishes without changing the physics. Also, one can see from eqn. 3.4 that any change in **A** requires a compensating change in ϕ in order to keep **E** (and hence the physics) unchanged.

Recall that a curl-free field must be the gradient of a scalar field; hence we write as before $\mathbf{A}' = \mathbf{A} + \nabla\Lambda$. Let us denote the correspondingly changed potential as ϕ'. The magnetic induction field is invariant under this change and we can use eqn. 3.4 to express the electric field in terms of the changed, primed, potentials,

$$\begin{aligned} \mathbf{E} &= -\nabla\phi' - \frac{\partial \mathbf{A}'}{\partial t} \\ &= -\nabla\phi' - \frac{\partial}{\partial t}(\mathbf{A} + \nabla\Lambda) \\ &= -\nabla\left(\phi' + \frac{\partial\Lambda}{\partial t}\right) - \frac{\partial \mathbf{A}}{\partial t} \end{aligned} \tag{3.8}$$

or in terms of the unchanged potentials $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$. Comparison of the terms gives $\phi' = \phi - \partial\Lambda/\partial t$.

The pair of coupled transformations

$$\begin{aligned} \mathbf{A}' &= \mathbf{A} + \nabla\Lambda \\ \phi' &= \phi - \frac{\partial\Lambda}{\partial t} \end{aligned} \tag{3.9}$$

is called a **gauge transformation** and the invariance of the fields under such a transformation is called gauge invariance. The transformations are useful for recasting the somewhat awkward equations 3.5, 3.6 into a more elegant form. Although many different choices of gauge can be made, the **Coulomb gauge** and the **Lorenz gauge** are of particular use.

In statics it is usually best to choose Λ so that $\nabla \cdot \mathbf{A} = 0$, a choice known as the Coulomb gauge. With this choice, eqns. 3.5, 3.6 reduce to Poisson's equations

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

and

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{j} \; .$$

In electrodynamics, i.e. with time-dependence, it is also frequently useful to choose

$$\nabla \cdot \mathbf{A} = -\mu_0\,\epsilon_0\,\frac{\partial\phi}{\partial t}$$

a choice known as the Lorenz gauge. With this choice, eqns. 3.5, 3.6 take the form of wave equations:

$$\nabla^2\phi - \mu_0\,\epsilon_0\,\frac{\partial^2\phi}{\partial\,t^2} = -\frac{\rho}{\epsilon_0} \tag{3.10}$$

$$\nabla^2\mathbf{A} - \mu_0\,\epsilon_0\,\frac{\partial^2\mathbf{A}}{\partial\,t^2} = -\mu_0\,\mathbf{j} \tag{3.11}$$

It is clear that in the Lorenz gauge both potentials obey manifestly similar equations that fit naturally into a relativistic framework. This gauge symmetry has profound consequences in quantum electrodynamic since it permits the existence of a zero mass carrier of the electrodynamic field, the photon.

Chapter 4

EM waves and energy flux

Every changing magnetic field engenders an electric field and, conversely, a changing electric field generates a magnetic field. Taken together, Faraday's and Ampere's laws give a wave equation whose solution we know as electromagnetic waves.

The equations corresponding to Faraday's and Ampere's laws, eqns. 1.3, 1.4 represent coupled differential equations which can be decoupled fairly easily by differentiating eqn. 1.3 once more:

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t}$$

using eqn. A.6 from the appendix yields

$$\nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) = -\frac{\partial}{\partial t} \left(\mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \tag{4.1}$$

The same procedure works in decoupling the eqn. 1.4 for **B**.

$$\nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \mu_0 \nabla \times \mathbf{j} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} . \tag{4.2}$$

Shuffling all source terms on one side and field terms on the left then leaves

$$\begin{aligned} \nabla^2 \mathbf{E} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} &= \mu_0 \frac{\partial \mathbf{j}}{\partial t} + \nabla \frac{\rho}{\epsilon_0} \\ \nabla^2 \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} &= -\mu_0 \nabla \times \mathbf{j} \end{aligned} \tag{4.3}$$

4.1 Waves in vacuum

Eliminating all sources, i.e. $\mathbf{j} = 0$ and $\rho = 0$, we obtain the homogeneous wave equations, describing hence waves in vacuum:

$$\nabla^2 \mathbf{E} = \mu_0 \, \epsilon_0 \, \frac{\partial^2 \mathbf{E}}{\partial \, t^2} \tag{4.4}$$

and in similar fashion

$$\nabla^2 \mathbf{B} = \mu_0 \, \epsilon_0 \, \frac{\partial^2 \mathbf{B}}{\partial \, t^2} \tag{4.5}$$

From year one, it should be well known how to solve these but a repeat of this special case is quite instructive for the rest of the lecture. Plane waves are solutions of the equations above. However, this deserves a few remarks.

It is important first of all to recognise that the Laplacian, ∇^2, acting on a vector is not merely ∇^2 acting on each individual component. Only when the basis is independent of the coordinates can one make the simplification $(\nabla^2 \mathbf{F})_i = \nabla^2 \mathbf{F}_i$. Fortunately, this covers the important case of Cartesian coordinates (not spherical coordinates though). Thus eqns. 4.4 and 4.5 reduce to six identical, uncoupled scalar differential equations of the form

$$\nabla^2 \psi(\mathbf{r}, t) = \mu_0 \, \epsilon_0 \, \frac{\partial^2 \psi(\mathbf{r}, t)}{\partial \, t^2}$$

with $\psi(\mathbf{r}, t)$ representing any one of $E_x, \; E_y, \; E_z$ or $B_x, \; B_y, \; B_z$.

Among the solutions are plane-wave solutions of the form

$$\psi(\mathbf{r}, t) = \psi_0 \, \exp\left(i(\mathbf{k} \cdot \mathbf{r} \pm \omega \, t)\right) \tag{4.6}$$

Once back-substituted into the scalar wave equation one finds that with the help of

$$\nabla^2 \psi = -k^2 \psi$$

and

$$\frac{\partial^2 \psi}{\partial \, t^2} = -\omega^2 \psi$$

that

$$k^2 \psi = \omega^2 \, \mu_0 \, \epsilon_0 \, \psi \; .$$

One can conclude that both the phase velocity ω/k and the group velocity $\partial\omega/\partial k$ are equal to

$$c = \frac{1}{\sqrt{\mu_0 \, \epsilon_0}} \; .$$

In deriving eqns. 4.4, 4.5 the differentiation served to remove the coupling between **E** and **B**. The fields **E** and **B** are not independent though and one must restore the lost coupling by insisting that **E** and **B** satisfy not only the wave equations but also the Maxwell equations with $\rho = 0$ and $\mathbf{j} = 0$. Thus, noting that for a vector plane wave

$$\nabla\cdot \;\leftrightarrow i\mathbf{k}\cdot$$

and

$$\nabla\times \;\leftrightarrow i\mathbf{k}\times$$

one gets the following set of relations

$$\begin{aligned}
\nabla\cdot\mathbf{E} = 0 &\quad\Rightarrow\quad i\mathbf{k}\cdot\mathbf{E} = 0 \\
\nabla\cdot\mathbf{B} = 0 &\quad\Rightarrow\quad i\mathbf{k}\cdot\mathbf{B} = 0 \\
\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} &\quad\Rightarrow\quad i\mathbf{k}\times\mathbf{E} = \mp i\omega\mathbf{B} \\
\nabla\times\mathbf{B} = \mu_0\,\epsilon_0\,\frac{\partial\mathbf{E}}{\partial t} &\quad\Rightarrow\quad i\mathbf{k}\times\mathbf{B} = \pm i\omega\,\mu_0\,\epsilon_0\,\mathbf{E}
\end{aligned} \tag{4.7}$$

The first two of these expressions imply that **E** and **B** are each perpendicular to the propagation direction **k**, while the second two imply that **E** and **B** are perpendicular to each other. It is worth reiterating that these conclusions are valid only for infinite plane waves and do not apply to spherical waves or bounded waves.

4.2 Poynting's Theorem

The concept of energy of a charge distribution was covered to some extent in year one already. Now consider the energy required to produce a given electromagnetic field, this time less constrained by the construction of current loops or static charge distributions than last year. This will not only yield the energy density of the fields but also the rate at which energy is transported by the field.

Consider the work dW done by the electromagnetic field on the charge q contained in a small volume d^3r, moving through the field with velocity $\mathbf{v}$ when it is displaced through distance $d\mathbf{l}$. The work done on the charge is just the gain in mechanical energy (kinetic and potential) of the charge. If the field was the only agent acting on the charge, for instance accelerating it, then the field must have supplied this energy and therefore decreases its own energy, or else there must have been a corresponding external input of energy.

The work performed by the Lorentz force is

$$dW = q\,(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} = q\,(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\mathbf{v}\,dt$$

Replacing q with $\rho\, d^3r$, one finds the rate of change of energy of all the charge contained in some volume V as the volume integral

$$\frac{dW}{dt} = \int_V (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\, \rho\, d^3r = \int_V \mathbf{E} \cdot \mathbf{j}\, d^3r \tag{4.8}$$

replacing the current in the expression above using Ampere's law according to

$$\mathbf{j} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

the integrand in eqn. 4.8 can be re-arranged as follows (using eqn. A.10 from the appendix)

$$\begin{aligned} \mathbf{E} \cdot \mathbf{j} &= \mathbf{E} \cdot \left(\frac{1}{\mu_0} (\nabla \times \mathbf{B}) - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\ &= \frac{1}{\mu_0} \left[-\nabla \cdot (\mathbf{E} \times \mathbf{B}) + \mathbf{B} \cdot (\nabla \times \mathbf{E}) \right] - \epsilon_0\, \mathbf{E} \frac{\partial \mathbf{E}}{\partial t} \\ &= \frac{1}{\mu_0} \left(-\nabla \cdot (\mathbf{E} \times \mathbf{B}) - \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) - \epsilon_0\, \mathbf{E} \frac{\partial \mathbf{E}}{\partial t} \\ &= -\frac{1}{2} \frac{\partial}{\partial t} \left(\epsilon_0 |\mathbf{E}|^2 + \frac{|\mathbf{B}|^2}{\mu_0} \right) - \frac{1}{\mu_0} \nabla \cdot (\mathbf{E} \times \mathbf{B}) \end{aligned} \tag{4.9}$$

Using eqn. 4.9 and the divergence theorem for the volume integral, one can cast eqn. 4.8 into the convenient form

$$\frac{dW}{dt} = -\frac{d}{dt} \int_V \frac{1}{2} \left(\epsilon_0 |\mathbf{E}|^2 + \frac{|\mathbf{B}|^2}{\mu_0} \right) d^3r - \oint_{\partial V} \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}) d\mathbf{S}\,. \tag{4.10}$$

This is considered convenient since the first integral is nothing but the total energy density of the fields, covered last year (and hence reminding you here of the expressions).

