Introduction to Quantum Field Theory

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齊白石墨蝦圖
書：魚蝦負我短劍白石老人爲魚蝦所誤

Cover Image：Ink Shrimp by Qi Baishi，ca 1947
Inscription：Carry my dagger，fish and shrimp；much，alas，have I given to you．

## Preface

This series of ten one－hour lectures on Introduction to Quantum Field Theory（QFT） serves as the first of the three－part module，PX454：Theoretical Particle Physics． It is tailored for students in their final year pursuing a master＇s degree in physics at the University of Warwick during the spring of 2024 ．Within this brief timeframe， we aim to reach two primary objectives：

1．Enable students to perform basic field－theoretical calculations．
2．Facilitate a smooth transition for students to independently explore QFT with the aid of extensive literature．

With these goals in mind，we place an emphasis on calculations，and students are encouraged to attempt the questions in the Exercise sections，designed to guide them in comprehending the derivations presented．

As indicated by the title，our focus will be on QFT for particle physics．The system we consider here is one where methods in statistical mechanics do not apply． Consequently，our approach differs noticeably from a QFT course oriented towards condensed matter physics，where temperature plays a vital role．

One of the celebrated achievements of QFT is its ability to describe the mi－ croscopic world at the fundamental level through perturbation theories，wherein interactions are broken down in assending orders of coupling strength．Within two chapters，we delve into the cononical quantisation of scalar，spinor，and photon fields without interactions，trying to establish a foundational understanding at the zero－th order of perturbation．Chapter 1 elucidates the core ideas of canonical quantisation using the simplest field：real scalars in $1+1$ spacetime．After the introduction of the Lagrange－Hamilton formalism，these concepts find applications in Chapter 2，where we systematically quantise the physical fields one by one．

In preparing these notes，I am deeply grateful for the inspiration and suggestions from my PX454 co－lecturers，Prof．Paul Harrison and Prof．Bill Murray．Their support has been truly invaluable．

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## Chapter 1

## Quantum Theory of Free Fields

Fourier transform the momentum operator,

$$
\begin{aligned}
& \mathcal{F}[\hat{p} f]=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} d x\left(-i \frac{d}{d x} f\right) e^{-i p x} \\
& =\frac{-i}{\sqrt{2 \pi}}\left\{\left[f e^{-i p x}\right]_{-\infty}^{\infty}+i \int d x p f e^{-i p x}\right\} \\
& =\frac{1}{\sqrt{2 \pi}} \int d x p f e^{-i p x} \\
& =p \mathcal{F}[f]
\end{aligned}
$$

with a function that gracefully fades into oblivion at infinities.

### 1.1 Quantum Harmonic Oscillator

We use the quantum harmonic oscillator as the initial test case for methods employed in Quantum Field Theory (QFT).

In the position space, we have the position and momentum operators,

$$
\begin{align*}
& \hat{x}=x  \tag{1.1}\\
& \hat{p}=-i \frac{\mathrm{~d}}{\mathrm{~d} x} \tag{1.2}
\end{align*}
$$

which gives us the commutation relation (Exercise 11.10),

$$
\begin{equation*}
[\hat{x}, \hat{p}]=i \tag{1.3}
\end{equation*}
$$

The Hamiltonian of a harmonic oscillator is given by

$$
\begin{equation*}
\hat{H}=\frac{\hat{p}^{2}}{2 m}+\frac{1}{2} m \omega^{2} \hat{x}^{2} \tag{1.4}
\end{equation*}
$$

Define the ladder operators $\hat{a}$ and $\hat{a}^{\dagger}$ as linear combinations of $\hat{x}$ and $\hat{p}$ :

$$
\begin{align*}
\hat{a} & =\frac{1}{\sqrt{2}}\left(\sqrt{m \omega} \hat{x}+\frac{i}{\sqrt{m \omega}} \hat{p}\right),  \tag{1.5}\\
\hat{a}^{\dagger} & =\frac{1}{\sqrt{2}}\left(\sqrt{m \omega} \hat{x}-\frac{i}{\sqrt{m \omega}} \hat{p}\right) . \tag{1.6}
\end{align*}
$$