If one supposes for the moment that the surface integral in eqn. 4.10 vanishes then the equation states that the rate at which the test charges gain mechanical energy is just equal to the rate at which the fields loose energy. Now suppose that in spite of doing work on charges in the volume of interest, the fields remained constant. Clearly, one would need an inflow of energy to allow this. The surface integral in eqn. 4.10 has exactly this form; including the minus sign, it measures the total flux crossing the surface *into* the volume. Therefore

$$\mathbf{S} \equiv \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}) \tag{4.11}$$

is called the **Poynting vector** over which a surface integral measures the energy flux crossing the surface *out* of a volume (negative sign means into the volume). This translates also as the rate at which the electromagnetic field transports energy across a surface.

One particularly easy and intuitive example of that would be the energy flux (irradiance) of an electromagnetic plane wave with electric field amplitude $|\mathbf{E}|$: all it takes is to calculate the Poynting vector for this case and any concrete flux could then be calculated using a surface integral over that vector. Get the Poynting vector for this case using eqn. 4.11 and $\mathbf{B} = (\hat{\mathbf{k}} \times \mathbf{E})/c$ from the wave section above and get

$$\mathbf{S} = \frac{\mathbf{E} \times (\hat{\mathbf{k}} \times \mathbf{E})}{\mu_0\, c} = \frac{|\mathbf{E}|^2}{\mu_o\, c}\hat{\mathbf{k}}$$

which is the correct result for the instantaneous flux but measuring that is not realistic. One would actually measure the average value of the electric field, namely $\langle |\mathbf{E}|^2 \rangle = 1/2\, |\mathbf{E}|^2$.

Chapter 5

Static electromagnetic fields in matter

Here we aim at describing electromagnetic fields in matter, targeting the Maxwell equations in matter as displayed earlier. This should reveal the meaning of the new vector fields **D** and **H**.

5.1 The electric field due to a polarised dielectric

Let's begin the discussion with a fairly phenomenological consideration of the electric field arising from charges in matter. Consider a dielectric (a piece of matter which can contain fixed and mobile charges) having charges, electric dipoles, quadrupoles etc. distributed throughout the material. The charge density shall be denoted by ρ as before and the dipole moment density or simply the **polarisation**, by $\mathbf{P}(\mathbf{r})$. The potential at position $\mathbf{r}$ due to this distribution, truncated at the dipole moment distribution, is given by (short-cut: assuming you know the Coulomb potential and dipole potential from first year electrostatics)

$$\phi(\mathbf{r}) = \frac{1}{4\,\pi\epsilon_0} \int \left[\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \mathbf{P}(\mathbf{r}') \cdot \nabla' \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) \right] d^3r' \qquad (5.1)$$

This elementary expression of two independent contributions to the electric potential in a dielectric can now help to gain understanding of the fields when plugged into the Maxwell equations. First consider Gauss law on the divergence of the electric field, $\nabla \cdot \mathbf{E}$. Getting there, one needs to consider $\mathbf{E} = -\nabla\phi$ and $\nabla \cdot \mathbf{E} = -\nabla^2\phi$.

Thus

$$\begin{aligned}\nabla\cdot\mathbf{E} &= -\frac{1}{4\,\pi\epsilon_0}\,\nabla^2\int\left[\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}+\mathbf{P}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right]\,d^3r' \\ &= -\frac{1}{4\,\pi\epsilon_0}\int\left[\rho(\mathbf{r}')\,\nabla^2\,\frac{1}{|\mathbf{r}-\mathbf{r}'|}+\mathbf{P}(\mathbf{r}')\cdot\nabla'\left(\nabla^2\,\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right]\,d^3r' \qquad (5.2)\end{aligned}$$

Using the relation (short-cut: assuming this delta function expression is known, i.e. no derivation)

$$\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -4\,\pi\,\delta(\mathbf{r}-\mathbf{r}')$$

get

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\int\left[\rho(\mathbf{r}')\,\delta(\mathbf{r}-\mathbf{r}')+\mathbf{P}(\mathbf{r}')\cdot\nabla'\,\delta(\mathbf{r}-\mathbf{r}')\right]\,d^3r' \qquad (5.3)$$

The first term in eqn. 5.3 integrates easily, see below, but the second term needs some work. Let's expand it in Cartesian coordinates and see what happens:

$$\begin{aligned}\int\mathbf{P}(\mathbf{r}')\cdot\nabla'\,\delta(\mathbf{r}-\mathbf{r}')\,d^3r' &= \int P_x(\mathbf{r}')\,\frac{\partial}{\partial x'}\,[\delta(x-x')\,\delta(y-y')\,\delta(z-z')]\;dx'\,dy'\,dz' \\ &+\int P_y(\mathbf{r}')\,\frac{\partial}{\partial y'}\,\delta(\mathbf{r}-\mathbf{r}')\,d^3r'+\int P_z(\mathbf{r}')\,\frac{\partial}{\partial z'}\,\delta(\mathbf{r}-\mathbf{r}')\,d^3r' \\ & \qquad (5.4)\end{aligned}$$

Now something possibly not well known, i.e. using

$$\int_{-\infty}^{\infty} f(x)\,\frac{\partial}{\partial a}\,\delta(x-a)\,dx = -\frac{\partial}{\partial a}\,f(a)$$

one can integrate each term on the right hand side of eqn. 5.4. For the first integral get

$$\begin{aligned}\int P_x(\mathbf{r}')\,\frac{\partial}{\partial x'}\,\delta(x-x')\delta(y-y')\delta(z-z')dx'dy'dz' &= \int P_x(x',y,z)\,\frac{\partial}{\partial x'}\,\delta(x-x')dx' \\ &= -\frac{\partial P_x(x,y,z)}{\partial x} \\ &= -\frac{\partial P_x(\mathbf{r})}{\partial x} \qquad (5.5)\end{aligned}$$

In exactly the same fashion, get the remaining integrals as

$$-\frac{\partial P_y(\mathbf{r})}{\partial y} \quad ; \quad -\frac{\partial P_z(\mathbf{r})}{\partial z}$$

to give finally

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}-\frac{\nabla\cdot\mathbf{P}(\mathbf{r})}{\epsilon_0} \qquad (5.6)$$

In other words, $-\nabla\cdot\mathbf{P}$ acts as a source of electric field and is often given the name **bound charge**.

5.2 Bound charge at a boundary

At discontinuities in $\mathbf{P}$, as at the boundaries of dielectrics, one gets an accumulation of bound surface charge even for uniform polarisations. This observation is verified from eqn. 5.6 when setting the free charge to zero:

$$\nabla \cdot \mathbf{E} = -\frac{\nabla \cdot \mathbf{P}(\mathbf{r})}{\epsilon_0}$$

Applying Gauss' law to a small flat pill-box with top and bottom surface parallel to the dielectric surface, one finds

$$\int_V \nabla \cdot \mathbf{E} \, d^3r = -\int_V \frac{\nabla \cdot \mathbf{P}(\mathbf{r})}{\epsilon_0} \, d^3r$$

hence

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = -\frac{1}{\epsilon_0} \oint_S \mathbf{P}(\mathbf{r}) \cdot d\mathbf{S}$$

Noting that the surface of the pill-box inside the dielectric is directed opposite to the outward–facing normal $\mathbf{n}$, one finds (the trick of the pill-box is to let the width go to zero).

$$(\mathbf{E}_e - \mathbf{E}_i) \cdot \mathbf{S}_e = -\frac{1}{\epsilon_0} \mathbf{P} \cdot \mathbf{S}_i = \frac{1}{\epsilon_0} \mathbf{P} \cdot \mathbf{S}_e$$

requiring that

$$(\mathbf{E}_e - \mathbf{E}_i) \cdot \mathbf{n} = \frac{1}{\epsilon_0} \mathbf{P} \cdot \mathbf{n}$$

where $\mathbf{n}$ is the normal facing outward from the dielectric. This discontinuity in the field is exactly the effect that a surface charge, $\sigma = \mathbf{P} \cdot \mathbf{n}$, at the surface of the dielectric would have.

In total, the electric field produced by a polarised dielectric medium is identical to that produced by a bound charge density, $\rho_b = -\nabla \cdot \mathbf{P}$ and a bound surface charge density, $\sigma_b = \mathbf{P} \cdot \mathbf{n}$, on the surface. The resulting electric field may then be written

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{[-\nabla' \cdot \mathbf{P}(\mathbf{r}')] \, (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d^3r' + \frac{1}{4\pi\epsilon_0} \oint_S \frac{(\mathbf{P} \cdot \mathbf{n}) \, (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dS' \qquad (5.7)$$

5.3 Electric displacement field

The calculation of the microscopic field $\mathbf{E}$ arising from charges and molecular dipoles of the medium requires considerable care. It is frequently useful to think of the polarisation of the medium as merely a property of the medium rather than as a source of field. To do so requires the definition of the **electric displacement** field. The differential equation 5.6 is more conveniently written

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho \tag{5.8}$$

The quantity

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$$

is the electric displacement field. In terms of **D**, Maxwell's first equation becomes

$$\nabla \cdot \mathbf{D} = \rho \tag{5.9}$$

The dipoles of the medium are not a source for **D**, only the free charges act as sources (see also eqn. 1.11).

When **P** can be adequately *approximated* by $\mathbf{P} = \chi \, \epsilon_0 \, \mathbf{E}$, compare eqn. 1.17, we find that

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \chi \, \epsilon_0 \mathbf{E} = \epsilon_0 (1 + \chi) \mathbf{E} = \epsilon \, \mathbf{E}$$

The constant ϵ is called the **permittivity** of the dielectric. The **dielectric constant** κ is defined by

$$\kappa \equiv 1 + \chi = \frac{\epsilon}{\epsilon_0}$$

In general, because it takes time for dipoles to respond to applied fields, all three constants, χ, ϵ and κ are frequency dependent materials constants.

5.4 Magnetic induction field

Materials can just as well contain magnetic dipoles as electric dipoles. The vector potential due to a magnetic dipole, **m** at position $\mathbf{r}_j$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times (\mathbf{r} - \mathbf{r}_j)}{|\mathbf{r} - \mathbf{r}_j|^3} \tag{5.10}$$

Note that this expression follows from eqn. 3.3 when applied to a single closed current loop and Taylor expanded to first order, i.e. the dipole term. The magnetic dipole moment **m** is then defined as

$$\mathbf{m} \equiv I \int d\mathbf{S} \, ,$$

the integral over the area of the current loop, carrying the current I.

From here on the line of argument follows closely the previous discussion of polarisation for electric fields. Nevertheless it is worthwhile to follow through since a similar standard tool as for electrostatics, the Gauss pill-box, will feature, the Amperian loop, which is crucial for many calculations in magnetostatics and dynamics.

The equivalent concept to the electric dipole polarisation density is the **magnetisation M** consisting of n molecular magnetic dipoles per unit volume. The vector potential arising from both currents and magnetic dipoles in the material is then

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}')\, d^3r'}{|\mathbf{r}-\mathbf{r}'|} + \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, d^3r' \tag{5.11}$$

The second integral may be written (same trick as previously)

$$\int_V \frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, d^3r' = \int_V \mathbf{M}(\mathbf{r}')\times\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) d^3r' \; .$$

Using the identity, eqn. A.8 from the appendix the expression above can be cast into

$$\begin{aligned} &= \int_V \frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r' - \int_V \nabla'\times\left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) d^3r' \\ &= \int_V \frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r' + \oint_S \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\times d\mathbf{S} \end{aligned} \tag{5.12}$$

If $\mathbf{M}$ is localised to a finite region and the surface of integration lies outside this region, the second integral vanishes. The surface must be outside the region with $\mathbf{M}$ by construction since the volume of integration in eqn. 5.11 must include all $\mathbf{M}$ and $\mathbf{j}$. Therefore one has

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}')+\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r' \tag{5.13}$$

Thus according to eqn. 5.13, the magnetisation of the medium contributes to the vector potential like an effective current

$$\mathbf{j}_m = \nabla\times\mathbf{M} \; .$$

Getting back to the Maxwell equation for $\nabla\times\mathbf{B}$, one notes

$$\nabla\times\mathbf{B} = \nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \; .$$

Consider the Coulomb gauge (we are in the static section after all), setting $\nabla\cdot\mathbf{A} = 0$, one obtains

$$\begin{aligned} \nabla\times\mathbf{B} &= -\nabla^2\frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}')+\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r' \\ &= \mu_0 \int_V \left[\mathbf{j}(\mathbf{r}')+\nabla'\times\mathbf{M}(\mathbf{r}')\right]\, \delta(\mathbf{r}-\mathbf{r}')\, d^3r' \\ &= \mu_0\left[\mathbf{j}(\mathbf{r})+\nabla\times\mathbf{M}(\mathbf{r})\right] \end{aligned} \tag{5.14}$$

Again one finds that according to eqn. 5.14 the curl of the magnetisation behaves like a current density. Just as the polarisation contributes an effective surface charge at discontinuities, a discontinuity in magnetisation contributes an effective surface current.

5.5 Surface current at a boundary

Consider eqn. 5.14 for a uniformly magnetised bar with magnetisation $\mathbf{M} = M_z\,\mathbf{k}$ in z-direction only. Inside the magnet, have $\nabla \times \mathbf{M} = 0$, implying that $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$; the uniform magnetisation makes no contribution to the induction field. At the boundary one invokes the Amperian loop as the tool of choice - equivalent to the Gauss pill-box for electrostatics.

Draw a thin rectangular loop with long sides parallel to the magnetisation $\mathbf{M}$ straddling the boundary and integrate the curl equation over the area included in the loop

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \mu_0 \int_S (\nabla \times \mathbf{M}) \cdot d\mathbf{S}$$

where $\mathbf{j} = 0$ is assumed for simplicity. By means of Stokes theorem both surface integrals can be recast as line integrals along the loop

$$\oint_{\partial S} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \oint_{\partial S} \mathbf{M} \cdot d\mathbf{l}$$

which after letting the width shrink to zero (same trick as for the pill-box tool), gives

$$B_z^{int} - B_z^{ext} = \mu_0\, M_z\ .$$

A surface current $\mathbf{j} = M_z\,\hat{\mathbf{j}}$, i.e. in y-direction, would produce exactly this kind of discontinuity in $\mathbf{B}$. Thus, at the boundary, the effect of a discontinuity in the magnetisation is exactly the same as that of a surface current $\mathbf{j} = \mathbf{M} \times \mathbf{n}$.

5.6 Magnetic field intensity

As in the case of electric polarisation, it is frequently preferable to ascribe the magnetisation to the medium as an attribute rather than having it act as a source. Re-writing eqn. 5.14 in a more convenient form by collecting all curl terms on the left, one obtains

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) = \mathbf{j} \tag{5.15}$$

The quantity

$$\mathbf{H} \equiv \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

is called the **magnetic field intensity**. With $\mathbf{H}$ being directly proportional to the controllable variable $\mathbf{j}$, the relation between $\mathbf{B}$ and $\mathbf{H}$ is conventionally regarded as $\mathbf{B}$ being a function of $\mathbf{H}$ with $\mathbf{H}$ being the independent variable. This perspective leads to writing

$$\mathbf{B} = \mu_0 \left[\mathbf{H} + \mathbf{M}(\mathbf{H})\right]$$

In the linear isotropic approximation, write

$$\mathbf{B} = \mu_0 \left(\mathbf{H} + \mathbf{M}\right) = \mu_0 \left(1 + \chi_m\right) \mathbf{H} \equiv \mu \, \mathbf{H}$$

with the **magnetic susceptibility** χ_m and the **magnetic permeability** μ.

Materials with $\chi_m > 0$ are **paramagnetic** whereas those with $\chi_m < 0$ are **diamagnetic**. For most materials, both paramagnetic and diamagnetic, $|\chi_m| << 1$. If $\chi_m >> 1$, the material is called ferromagnetic. In this latter case, the relation between $\mathbf{B}$ and $\mathbf{H}$ is still valid but $\mathbf{M}$ is usually a very complicated, nonlinear function of $\mathbf{H}$.

It is worth pointing out that to this point $\mathbf{H}$ and $\mathbf{D}$ appear to be nothing but mathematical constructs derivable from the fields $\mathbf{B}$ and $\mathbf{E}$. However, what is a more useful interpretation is that $\mathbf{B}$ and $\mathbf{E}$ are regarded as the fields responsible for forces on charged particles, whereas $\mathbf{H}$ and $\mathbf{D}$ are the fields generated by the sources. Consider source and force fields as distinct.

5.7 Boundary conditions for the static fields

Boundary conditions were discussed above already in order to aid an understanding of the source fields and their meaning. Here, all boundary conditions are collected in one place since they form a crucial part of electromagnetism in media, i.e. working with source fields. Later, when examining time-dependent fields, i.e. waves, we'll need these again.

The Maxwell equations in matter when simplified to time-independent fields read (compare eqns. 1.11 to 1.14):

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{j} \end{aligned} \tag{5.16}$$

where ρ and $\mathbf{j}$ are the free charges and currents (dropped the index to indicate 'free' since now it is clear what these are). In addition, the following relations have been worked out so far:

$$\begin{aligned} \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{B} &= \mu_0 (\mathbf{H} + \mathbf{M}) \end{aligned} \tag{5.17}$$

The first and the last Maxwell equation in eqns. 5.16 can be integrated to give Gauss' law

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, d^3 r \tag{5.18}$$

and Ampere's law

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{j} \cdot \, d\mathbf{S} \tag{5.19}$$

in the presence of matter. Getting at the boundary conditions, one intergrates each of the equations 5.16 and for ease of referring to various directions, assume the $x-y$ plane to be tangential to the dielectric interface.

Consider $\nabla \cdot \mathbf{D}$ inside a thin pill-box of width ϵ whose flat sides lie on the opposite sides of a dielectric interface. Let the charge density be described by

$$\rho = \rho_V(x, y, z) + \sigma(x, y)\delta(z) \; ,$$

where ρ_V is a volume charge density and σ a surface charge density confined to the interface. Integrate over the pill-box to get

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \bar{\rho}_V \, V + \int_S \sigma \, dx \, dy \; ,$$

where $\bar{\rho}_V \, V$ is the mean volume charge density times the volume. Now break the surface integral of the electric displacement into integrals over each of the three surfaces of the pill-box to write

$$\int_S D_z^I(x, y, \epsilon/2) \, dx \, dy - \int_S D_z^{II}(x, y, -\epsilon/2) \, dx \, dy + \int_{rim} \mathbf{D} \cdot d\mathbf{S} = \bar{\rho}_V \, S \, \epsilon + \int_S \sigma \, dx \, dy$$

In the limit of vanishing ϵ this becomes

$$\int_S (D_z^I - D_z^{II}) \, dx \, dy = \int_S \sigma(x, y) \, dx \, dy$$

which can only hold true for arbitrary surfaces, S, if $D_z^I - D_z^{II} = \sigma$, or to make this conclusion coordinate independent

$$\left(\mathbf{D}^I - \mathbf{D}^{II} \right) \cdot \mathbf{n} = \sigma$$

where $\mathbf{n}$ is as usual the unit normal vector to the interface pointing from region II to region I. In conclusion, *the perpendicular component of* $\mathbf{D}$ *is discontinuous by* σ.

The same argument can be applied to $\nabla \cdot \mathbf{B} = 0$ and results in

$$\left(\mathbf{B}^I - \mathbf{B}^{II}\right) \cdot \mathbf{n} = 0$$

or *the perpendicular component of* $\mathbf{B}$ *is continuous* across the interface.

Thirdly, look at $\nabla \times \mathbf{E} = 0$: One obtains the boundary condition on the tangential component of $\mathbf{E}$ by integrating $\nabla \times \mathbf{E}$ over the area of the thin loop whose two long sides lie on opposite sides of the interface. This works out as the following

$$\begin{aligned} 0 &= \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \oint_C \mathbf{E}\, d\mathbf{l} \\ &= \int_{x_0}^{x_0+L} \left(E_x^{II} - E_x^I\right) dx + \epsilon \left(\bar{E}_z(x_0+L) - \bar{E}_z(x_0)\right) . \end{aligned}$$

Taking the limit $\epsilon \to 0$, we require

$$\int_{x_0}^{x_0+L} \left(E_x^{II} - E_x^I\right) dx = 0 ,$$

which can only hold for all L if $E_x^{II} = E_x^I$ with the same conclusion valid also for the $y-$component. Generally, *the tangential component of* $\mathbf{E}$ *is continuous across the interface.*

Finally, to $\nabla \times \mathbf{H} = \mathbf{j}$: Again, the loop geometry helps the argument. Consider the same Amperian loop as discussed above and allow for a body current density $\mathbf{J}(x,y,z)$ and a surface current density $\mathbf{j}(x,y)\delta(z)$ confined to the interface. As $\mathbf{j}$ lies in the $x-y-$plane, have $j_z = 0$. Integrating $\nabla \times \mathbf{H}$ over the surface included by the loop gives

$$\int_S \mathbf{j} \cdot\, d\mathbf{S} = \int_S -J_y dS + \int_S -j_y\, \delta(z)\, dx\, dz$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = -J\epsilon\, L - \int_{x_0}^{x_0+L} j_y\, dx$$

Again, taking the limit $\epsilon \to 0$, get

$$\int_{x_0}^{x_0+L} \left(H_x^{II} - H_x^I\right) dx = -\int_{x_0}^{x_0+L} j_y\, dx$$

Equating the integrands gives $H_x^I - H_x^{II} = j_y$. Similarly, placing the loop in the $z-y-$plane, one obtains a similar result: $H_y^I - H_y^{II} = -j_x$ (the sign comes from the x-axis pointing out of the paper as opposed to into the paper as before for the y-axis). This generalizes to

$$\mathbf{n} \times \left(\mathbf{H}^{I} - \mathbf{H}^{II}\right) = \mathbf{j}$$

or, *the parallel component of* **H** *is discontinuous by* **j**. A summary of the boundary conditions is given below, labelling the components perpendicular and parallel to the surface, the interface, by $\perp$ and $||$, respectively.

$(\mathbf{D})_{\perp}$ is discontinuous by σ.
$(\mathbf{E})_{\|\|}$ is continuous.
$(\mathbf{B})_{\perp}$ is continuous.
$(\mathbf{H})_{\|\|}$ is discontinuous by $\mathbf{j}$.