We have the commutation relation,

$$
\begin{equation*}
\left[\hat{a}, \hat{a}^{\dagger}\right]=1 \tag{1.7}
\end{equation*}
$$

The Hamiltonian can then be rewritten in terms of $\hat{a}$ and $\hat{a}^{\dagger}$ :

$$
\begin{equation*}
\hat{H}=\frac{1}{2}\left(\hat{a}^{\dagger} \hat{a}+\hat{a} \hat{a}^{\dagger}\right) \omega=\left(\hat{a}^{\dagger} a+\frac{1}{2}\right) \omega \tag{1.8}
\end{equation*}
$$

This leads to the following observations. First,

$$
\begin{equation*}
\left[\hat{H}, \hat{a}^{\dagger}\right]=\omega \hat{a}^{\dagger} \tag{1.9}
\end{equation*}
$$

In the Heisenberg picture, an operator evolves with time as

$$
\begin{equation*}
\hat{\mathcal{O}}(t)=e^{i H t} \hat{\mathcal{O}}(0) e^{-i H t} \tag{1.10}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\frac{\mathrm{d} \hat{\mathcal{O}}}{\mathrm{~d} t}=i[H, \hat{\mathcal{O}}] \tag{1.11}
\end{equation*}
$$

Therefore, we arrive at the time evolution of $\hat{a}^{\dagger}$,

$$
\begin{equation*}
\hat{a}^{\dagger}(t)=e^{i \omega t} \hat{a}^{\dagger}(0) \tag{1.12}
\end{equation*}
$$

Similarly, we have

$$
\begin{align*}
{[\hat{H}, \hat{a}] } & =-\omega \hat{a}  \tag{1.13}\\
\hat{a} & =e^{-i \omega t} \hat{a}(0) \tag{1.14}
\end{align*}
$$

The seemingly time-dependent phases, $e^{ \pm i \omega t}$, actually mean that the system is timetranslation invariant, i.e., time-independent.

Second, the energy eigenstate of the harmonic oscillator at level $n$ is $|n\rangle$ with eigenvalue $E_{n}$,

$$
\begin{equation*}
\hat{H}|n\rangle=E_{n}|n\rangle \tag{1.15}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\hat{H}\left(\hat{a}^{\dagger}|n\rangle\right)=\left(E_{n}+\omega\right)\left(\hat{a}^{\dagger}|n\rangle\right)=E_{n+1}\left(\hat{a}^{\dagger}|n\rangle\right) \tag{1.16}
\end{equation*}
$$

That is

$$
\begin{equation*}
\hat{a}^{\dagger}|n\rangle=|n+1\rangle \tag{1.17}
\end{equation*}
$$

up to a normalisation factor. Similarly,

$$
\begin{equation*}
\hat{a}|n\rangle=|n-1\rangle \tag{1.18}
\end{equation*}
$$

which lowers the energy level of the system; however, this cannot continue forever, as it will eventually hit the ground state, which means

$$
\begin{equation*}
\hat{a}|0\rangle=0 . \tag{1.19}
\end{equation*}
$$

As a result, we have the zero-point energy of the system,

$$
\begin{equation*}
\langle 0| \hat{H}|0\rangle=\langle 0|\left(\hat{a}^{\dagger} a+\frac{1}{2} \omega\right)|0\rangle=\frac{1}{2} \omega \tag{1.20}
\end{equation*}
$$

These ladder operators are called annihilation and creation operators.

### 1.1.1 Mode Expansion

The definitions of the ladder opperators, Eqs. 1.5 and 1.6, could seem unnatural at first glance. In fact, the essence is that they are liner transformations of $\hat{x}$ and $\hat{p}$ and form a ladder up and down the energy levels; all other features are derivative.