5.8 Conduction

This subsection serves to introduce a final and rather important constitutive relation for the Maxwell equations in matter, i.e. **Ohm's Law**. It will be needed at several points in the subsequent lectures and is noteworthy on its own hence this separate section.

Like all constitutive relations also Ohm's law defines a new material property, called **conductivity, g**. This symbol is not exactly as standardized as other symbols met so far. Often textbooks will use σ which is used already for the surface charge density in this lecture (and in textbooks). What is the special status of conductivity to warrant an entire new law? Just like before, for the susceptibilities in eqns. 1.17, 1.18, conductivity summarises fundamental physical processes of charges in matter. A short discussion of these processes will reveal a few additionally worthwhile concepts to learn.

Ohm's law is expressed as

$$\mathbf{j} = g\,\mathbf{E}\ . \tag{5.20}$$

A simple law which might or might not look surprising. What could be the reason for this direct proportionality between a current and an electric field? It could have been any relation after all since it is not directly determined by the Maxwell equations but comes in as an 'add-on', a constitutive relation.

First step is to observe that for a static charge distribution, there is no current (continuity equation) independent of matter, geometry or anything else. For any non-zero current to flow there has to be a time-dependence of the charge density. This can only be achieved by applying a force hence in electrodynamics this means the Lorentz force. For static charges though the Lorentz force due to a magnetic field is zero (the $\mathbf{v} \times \mathbf{B}$ term), ergo remains the electric field as the sole provider (one could introduce mechanical or thermal forces but that leads us away from electrodynamics).

The process of interest for any current in matter consists of charges accelerating (force applied) due to an electric field, either external or internal, variable or static. What puts conductivity outside Maxwell's equations is the observation that the charge transport, the kinematics of accelerating charges in matter, any matter, is governed by **charge scattering**.

Matter 'resists' to a certain degree the free movement of charge. *Conductivity describes the impediment to charge motion through matter*: the higher the conductvity, the farther an electron may move on average before undergoing a collision. Therefore, the inverse of conductivity is called **resistivity**, $\eta = 1/g$.

This idea of charge scattering when pulled leads immediately to a very useful new concept that should be known from mechanics: terminal velocity. Consider scattering as a speed–dependent process (this is an assumption) working against the acceleration (like air resistance against free fall) and look at the steady state solution, the long time behaviour and get

$$m\,\frac{d\mathbf{v}}{dt} = q\,\mathbf{E} - \frac{m\,\mathbf{v}}{\tau} \to 0$$

where the last relation results from achieving a steady state, terminal speed. The τ symbol indicates the average time it takes between collisions. Then the average velocity achieved by charges in matter would be

$$< \mathbf{v} >= \frac{q\,\tau}{m}\,\mathbf{E}$$

with $\frac{q\,\tau}{m}$ the **mobility** and the velocity from above the **drift velocity**. Computing the net current density as

$$\mathbf{j} = n\,q\,< \mathbf{v} >= \frac{n\,q^2\,\tau}{m}\mathbf{E}$$

with n the carrier number density, one can express the conductivity as

$$g = \frac{n\,q^2\,\tau}{m}$$

For most materials of interest, the conductivity is indeed independent of the electric field, a materials constant. This is fortuitous since a lot of technical applications rely on characterising charge transport with high precision and this way a measurement of the conductivity and drift speed is often all that is needed. Even microscopic physics can be explored by turning the argument above around: measuring mobility relates to information about elementary charge scattering processes in a material, something very useful to know for technical and fundamental applications.

Another useful observation can also be made at this point. What happens to charge inserted into a bulk of matter that obeys Ohm's law? An initial static charge density ρ_0 is introduced at time $t = 0$. The charge density must follow the continuity equation but now we introduce Ohm's law for the body of matter to get

$$g\,\nabla \cdot \mathbf{E} = -\frac{\partial \rho}{\partial t}\ .$$

Using Gauss' law, eliminate $\nabla \cdot \mathbf{E}$ to get

$$\frac{g}{\epsilon}\,\rho = -\frac{\partial \rho}{\partial t}\ .$$

Solving this differential equation with the boundary condition $\rho(t = 0) = \rho_0$ yields

$$\rho(t) = \rho_0 \exp\left(-\frac{g}{\epsilon}\,t\right)\ .$$

The charge density inside hence decays exponentially in time. Of course, the total charge must be conserved and thus charge within the body travels to the surface where it distributes itself in such a way that the field internal to the body approaches zero at equilibrium.

The rate at which a volume charge dissipates is determined by the **relaxation time** ϵ/g; for copper (a good conductor) this is merely 10^{-19} s. Even distilled water, a relatively poor conductor, has a relaxation time of 10^{-6} s. Thus one can see how rapidly static equilibrium is achieved in matter.

Chapter 6

EM waves in a medium

Waves in vacuum were discussed in section 4.1 where the wave equations for the fields, eqns. 4.3, thankfully reduced to simple form after setting all sources equal to zero. This will not be possible anymore when bringing matter into play. Waves in vacuum were a revolutionary concept at the time since waves by definition at the time required a 'medium' to give substance to waves, no medium, no waves. However, apart from this conceptually important insight, they offer little more to deal with unless we leave the topic of electromagnetism and enter the special and general relativity theories (which we won't). In case you wonder, the topic of radiation deals with the production of waves from sources and is a big topic on its own but also goes beyond purely considering waves in vacuum.

At this point bringing matter into consideration, the task is more about what happens to these stand-alone field oscillations, Maxwell's waves, when they encounter matter with its own electromagnetic properties. Two main arguments will have to be pursued: (a) what happens at the interfaces between a wave in a medium, typically vacuum, and a material body and (b) how do waves behave inside matter as opposed to inside vacuum. All these examinations will in the end lead us to classical optics, the study of light waves in media and at media boundaries.

6.1 Boundary conditions for oscillating fields

In contrast to section 5.7 one has to consider the full set of Maxwell equations, eqns. 1.11 to 1.14. Other than that, the chain of arguments is very similar to what we have done already. First of all, there is simply no change to eqns. 1.11, 1.12, the two divergence equations. As a consequence, the static boundary conditions remain, i.e. $\mathbf{B}_\perp$ is continuous and $\mathbf{D}_\perp$ is discontinuous by the (time-dependent) surface charge σ. Only the curl equations could be altered. Let's take a look.

Integrating the equation for $\nabla\times\mathbf{E}$ over the area of the Amperian loop spanning the interface between two media, say vacuum and matter, one has

$$\int_S (\nabla\times\mathbf{E})\cdot d\mathbf{S} = -\int_S \frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S}\ .$$

Application of Stokes theorem to the left integral gives

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\int_S \frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S}$$

and as the width of the loop shrinks to zero, the surface integral on the right could vanish just like it did for the static case in section 5.7 for the body current density. The same line of argument also recovers exactly the earlier results for the $\nabla\times\mathbf{H}$ equation, so what is going wrong? It shouldn't be identical to the static case. These time-dependent contributions from $\frac{\partial\mathbf{B}}{\partial t}$ and $\frac{\partial\mathbf{D}}{\partial t}$ should make themselves felt at some point and they do, note the hypothetical 'could vanish' in context of the surface integral above.

The new element we overlooked so far is that any oscillating field at a boundary will move charges, i.e. produce a time dependent surface charge density at the boundary. When fields oscillate, it is generally not possible to maintain a zero or otherwise constant surface charge density as is seen from the continuity equation, eqn. 1.5. Integrating eqn. 1.5 over the volume of the Gauss' pill-box with, as usual,

$$\rho = \rho_V(x,y,z) + \sigma(x,y)\delta(z)\ ,$$

one obtains

$$\int_V \nabla\cdot\mathbf{j}\, d^3r = -\frac{\partial}{\partial t}\left[\int_V \rho_v\, d^3r + \int_S \sigma(x,y)\, dx\, dy\right]$$

or

$$\oint_S \mathbf{j}\cdot d\mathbf{S} = -\frac{\partial}{\partial t}\left[\bar{\rho}_V\, S\,\epsilon + \int_S \sigma(x,y)\, dx\, dy\right]\ ,$$

where $\bar{\rho}_V$ is the mean charge density in the volume of integration. As the thickness of the pill-box shrinks to zero,

$$\int_S \left(\mathbf{j}^I\cdot\mathbf{n} - \mathbf{j}^{II}\cdot\mathbf{n}\right)\, dx\, dy = -\frac{\partial}{\partial t}\int_S \sigma(x,y)\, dx\, dy\ ,$$

which leads to

$$\left(\mathbf{j}^I - \mathbf{j}^{II}\right)\cdot\mathbf{n} = -\frac{\partial\sigma}{\partial t}$$

So far, nothing new, same argument as in the static case. Now, however, consider a wave arriving and let the surface charge density change in time and the medium be an Ohmic conductor, i.e.

$$\sigma = \sigma_0 \, \exp\left(-i\omega t\right)$$

and $\mathbf{j} = g\,\mathbf{E}$, then the above becomes

$$g_1\, E_n^I - g_2\, E_n^{II} = i\omega\sigma \ .$$

On the other hand, from the divergence equation for $\mathbf{D}$ we have

$$D_n^I - D_n^{II} = \sigma$$

Eliminating σ from both equations, one obtains the following

$$\left(\epsilon_1 + \frac{ig_1}{\omega}\right)\, E_n^I = \left(\epsilon_2 + \frac{ig_2}{\omega}\right)\, E_n^{II}$$

Three special cases may now be distinguished. In all cases, assume the medium to be Ohmic and the standard constitutive relations eqns. 1.15, 1.16 as well as a simple wave, $\mathbf{H} = \mathbf{H}_0\, e^{-i\omega t}$ and $\mathbf{E} = \mathbf{E}_0\, e^{-i\omega t}$.

1. Both materials are non-conductors, i.e. no sustained surface current nor a surface charge density, i.e. $\mathbf{E}_{||}$, $\mathbf{H}_{||}$, $\mathbf{D}_{\perp}$ and $\mathbf{B}_{\perp}$ are all continuous.

2. Both materials have non-zero, finite conductivity: surface currents cannot be sustained in a less-than-perfect conductor, hence $\mathbf{E}_{||}$, $\mathbf{H}_{||}$, $\left(\epsilon + \frac{ig}{\omega}\right)\, \mathbf{E}_{\perp}$ and $\mathbf{B}_{\perp}$ are all continuous.

3. One of the media is a perfect conductor, say $g_2 \to \infty$ which is frequently a good approximation for a metal boundary. Using

$$\nabla \times \mathbf{H}^{II} = \mathbf{j}^{II} + \frac{\partial \mathbf{D}^{II}}{\partial t} = \left(g_2 - i\epsilon_2\omega\right)\, \mathbf{E}^{II}$$

we obtain, keeping the curl term finite, that is $\mathbf{E}^{II} = 0$. Next using

$$\nabla \times \mathbf{E}^{II} = -\frac{\partial \mathbf{B}^{II}}{\partial t} = i\mu\omega \mathbf{H}^{II} = 0$$

get $\mathbf{H}^{II} = 0$. The boundary conditions become

$$\mathbf{E}^{II} = \mathbf{H}^{II} = 0, \quad \mathbf{B}_{\perp}^{I} = \mathbf{B}_{\perp}^{II} = 0, \quad \mathbf{E}_{||}^{I} = \mathbf{E}_{||}^{II} = 0$$

$$\epsilon_1\, \mathbf{E}_{\perp}^{I} = \sigma\,, \quad \mathbf{n} \times \mathbf{H}^{I} = \mathbf{j}$$

where $\mathbf{n}$ is the normal pointing outward from the perfect conductor.