To see how this actually happens, let's consider canonical coordinates $\hat{q}$ and $\hat{p}$, which have been promoted to operators (i.e., quantised) with $[\hat{q}, \hat{p}]=i$. Expand them in $\hat{a}$ and $\hat{b}$ via a general linear transform,

$$
\begin{align*}
& \hat{q}=c_{1} \hat{a}+c_{2} \hat{b}  \tag{1.21}\\
& \hat{p}=\lambda\left(c_{1} \hat{a}-c_{2} \hat{b}\right), \tag{1.22}
\end{align*}
$$

where $c_{1}, c_{2}$, and $\lambda$ are coefficients to be determined. We can obtain the inverse transform straightforwardly:

$$
\begin{align*}
& \hat{a}=\frac{\hat{q}+\hat{p} / \lambda}{2 c_{1}}  \tag{1.23}\\
& \hat{b}=\frac{\hat{q}-\hat{p} / \lambda}{2 c_{2}} \tag{1.24}
\end{align*}
$$

and the commutation relation,

$$
\begin{equation*}
[\hat{a}, \hat{b}]=-\frac{[\hat{q}, \hat{p}]}{2 \lambda c_{1} c_{2}}=-\frac{i}{2 \lambda c_{1} c_{2}} \tag{1.25}
\end{equation*}
$$

Assume both $\hat{p}$ and $\hat{q}$ are Hermitian and choose $\hat{b}=\hat{a}^{\dagger}$ (Exercise 1.2 d ), then $c_{1}=c_{2}^{*}$ and $\lambda$ is a pure imaginary number.

Further assume a general form of a Hamiltonian, which is by definition Hermitian, with real coefficients $h_{1}$ and $h_{2}$ to be given for a physical system, and then expand and collect terms of $\hat{a}$ and $\hat{a}^{\dagger}$ :

$$
\begin{align*}
\hat{H} & =h_{1} \hat{p}^{2}+h_{2} \hat{q}^{2}  \tag{1.26}\\
& =A \hat{a} \hat{a}+B \hat{a}^{\dagger} \hat{a}^{\dagger}+C\left(\hat{a}^{\dagger} \hat{a}+\hat{a} \hat{a}^{\dagger}\right) \tag{1.27}
\end{align*}
$$

with

$$
\begin{align*}
& A=\left(h_{1} \lambda^{2}+h_{2}\right) c_{1}^{2}  \tag{1.28}\\
& B=\left(h_{1} \lambda^{2}+h_{2}\right) c_{2}^{2}  \tag{1.29}\\
& C=\left(-h_{1} \lambda^{2}+h_{2}\right) c_{1} c_{2} \tag{1.30}
\end{align*}
$$

If

$$
\begin{equation*}
h_{1} \lambda^{2}+h_{2}=0 \tag{1.31}
\end{equation*}
$$

then both $A$ - and $B$-terms vanish, so that we are left with

$$
\begin{equation*}
\hat{H}=C\left(\hat{a}^{\dagger} \hat{a}+\hat{a} \hat{a}^{\dagger}\right) \tag{1.32}
\end{equation*}
$$

which gives us

$$
\begin{align*}
& {\left[\hat{H}, \hat{a}^{\dagger}\right] \sim \hat{a}^{\dagger}}  \tag{1.33}\\
& {[\hat{H}, \hat{a}] \sim \hat{a}} \tag{1.34}
\end{align*}
$$

The relation, Eq. 1.31, can be viewed as a dispersion relation for reasons that will become clear in Sec. 1.2.1.

Define $D$ as follows:

$$
\begin{equation*}
\left[\hat{H}, \hat{a}^{\dagger}\right] \equiv D \hat{a}^{\dagger}, \quad D>0 \tag{1.35}
\end{equation*}
$$

which implies $[\hat{H}, \hat{a}]=-D \hat{a}$. It can be shown that

$$
\begin{align*}
& D=2 \sqrt{\left|h_{1} h_{2}\right|}  \tag{1.36}\\
& C\left[\hat{a}, \hat{a}^{\dagger}\right]=D \tag{1.37}
\end{align*}
$$

Given the dispersion relation, the Hamiltonian, Eq. 1.26, fully determines the dynamics of the system (Eq. 1.36), regardless of $c_{1,2}$ and $\lambda$. As a result, while the conventions, like $c_{1,2}$, enter the expressions of $C$ and $\left[\hat{a}, \hat{a}^{\dagger}\right]$, they cancel each other such that $D$ is independent of them. In the case of the harmonic oscillator,

$$
\begin{align*}
h_{1} & =\frac{1}{2 m}  \tag{1.38}\\
h_{2} & =\frac{1}{2} m \omega^{2} \tag{1.39}
\end{align*}
$$

It can be shown that

$$
\begin{align*}
\lambda & =-i m \omega  \tag{1.40}\\
D & =\omega \tag{1.41}
\end{align*}
$$

By requiring that $\left[\hat{a}, \hat{a}^{\dagger}\right]=1$, we have

$$
\begin{equation*}
C=\frac{\omega}{2} \tag{1.42}
\end{equation*}
$$

and also $c_{1,2}$ can be fixed up to an arbitrary phase - in this sense, the harmonic oscillator case is fully recovered (Exercise 1.2).

### 1.2 Quantum Field Expansion

Consider a real scalar field, $\phi(x, t)$, that is a generalised coordinate in $1+1$ spacetime, and its real canonical momentum,

$$
\begin{equation*}
\pi(x, t) \tag{1.43}
\end{equation*}
$$

Promote them into operators (second quantisation), while $x$ and $t$ are now considered labels only (namely, the dynamics of $x$, such as $\dot{x}$, do not concern us anymore). We have the canonical equal-time commutators,

$$
\begin{align*}
{[\phi(x, t), \pi(y, t)] } & =i \delta(x-y)  \tag{1.44}\\
{[\phi(x, t), \phi(y, t)] } & =[\pi(x, t), \pi(y, t)]=0 \tag{1.45}
\end{align*}
$$

Expand them in $\hat{a}$ and $\hat{a}^{\dagger}$ via Fourier transforms (Exercise 1. 1b):

$$
\begin{align*}
& \hat{\phi}(x, t)=\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} k\left(c_{1} \hat{a} e^{i k x}+c_{2} \hat{a}^{\dagger} e^{-i k x}\right)  \tag{1.46}\\
& \hat{\pi}(x, t)=\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} k\left(-i \omega c_{1} \hat{a} e^{i k x}+i \omega c_{2} \hat{a}^{\dagger} e^{-i k x}\right) \tag{1.47}
\end{align*}
$$

where the $x$-dependence is fully encapsulated in the phases, while the $t$-dependence is in $\hat{a}$ and $\hat{a}^{\dagger}$. The operators, $\hat{a}$ and $\hat{a}^{\dagger}$, and the coefficients, $c_{1,2}\left(c_{1}=c_{2}^{*}\right)$ and $\omega$ (real number), all depend on $k$.

To simplify the notation, from now on we drop the ^-notation for operators and use the following compact forms for Fourier transforms:

$$
\begin{align*}
& \tilde{f}_{\eta}(k)=\mathcal{F}_{\eta}[f]=\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} x f(x) e^{\eta i k x},  \tag{1.48}\\
& \tilde{s}_{\eta}(x)=\mathcal{K}_{\eta}[s]=\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} k s(k) e^{\eta i k x} \tag{1.49}
\end{align*}
$$

where $\eta$ stands for the sign + or - . Equations 1.46 and 1.47 become

$$
\begin{align*}
\phi & =\mathcal{K}_{+}\left[c_{1} a\right]+\mathcal{K}_{-}\left[c_{2} a^{\dagger}\right]  \tag{1.50}\\
\pi & =\mathcal{K}_{+}\left[-i \omega c_{1} a\right]+\mathcal{K}_{-}\left[i \omega c_{2} a^{\dagger}\right] \tag{1.51}
\end{align*}
$$