6.2 Plane waves in material media

Let's start like in the previous waves section but now with the Maxwell equations in matter, i.e. take the curl of eqn.1.13

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu \frac{\partial}{\partial t}\left(\mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}\right) .$$

Expanding the left side and using the constitutive relations for linear, ohmic materials to replace $\mathbf{j}$ with $g\mathbf{E}$ and $\partial \mathbf{D}/\partial t$ with $\epsilon\, \partial \mathbf{E}/\partial t$, we find

$$\nabla\left(\nabla \cdot \mathbf{E}\right) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}\left(\mu\, g\, \mathbf{E} + \mu\, \epsilon \frac{\partial \mathbf{E}}{\partial t}\right) .$$

In the absence of free charge, $\nabla \cdot \mathbf{E} = 0$ and assuming harmonically varying fields

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{-i\omega\, t}$$

we may replace $\frac{\partial}{\partial t}$ with $-i\omega$ to obtain

$$\nabla^2 \mathbf{E}_0 + \mu\, \epsilon\, \omega^2 \left(1 + \frac{ig}{\omega \epsilon}\right) \mathbf{E}_0 = 0 \tag{6.1}$$

Note that picking harmonically varying fields is really no restriction as we could always frequency analyze the temporal variations and apply the formula to Fourier components (not examinable, just a side-remark).

A similar argument works for $\mathbf{B}$

$$\nabla \times (\nabla \times \mathbf{H}) = \nabla \times \left(\mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}\right) = \nabla \times \left(g\, \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}\right)$$

and, following the same steps as above

$$\frac{1}{\mu}\left[\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}\right] = -\frac{\partial}{\partial t}\left(g\, \mathbf{B} + \epsilon \frac{\partial \mathbf{B}}{\partial t}\right)$$

leads to

$$\nabla^2 \mathbf{B}_0 + \mu\, \epsilon\, \omega^2 \left(1 + \frac{ig}{\omega \epsilon}\right) \mathbf{B}_0 = 0 \tag{6.2}$$

Note how the first-order derivative introduces a complex term in the wave equations. This will have consequences for conducting media, see section 6.4.

6.3 Plane waves in linear, isotropic dielectrics

In non-conducting ($g = 0$) dielectrics, the fields satisfy the homogeneous wave equations

$$\nabla^2 \mathbf{E} - \mu\,\epsilon\,\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0\ ; \quad \nabla^2 \mathbf{H} - \mu\,\epsilon\,\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \tag{6.3}$$

Considering $\mathbf{H}$ instead of $\mathbf{B}$ is easier because of the more parallel boundary conditions ($\mathbf{D}_\perp$ and $\mathbf{B}_\perp$ are continuous and $\mathbf{E}_{||}$ and $\mathbf{H}_{||}$ are continuous). This pair of equations is satisfied for instance by plane wave solutions

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega\,t}\ ; \quad \mathbf{H}(\mathbf{r},t) = \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega\,t}$$

The two solutions above are not independent but are linked by Maxwell's equations just as in section 4.1. The divergence equations give for the plane waves

$$\mathbf{k}\cdot\mathbf{D}_0 = 0\ ; \quad \mathbf{k}\cdot\mathbf{B}_0 = 0$$

implying that the wave vector $\mathbf{k}$ is perpendicular to each of $\mathbf{D}$ and $\mathbf{B}$. The curl equations give

$$\mathbf{k}\times\mathbf{E}_0 = \omega\,\mu\,\mathbf{H}_0\ ; \quad \mathbf{k}\times\mathbf{H}_0 = -\omega\,\epsilon\mathbf{E}_0$$

implying that $\mathbf{E}$ and $\mathbf{H}$ are perpendicular to each other. For isotropic media $\mathbf{E}$ is parallel to $\mathbf{D}$, and $\mathbf{B}$ is parallel to $\mathbf{H}$. We may therefore conclude that the wave vector $\mathbf{k}$, the normal to the surfaces of constant phase, is parallel to the Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{H}$. In other words, for isotropic media the direction of energy propagation is along $\mathbf{k}$.

Substituting the plane wave solutions into the homogeneous wave equations from above immediately gives the **dispersion relation**

$$-k^2 + \mu\,\epsilon\,\omega^2 = 0$$

from which one concludes that the phase velocity ω/k of the wave is given by $1/\sqrt{\mu\,\epsilon}$. The group velocity $\partial\omega/\partial k$ for isotropic media will generally be different from the phase velocity.

Other important terms introduced at this point represent other combinations of permittivity and permeability that often occur in this context, i.e. the **index of refraction**, $n = c/v_{phase} = c\,\sqrt{\mu\,\epsilon}$ and the **intrinsic impedance**, $Z = \sqrt{\mu/\epsilon}$, i.e. $Z_0 = \sqrt{\mu_0/\epsilon_0} = 377\ \Omega$ in vacuum. The index of refraction is dimensionless and takes on the value $n = 1$ in vacuum.

At this point one could turn to optics, see chapter 7, and consider how electromagnetic wave properties as discussed above determine well known basic relations in optics. After all, most of optics deals with non-conducting dielectrics like glass. However, in order to conclude this section of the lecture and postpone the optics to the end, we turn our attention to conducting matter and waves.

6.4 Plane waves in isotropic, linear conducting matter

This type of matter is defined by constant μ, ϵ and a finite conductivity g. More complicated, non-linear matter is beyond syllabus. The simplification of the previous section, $g = 0$, is not possible anymore and we recall the pathway to the eqns. 6.1, 6.2. Before plugging in the harmonically varying waves as solutions, both relations for **E** and **H** (replacing **B** in the equation before eqn. 6.2 as before) turn out to be formally identical and can be written concisely as

$$\left(\nabla^2 - \mu\, g\,\frac{\partial}{\partial t} - \mu\,\epsilon\,\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = 0 \tag{6.4}$$

which just says two identical equations, one for **E**, one for **H**. However, in contrast to the non-conducting case the first order derivative is still present, multiplied by conductivity. So that makes things a little different.

Considering the wave equations for harmonically oscillating waves, eqn.6.1 and eqn.6.2, one can make the educated guess that complex numbers might become important and try the ansatz for a generic solution as a complex plane wave

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i\hat{\mathbf{K}}\cdot\mathbf{r}-i\omega\,t}\ ; \quad \mathbf{H}(\mathbf{r},t) = \mathbf{H}_0 e^{i\hat{\mathbf{K}}\cdot\mathbf{r}-i\omega\,t}$$

(using the 'hat' symbol to distinguish new complex terms from the previous) where $\hat{\mathbf{K}}$ is the complex wave vector, i.e. a vector whose components are complex numbers. Maxwell's equations applied to these expressions give the relations between the fields:

$$\hat{\mathbf{K}}\cdot\mathbf{E}_0 = 0\ ; \quad \hat{\mathbf{K}}\cdot\mathbf{H}_0 = 0$$

and

$$\hat{\mathbf{K}}\times\mathbf{E}_0 = \omega\,\mu\,\mathbf{H}_0\ ; \quad \hat{\mathbf{K}}\times\mathbf{H}_0 = -\omega\left(\epsilon + \frac{i\,g}{\omega}\right)\mathbf{E}_0.$$

These relations combined with the plane wave solutions, inserted in eqns.6.1 and 6.2 then result in the new **dispersion relation**:

$$\hat{\mathbf{K}}^2 = \mu\,\omega^2\left(\epsilon + \frac{i\,g}{\omega}\right). \tag{6.5}$$

If one assumes(!) the real and imaginary parts of the vector point in the same direction for simplicity then substituting $\hat{\mathbf{K}} \equiv \mathbf{k} + i\,\vec{\alpha}$ (choose $\vec{\alpha}$ as vector symbol since greek letters don't show up as bold font) into the plane harmonic wave solution for the electric field reveals a new physical effect for the electric field in a conductor:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0\, e^{-\vec{\alpha}\cdot\mathbf{r}}\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega\,t}$$

The wave decays exponentially along its path. The penetration length $\delta = 1/\alpha$ is known as the **skin depth**, its inverse is often called absorption coefficient. This number can be made more practical by connecting it to various material constants.

Combining the above with eqn. 6.5, we can read off for the complex wave vector

$$\hat{\mathbf{K}}^2 = k^2 - \alpha^2 + i\,2\,k\,\alpha = \mu\,\omega^2\left(\epsilon + \frac{i\,g}{\omega}\right) .$$

Assuming that g and ϵ are real (often ϵ is not real), we equate real and imaginary parts to obtain

$$k^2 - \alpha^2 = \mu\,\epsilon\omega^2\ ; \quad 2\,\alpha\,k = \mu\,\omega\,g$$

This pair of relations can be solved for k and α to give

$$k^2 = \frac{\mu\,\epsilon\,\omega^2}{2}\left[1 + \sqrt{1 + \left(\frac{g}{\epsilon\omega}\right)^2}\right] \equiv \frac{\mu\,\epsilon\,\omega^2}{2}\,\beta \tag{6.6}$$

and

$$\alpha^2 = \frac{\mu^2\,g^2\,\omega^2}{4\,k^2} = \frac{\mu\,g^2}{2\,\epsilon\,\beta} \tag{6.7}$$

where

$$\beta \equiv 1 + \sqrt{1 + \left(\frac{g}{\epsilon\omega}\right)^2} .$$

The wave properties for waves in a homogeneous conductor can now be expressed in terms of the medium's properties as

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega\sqrt{\mu\,\epsilon}}\sqrt{\frac{2}{\beta}} \approx 2\pi\sqrt{\frac{2}{\mu\,g\,\omega}} \tag{6.8}$$

$$v_{phase} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\,\epsilon}}\sqrt{\frac{2}{\beta}} \approx \sqrt{\frac{2\omega}{\mu\,g}} \tag{6.9}$$

$$\delta = \frac{1}{\alpha} = \frac{2}{g}\sqrt{\frac{\epsilon}{\mu}}\sqrt{\frac{\beta}{2}} \approx \sqrt{\frac{2}{g\,\mu\,\omega}} \tag{6.10}$$

$$n = \frac{c}{v_{phase}} = c\,\sqrt{\mu\,\epsilon}\sqrt{\frac{\beta}{2}} \approx c\sqrt{\frac{\mu\,g}{2\,\omega}} \tag{6.11}$$

where the near equality in each of the four expressions above holds for **good conductors** only ($g \gg \epsilon\omega$).

This set of formulae for basic properties is quite important, hence look at an example: given a wave of angular frequency $\omega = 2\pi \times 10^{10}$ s^{-1} propagating through a good conductor like aluminium ($g = 3.53{\times}10^7\ \Omega^{-1}\mathrm{m}^{-1}$) and like for most conducting materials, $\mu \approx \mu_0 = 4\pi \times 10^{-7}$, we get interesting properties like the wavelength

$$\lambda = 2\pi \sqrt{\frac{2}{4\pi \times 10^{-7} \times 3.53 \times 10^7 \times 2\pi \times 10^{10}}} = 5.32 \times 10^{-6}\mathrm{m} = 5.32\,\mu\mathrm{m}$$

in aluminium and the skin depth as

$$\delta = \frac{\lambda}{2\pi} = 0.85\,\mu\mathrm{m}\ .$$

The refractive index may be found from

$$n = \frac{c}{v_{phase}} = \frac{2\pi\, c}{\lambda\omega} = 5.64 \times 10^3$$

Therefore, in a good conductor, electromagnetic waves have a very large refractive index, very small wavelength, and a very small skin depth.

If that were all that can happen to a wave encountering a conducting medium then a complex wave vector could be filed under curiosities and be done with it. However, the consequences of the treatment above are far reaching and very important and hence deserve to have another, closer look at what they are.

6.5 Drude-Lorentz model and applications

A general feature of a complex wave vector being required for solving the wave equations 6.1 and 6.2 appears to be the damped plane wave

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0\, e^{-\vec{\alpha}\cdot\mathbf{r}}\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega\, t}$$

where the damping results from the imaginary part of the wave vector. This requires an explanation and the first to successfully come up with a physical model were Drude and Lorentz, independently from each other.

Consider a dielectric composed of heavy positive ions surrounded by electrons bound to the ions by a harmonic potential, $V = \frac{1}{2}\, k\, r^2$. The equations of motions for such an electron, mass m, is

$$m\,\frac{d^2\mathbf{r}}{dt^2} + m\,\gamma\,\frac{d\mathbf{r}}{dt} + k\,\mathbf{r} = q\,\mathbf{E}(\mathbf{r}, t)$$

for slow motion electrons where the magnetic field contribution to the force can be neglected. The damping term containing the damping constant, γ, should account for energy loss to the rest of the material. Allowing for a harmonic wave to drive the oscillator means setting $\mathbf{E} = \mathbf{E}_0\, e^{-i\omega t}$ and $\mathbf{r} = \mathbf{r}_0\, e^{-i\omega t}$, then the equations of motion become, with $\omega_0^2 \equiv k/m$:

$$\left(-\omega^2 - i\,\omega\gamma + \omega_0^2\right)\,\mathbf{r} = \frac{q}{m}\,\mathbf{E}$$

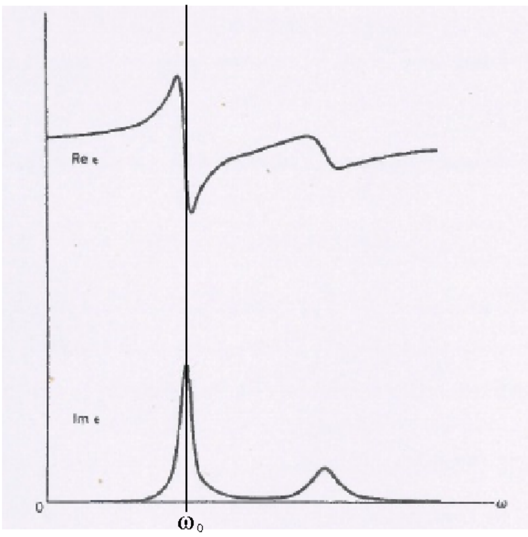

Figure 6.1: Real (top) and imaginary (bottom) part of the complex permittivity showing the dispersion curve and the absorption curve for waves in a conducting medium, respectively.

In order to connect this simple oscillator model to conducting matter, consider the pair of electron and ion as a dipol. A collection of dipoles in a medium makes up the polarization of a medium. If there are N such oscillators per unit volume then get for the polarization $\mathbf{P} = N\,q\, < \mathbf{r} >= \chi\,\epsilon_0\,\mathbf{E}$, with χ the susceptibility. This implies

$$\chi = \frac{N\,q^2}{m\,\epsilon_0\,\left(\omega_0^2 - \omega^2 - i\,\omega\gamma\right)}$$

Rationalizing the denominator gives

$$\chi = \frac{N\,q^2\,\left(\omega_0^2 - \omega^2 + i\,\omega\gamma\right)}{m\,\epsilon_0\,\left[\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2\right]}$$

which finally gives the complex and frequency-dependent permittivity

$$\hat{\epsilon} = \epsilon_0(1+\chi) = \epsilon_0 + \frac{N\,q^2\,\left(\omega_0^2 - \omega^2\right)}{m\,\left[\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2\right]} + \frac{i\,N\,q^2\,\omega\gamma}{m\,\left[\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2\right]}. \tag{6.12}$$

The frequency dependence of $\hat{\epsilon}(\omega)$ means that even a 'simple' conductor is a dispersive medium, see Fig. 6.1, i.e. waves with different frequency propagate with different phase velocity.

More importantly for applications of this observation though is to conclude that a complex permittivity implies a complex refractive index

$$\hat{n} = c\,\sqrt{\mu\,\hat{\epsilon}}$$

Thankfully, the interpretation of each part of that complex refractive index, the real and the imaginary part, is fairly straightforward using the complex wave vector. Recall that any wave vector is directly proportional to the refractive index using the phase velocity of a wave in the medium, see section 6.3.

The real part of the complex wave vector descibes the wave vector of a plane wave, while the imaginary part quantifies the damping, exponential decay or extinction of the wave in a conductor. Therefore the interpretation of the complex refractive index follows: the real part is the refractive index, describing refraction of a wave in a medium while the imaginary part quantifies **absorption** of a wave in a medium. As explicitly shown above, all of these properties are now frequency dependent, i.e. dispersive.

The dispersion curve in Fig. 6.1 shows some non-trivial behaviour around the resonance, ω_0:

- Rising permittivity as a function of frequency is called **normal** dispersion. The imaginary part will be much smaller than the real part. The classic example for this is that blue light is refracted with a larger refractive index than red light in a glass prism.
- Close to the resonance, ω_0, the permittivity falls with frequency (negative slope) which is called **anomalous** dispersion. As indicated in Fig. 6.1 the imaginary part has a maximum at that point and that means light is absorbed most strongly, i.e. resonance absorption.

The idea is analogous to a mechanical oscillator picture, i.e. the charge dipoles, from the induced polarization, follow the wave at low frequencies while at high frequencies the induced polarization oscillates out of phase with the wave but otherwise again shows normal dispersion also at high frequencies.

A first application to look at is the tenuous or thin plasma. Further applications will have to wait until the end of the final chapter, i.e. after optics.

Consider eqn. 6.12 and make two justifiable but apparently quite crude approximations:

- Let the resonance frequency of the system tend to zero, $\omega_0 \to 0$. This describes a system of **free** charges, here in a plasma. Remember that ω_0 results from the assumption of a harmonic oscillator potential binding the charges hence an ever smaller resonance frequency corresponds to a more and more shallow potential, less and less bound charges.

- Also let the damping factor tend to zero, $\gamma \to 0$, which justifies the term 'tenuous' plasma, i.e. practically non-interacting charges.

What is left in eqn. 6.12 is

$$\hat{\epsilon} = \epsilon_0 \left(1 + \frac{-N\,q^2}{m\,\epsilon_0 \omega^2}\right) = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right).$$

where $\omega_p^2 = \frac{N\,q^2}{m\,\epsilon_0}$ is a regular technical term in plasma physics, the plasma frequency.

One way of gaining some insight into what this form of permittivity means is to relate it to the general dispersion relation eqn. 6.5 in its simpler form

$$K^2 = \omega^2 \mu \hat{\epsilon}(\omega) = \frac{\omega^2}{c^2} \left(1 - \frac{\omega_p^2}{\omega^2}\right). \tag{6.13}$$

where as you recall, K^2 was the complex wave vector magnitude squared and $\hat{\epsilon}$ is the complex permittivity. Considering the wave vector in this manner then permits to use what we've seen above.

It is clear that K^2 tends to zero as ω tends to ω_p. When ω becomes smaller than ω_p then K becomes imaginary. An imaginary wave vector though means exponential damping of a wave in a medium, here the plasma. This in turn means that all of the wave is reflected, no transmission can take place. Transmission and reflection are properly dealt with in the next chapter. Let's just accept them for now.

The numerical example would be for instance the ionosphere around Earth. It has typically a charge density of $N = 10^{15}$ electrons per m^3 hence a plasma frequency around $f_p = 9 \times 10^6$ Hz. Therefore radio waves below several MHz are totally reflected while higher frequency waves like for television and UHF (larger than 80 MHz) are transmitted by the ionosphere. This all can change during solar flares for instance when significant changes to the electron density of the plasma can occur.

Chapter 7

Selected topics in Optics

This chapter will mostly focus on fundamental topics of geometric optics as explained with electromagnetic waves in Maxwell's theory.

7.1 Reflection and refraction - Snell's law

Consider a plane wave with wave vector $\mathbf{k}_i$ incident on a plane interface giving rise to a reflected wave with wave vector $\mathbf{k}_r$ and a transmitted wave with wave vector $\mathbf{k}_t$, see Fig. 7.1. These two assumptions, reflected and transmitted, are not strictly necessary at this point but will lead to two separate conclusions, both of which are important.

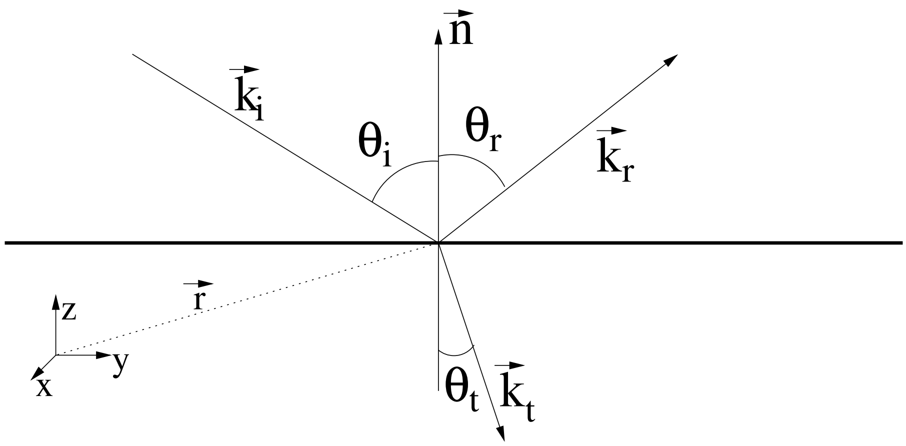

Figure 7.1: The wave vectors $\mathbf{k}_i$, $\mathbf{k}_r$ and $\mathbf{k}_t$ all lie in the plane of incidence.

At a point $\mathbf{r}$ on the interface, the parallel components of the electric field must be the same on both sides of the interface (compare boundary conditions in section 5.7). Therefore

$$E_{0,i}^{||}\, e^{i(\mathbf{k}_i\cdot\mathbf{r}-\omega\, t)} + E_{0,r}^{||}\, e^{i(\mathbf{k}_r\cdot\mathbf{r}-\omega\, t+\phi_r)} = E_{0,t}^{||}\, e^{i(\mathbf{k}_t\cdot\mathbf{r}-\omega\, t+\phi_t)}$$

(a phase change on transmission or reflection is at least a possibility). If this is to be satisfied at all $\mathbf{r}$ on the interface, the arguments of the exponentials must all have the same functional dependence on $\mathbf{r}$ and t, i.e.

$$\mathbf{k}_i \cdot \mathbf{r} - \omega\, t = \mathbf{k}_r \cdot \mathbf{r} - \omega\, t + \phi_r = \mathbf{k}_t \cdot \mathbf{r} - \omega\, t + \phi_t \tag{7.1}$$

From the first equality, get $(\mathbf{k}_i - \mathbf{k}_r)\cdot\mathbf{r} = \phi_r$. The surface defined by this equation, the interface, is perpendicular to $(\mathbf{k}_i - \mathbf{k}_r)$. In other words, the vector $(\mathbf{k}_i - \mathbf{k}_r)$ lies in the incidence plane which is perpendicular to the interface plane, described by its normal vector $\mathbf{n}$. Hence $\mathbf{n}$ lies in the incidence plane and the cross product of $(\mathbf{k}_i - \mathbf{k}_r)$ with $\mathbf{n}$ is zero:

$$(\mathbf{k}_i - \mathbf{k}_r) \times \mathbf{n} = 0\ ,$$

implying that

$$k_i\, \sin\theta_i = k_r\, \sin\theta_r$$

or, since the incident and reflected waves are both in the same medium, the magnitudes are equal, hence

$$\sin\theta_i = \sin\theta_r\ \Rightarrow \theta_i = \theta_r$$

Therefore the angle of incidence is equal to the angle of reflection.

Using the second half of eqns. 7.1 gives

$$(\mathbf{k}_i - \mathbf{k}_t)\cdot\mathbf{r} = \phi_t\ .$$

The same argument now yields

$$(\mathbf{k}_i - \mathbf{k}_t) \times \mathbf{n} = 0\ ,$$

leading to

$$k_i\, \sin\theta_i = k_t\, \sin\theta_t$$

which, after multiplying both sides with c/ω, becomes

$$n_i\, \sin\theta_i = n_t\, \sin\theta_t$$

where the refractive index appears. It is apparent that **Snell's law** is a consequence only of the plane wave nature of the disturbance and the requirement of continuity. It therefore finds plenty of applications outside optics. The three observations made in this section, wave vectors all in the plane of incidence, equal angles of incidence and reflection and Snell's law are the three fundamental laws of geometric optics.

7.2 Fresnel equations

The previous discussion of Snell's law dealt purely with angles of wave vectors because of the cancelling of amplitudes of waves due to the boundary condition on the parallel (to the interface) components of the electric field. If one wants to obtain the full amplitudes $E_{0,i}$, $E_{0,r}$ and $E_{0,t}$ of the incident, reflected and transmitted wave, one must use the boundary conditions in more detail.

First, one has to resolve the full vector **E** into two components, one perpendicular, normal, to the plane of incidence (not the interface, beware) and one embedded in the plane of incidence, parallel to it. Let's call the first $\mathbf{E}_n$ and show it in a drawing with the usual circle with a cross like the back of an arrow into the paper, and the second $\mathbf{E}_p$ (drawn obviously on paper as an arrow since we can in this case). As a reminder, the plane of incidence is the plane containing the incident and reflected rays.

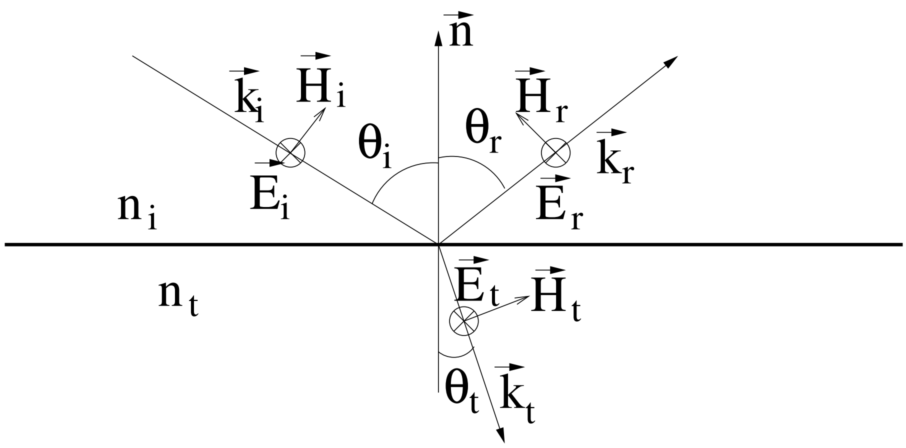

Figure 7.2: The electric vector of normal polarized waves is perpendicular to the plane of incidence. The vectors illustrated refer to the fields at the vertex of the rays.

7.2.1 Normal polarization

An n-polarized wave has its electric field **E** perpendicular to the plane of incidence, meaning that **H** lies in the plane of incidence. Assuming that $\mathbf{E}_i$, $\mathbf{E}_r$ and $\mathbf{E}_t$ all point in the same direction (say, into the paper), we obtain for points on the interface for the magnitudes

$$E_i + E_r = E_t \tag{7.2}$$

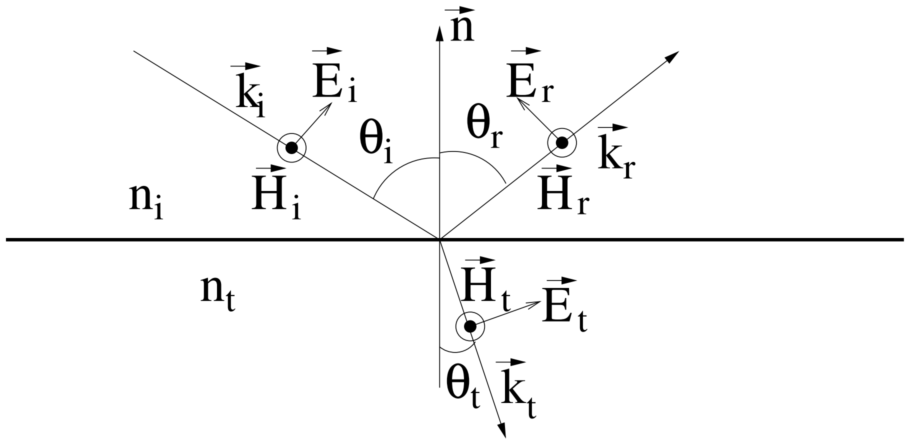

Figure 7.3: The electric vector of parallel polarized waves is parallel to the plane of incidence.

Applying the requirement that $H_{||}$ be continuous (boundary condition), we have also from Fig. 7.2

$$H_i \cos\theta_i - H_r \cos\theta_r = H_t \cos\theta_t \tag{7.3}$$

In the medium, the magnitudes of **H** and **E** are related by (remember, **E** is normal to the propagation direction of the wave here by construction)

$$k\,E = \omega\,\mu\,H \ ,$$

compare section 6.4 and take the magnitude. Using the dispersion relation in matter

$$k^2 = \omega^2\,\epsilon\,\mu$$

we get for the magnitudes (we want the index of refraction since it is easier to measure and discuss for each material in the process)

$$H = \sqrt{\frac{\epsilon}{\mu}}\,E = \frac{n}{c\,\mu}\,E \ .$$

Multiplying eqn.7.3 by c and substituting for H, get

$$\frac{n_i}{\mu_i}\,(E_i - E_r)\,\cos\theta_i = \frac{n_t}{\mu_t}\,E_t \cos\theta_t \tag{7.4}$$

where we used Snell's law already and the fact that reflected and incident ray are in the same material by definition.

Expressions eqn. 7.2 and eqn. 7.4 may be solved for E_r and E_t to obtain

$$\left(\frac{E_r}{E_i}\right)_n = \frac{\frac{n_i}{\mu_i}\cos\theta_i - \frac{n_t}{\mu_t}\cos\theta_t}{\frac{n_i}{\mu_i}\cos\theta_i + \frac{n_t}{\mu_t}\cos\theta_t} \equiv r_n \tag{7.5}$$

and

$$\left(\frac{E_t}{E_i}\right)_n = \frac{2\,\frac{n_i}{\mu_i}\cos\theta_i}{\frac{n_i}{\mu_i}\cos\theta_i + \frac{n_t}{\mu_t}\cos\theta_t} \equiv t_n \tag{7.6}$$

7.2.2 Parallel polarization

When the wave is p-polarized, the electric field is parallel to the plane of incidence, while **B** (and therefore **H**) is perpendicular. Assuming that **H** has the same direction for all three waves (for instance, out of the paper), we construct Fig. 7.3. Again using continuity of $H_{||}$ and $E_{||}$, we have for the magnitudes

$$H_i + H_r = H_t$$

and

$$E_i\cos\theta_i - E_r\cos\theta_r = E_t\cos\theta_t \tag{7.7}$$

Again replace H with $\sqrt{\epsilon/\mu}\,E$ to re-write the first equation above as

$$\frac{n_i}{\mu_i}\,(E_i + E_r) = \frac{n_t}{\mu_t}\,E_t \tag{7.8}$$

Solving eqn. 7.7 and eqn. 7.8 for E_r and E_t, we find

$$\left(\frac{E_r}{E_i}\right)_p = \frac{\frac{n_t}{\mu_t}\cos\theta_i - \frac{n_i}{\mu_i}\cos\theta_t}{\frac{n_i}{\mu_i}\cos\theta_t + \frac{n_t}{\mu_t}\cos\theta_i} \equiv r_p \tag{7.9}$$

and

$$\left(\frac{E_t}{E_i}\right)_p = \frac{2\,\frac{n_i}{\mu_i}\cos\theta_i}{\frac{n_i}{\mu_i}\cos\theta_t + \frac{n_t}{\mu_t}\cos\theta_i} \equiv t_p \tag{7.10}$$

This set, eqns. 7.5, 7.6, 7.9, 7.10 are the **Fresnel equations**. They are just not very practical in their current form. Two better ways to display them involve assumptions, very general ones but still assumptions.

Assume that all permeabilities are identical. This is very often entirely justified in optics applications except for exotic dielectrics like ferrites in microwave systems for instance. That gives $\mu_i = \mu_t = \mu_0$ and using Snell's law to eliminate ratios of refractive indices (bringing in the sin functions) one obtains Fresnel equations in the form

$$r_n = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}\ ; \quad r_p = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \tag{7.11}$$

$$t_n = \frac{2\,\cos\theta_i\sin\theta_t}{\sin(\theta_i + \theta_t)}\ ; \quad t_p = \frac{2\,\cos\theta_i\sin\theta_t}{\sin(\theta_i + \theta_t)\,\cos(\theta_i - \theta_t)} \tag{7.12}$$

A second alternative way of expressing the Fresnel equations is found by eliminating the angle of transmission, θ_t, which often is difficult to measure. Using Snell's law extensively and letting

$$n = \frac{\frac{n_t}{\mu_t}}{\frac{n_i}{\mu_i}}$$

and (Snell)

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \frac{1}{n}\sqrt{n^2-\sin^2\theta_i}$$

we get (more practical, not necessarily looking nicer)

$$r_n = \frac{\cos\theta_i - n\,\cos\theta_t}{\cos\theta_i + n\,\cos\theta_t} = \frac{\cos\theta_i - \sqrt{n^2-\sin^2\theta_i}}{\cos\theta_i + \sqrt{n^2-\sin^2\theta_i}} \tag{7.13}$$

$$t_n = \frac{2\,\cos\theta_i}{\cos\theta_i + \sqrt{n^2-\sin^2\theta_i}} \tag{7.14}$$

$$r_p = \frac{n\,\cos\theta_i - \cos\theta_t}{n\,\cos\theta_i + \cos\theta_t} = \frac{n^2\,\cos\theta_i - \sqrt{n^2-\sin^2\theta_i}}{n^2\,\cos\theta_i + \sqrt{n^2-\sin^2\theta_i}} \tag{7.15}$$

$$t_p = \frac{2\,n\,\cos\theta_i}{n^2\,\cos\theta_i + \sqrt{n^2-\sin^2\theta_i}} \tag{7.16}$$

7.3 Additional observations

Something easier to remember follows directly from the above, i.e. at normal incidence the amplitude reflection coefficients reduce to

$$r_n = \frac{1-n}{1+n}\,; \quad r_p = -\frac{1-n}{1+n} \tag{7.17}$$

Recall that for this section we use $\mu_i = \mu_t = \mu_0$ and

$$n = \frac{n_t}{n_i}$$

Other important consequence can be drawn for special cases for n, each of which is discussed below:

$n = \tan\theta_i$ **Brewster angle**.

$n < 1$ The medium can show **total internal reflection**.

The Brewster angle is defined as

$$n = \tan\theta_i \ . \tag{7.18}$$

What happens is that

$$r_p = \frac{\frac{\sin^2\theta_i}{\cos\theta_i} - \sqrt{\frac{\sin^2\theta_i}{\cos^2\theta_i} - \sin^2\theta_i}}{\frac{\sin^2\theta_i}{\cos\theta_i} + \sqrt{\frac{\sin^2\theta_i}{\cos^2\theta_i} - \sin^2\theta_i}} = \frac{\frac{\sin^2\theta_i}{\cos\theta_i} - \frac{\sin^2\theta_i}{\cos\theta_i}}{\frac{\sin^2\theta_i}{\cos\theta_i} + \frac{\sin^2\theta_i}{\cos\theta_i}} = 0$$

and all other components staying finite, non-zero. That means at the Brewster angle you expect to see perfectly polarized reflected waves (transmission still carries both polarizations) containing the electric field amplitude purely in the normal (to the plane of incidence) direction.

For the second case, $n < 1$, one can define again a special angle. This case corresponds to boundary transitions from a higher to a lower refractive index material, e.g. from glass to air. Consider

$$\cos\theta_t = \frac{1}{n}\sqrt{n^2 - \sin^2\theta_i}$$

which can become complex in this case. The **critical angle**, θ_c, would be

$$\sin\theta_c = n$$

at which there is no transmitted (refracted) wave anymore. That, however, implies that all the light is reflected, i.e. total internal reflection takes place.

After these standard optics effects, let's finish the chapter and the lecture with some more optics in conducting matter. This requires to recall some relations from chapter 6, especially sect. 6.5. Waves in conducting matter led to complex and frequency dependent refractive indices, permittivities and wave vectors.

First, picking up on total internal reflection from above, take a look at the complex refractive index in the high-frequency, free charges limit, see eqn. 6.13:

$$\hat{n}^2 = c^2\mu\hat{\epsilon}(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2}\right) .$$

At high frequencies, well beyond any plasma frequency, such as x-ray light waves, the refractive index of any(!) material is smaller than 1. That means, however, x-rays at a shallow angle of incidence, see the critical angle above, from vacuum or any medium with $n \geq 1$ are totally reflected at any interface, whatever the second medium. They show **total external reflection** a fact that is being used for building x-ray telescopes for instance.

Finally, consider reflection from any good conductor. Have the first medium non-conducting, $\hat{n}_1 = n_1$ and the second medium a good conductor. The 'good conductor' condition was defined as $(g \gg \epsilon\omega)$. When applied to eqns. 6.6 and 6.7 one can see that

$$\beta \rightarrow \frac{g}{\epsilon\,\omega}$$

and hence for eqn. 6.6 one gets

$$k^2 \rightarrow \frac{1}{2}\,\mu\,\epsilon\,\omega^2\,\frac{g}{\epsilon\,\omega} = \frac{1}{2}\,\mu\,\omega\,g$$

and for eqn. 6.7

$$\alpha^2 \rightarrow \frac{\mu\,g^2\,\epsilon\,\omega}{2\,\epsilon\,g} = \frac{1}{2}\,\mu\,\omega\,g$$

Therefore for a good conductor, the real and the imaginary part of the wave vector magnitude and hence the refractive index become identical.

$$\mathrm{Re}(\hat{n}) = \mathrm{Im}(\hat{n}) = c\,\sqrt{\frac{\mu\,g}{2\,\omega}} \gg 1 \qquad (7.19)$$

What can we learn from that? The most straightforward consequence is to note that the Fresnel equations remain valid also for complex refractive indices! Nothing in their derivation required the refractive indices to be real numbers. Replacing real number with complex numbers in the Fresnel equations will imply that phase changes of the wave are now included but that is of no concern here.

The quick revelation to work out here is to see what happens if a light wave impinges on a good conductor in terms of reflection. Make it simple by looking at eqn. 7.17, i.e. normal incidence light. Assuming light comes from an air (or vacuum) interface $\hat{n}_i = n_i = 1$ and hits a good conductor with $\hat{n}_2 = \hat{n}_t \gg n_i$. At normal incidence the reflection coeficients, eqns. 7.5, 7.9, become identical apart from a sign change which is a consequence of the definition of the incoming and reflected wave vector. In any case, for a good conductor the reflection magnitude at normal incidence

$$r = \frac{\hat{n}_t - \hat{n}_i}{\hat{n}_i + \hat{n}_t}$$

becomes $r \rightarrow 1$ with the assumptions from above. A wave hence impinging on a good conductor results in strong reflection, i.e. the reason why metals appear shiny in visible light in general.

Now note the contrasting observations for waves on a conducting medium. On the one hand we got the transparency to x-rays and other high-frequency waves in eqn.6.13, on the other the strong reflection in the good conductor limit. These are seemingly unrelated assumptions but both feature a frequency dependence of the refractive index. The good conductor assumption and consequence, eqn. 7.19, only carries on to the point that the frequency in the denominator starts to dominate. Then the refractive index will decrease to the point that it is close to a value of one and remain there, eqn. 6.13 which in turn implies, see above, that even a good conductor will become transparent to waves. This transition will show up most conveniently in reflection measurements on metals as a function of wave frequency and it is used to measure properties of the free electrons in metals for instance their effective mass entering the conductivity.

In summary, we barely scratched the surface of what Maxwell's equations can do for real life applications like in optics and plasma physics. The intention is to give a first taster session of what is still to come in the corresponding modules next year. These are PX384, Electrodynamics, PX392, Plasma Electrodynamics and even PX385, Condensed matter physics (anisotropic media or crystal optics). Recent textbooks like [4] will go into more detail and display experimental data as comparison. The vast number of applications and validations of this old theory is astonishing and makes it more relevant today even than it was a hundred years ago.

Appendix A

Resources

A.1 Vector identities

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \tag{A.1}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c} \tag{A.2}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})\, (\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})\, (\mathbf{b} \cdot \mathbf{c}) \tag{A.3}$$

$$\nabla \times \nabla\, \phi = 0 \tag{A.4}$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0 \tag{A.5}$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla\, (\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a} \tag{A.6}$$

$$\nabla \cdot (\phi\, \mathbf{a}) = \mathbf{a} \cdot \nabla \phi + \phi\, \nabla \cdot \mathbf{a} \tag{A.7}$$

$$\nabla \times (\phi\, \mathbf{a}) = \nabla\, \phi \times \mathbf{a} + \phi \nabla \times \mathbf{a} \tag{A.8}$$

$$\nabla\, (\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) \tag{A.9}$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \tag{A.10}$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}\, (\nabla \cdot \mathbf{b}) - \mathbf{b}\, (\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\, \mathbf{a} - (\mathbf{a} \cdot \nabla)\, \mathbf{b} \tag{A.11}$$

A.2 Integral identities

$$\int_V \nabla \cdot \mathbf{a}\, d^3 r = \int_S \hat{\mathbf{n}} \cdot \mathbf{a}\, dS \tag{A.12}$$

$$\int_V \nabla\, \phi\, d^3 r = \int_S \hat{\mathbf{n}}\, \phi\, dS \tag{A.13}$$

$$\int_V \nabla \times \mathbf{a} \, d^3 r = \int_S \hat{\mathbf{n}} \times \mathbf{a} \, dS \tag{A.14}$$

$$\int_S \hat{\mathbf{n}} \cdot \nabla \times \mathbf{a} \, dS = \oint_C \mathbf{a} \cdot d\mathbf{l} \tag{A.15}$$

$$\int_S \hat{\mathbf{n}} \times \nabla \phi \, dS = \oint_C \phi \, d\mathbf{l} \tag{A.16}$$

Appendix B

Glossary

Vector symbols are given in bold font, $\mathbf{A}$, unit vectors are indicated by a hat symbol, $\hat{\mathbf{r}}$. Only in section 6.4 'hat' indicates a complex term.

$\mathbf{A}$ vector potential

$\mathbf{B}$ magnetic induction field

$\mathbf{D}$ electric displacement field

d^3r, dV element of volume

$d\mathbf{S}$ element of surface, area integration element

$d\mathbf{l}$ line element, line integral element

$\delta(x)$ Dirac delta function

∂S boundary curve to area S

∂V boundary area to volume V

$\mathbf{E}$ electric field

e elementary charge ($\approx 1.6 \times 10^{-19}$ Coulomb)

ϵ permittivity

g conductivity

∇ gradient operator

$\mathbf{H}$ magnetic field intensity

I current

$\mathbf{j}$ current density

$\mathbf{k}$ wave vector

κ dielectric constant $= \epsilon/\epsilon_0$

λ wavelength

$\mathbf{m}$ magnetic dipole moment

$\mathbf{M}$ magnetization

μ permeability

n index of refraction

$\mathbf{n}$ normal vector

ϕ electric potential

$\mathbf{p}$ momentum

$\mathbf{P}$ polarization

q, Q electric charge

$\mathbf{r}$ position vector

ρ charge density

$\mathbf{S}$ Poynting vector

σ surface charge density

ω angular frequency

$\mathbf{v}$ velocity

$\chi_{e,m}$ electric, magnetic susceptibility

W work

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