Note that (Exercise 1.3a),

$$
\begin{align*}
\mathcal{F}_{+}\left[s \tilde{t}_{+}\right] & =s(k) t(-k)  \tag{1.52}\\
\mathcal{F}_{+}\left[s \tilde{t}_{-}\right] & =s(k) t(k) \tag{1.53}
\end{align*}
$$

We have

$$
\begin{align*}
\mathcal{F}_{+}\left[i \omega_{k} \phi\right] & =i \omega_{k} c_{1,-k} a_{-k}+i \omega_{k} c_{2, k} a_{k}^{\dagger}  \tag{1.54}\\
\mathcal{F}_{+}[\pi] & =-i \omega_{-k} c_{1,-k} a_{-k}+i \omega_{k} c_{2, k} a_{k}^{\dagger} \tag{1.55}
\end{align*}
$$

from which we can solve $a^{\dagger}$ :

$$
\begin{equation*}
a_{k}^{\dagger}=\frac{\mathcal{F}_{+}\left[i \omega_{k} \phi+\pi\right]}{2 i \omega_{k} c_{2, k}} \tag{1.56}
\end{equation*}
$$

where we have assumed

$$
\begin{equation*}
\omega_{k}=\omega_{-k} \tag{1.57}
\end{equation*}
$$

Similarly, we have

$$
\begin{equation*}
a_{k}=\frac{\mathcal{F}_{-}\left[-i \omega_{k} \phi+\pi\right]}{-2 i \omega_{k} c_{1, k}} \tag{1.58}
\end{equation*}
$$

Furthermore, note that (Exercise 1. 3 b ), if

$$
\begin{equation*}
[f(x), g(y)]=r \delta(x-y) \tag{1.59}
\end{equation*}
$$

then

$$
\begin{align*}
& {\left[\tilde{f}_{-}(k), \tilde{g}_{+}\left(k^{\prime}\right)\right]=r \delta\left(k-k^{\prime}\right)}  \tag{1.60}\\
& {\left[\tilde{f}_{-}(k), \tilde{g}_{-}\left(k^{\prime}\right)\right]=r \delta\left(k+k^{\prime}\right)}  \tag{1.61}\\
& {\left[\tilde{f}_{+}(k), \tilde{g}_{+}\left(k^{\prime}\right)\right]=r \delta\left(k+k^{\prime}\right)} \tag{1.62}
\end{align*}
$$

Therefore,

$$
\begin{align*}
{\left[a_{k}, a_{k^{\prime}}^{\dagger}\right] } & \sim\left[\mathcal{F}_{-}\left[-i \omega_{k} \phi\right], \mathcal{F}_{+}[\pi]\right]+\left[\mathcal{F}_{-}[\pi], \mathcal{F}_{+}\left[i \omega_{k^{\prime}} \phi\right]\right]  \tag{1.63}\\
& =\omega_{k} \delta\left(k-k^{\prime}\right)+\omega_{k^{\prime}} \delta\left(k-k^{\prime}\right)  \tag{1.64}\\
& =2 \omega_{k} \delta\left(k-k^{\prime}\right),  \tag{1.65}\\
{\left[a_{k}, a_{k^{\prime}}\right] } & \sim\left[\mathcal{F}_{-}\left[-i \omega_{k} \phi\right], \mathcal{F}_{-}[\pi]\right]+\left[\mathcal{F}_{-}[\pi], \mathcal{F}_{-}\left[-i \omega_{k^{\prime}} \phi\right]\right]  \tag{1.66}\\
& =\omega_{k} \delta\left(k+k^{\prime}\right)-\omega_{k^{\prime}} \delta\left(k+k^{\prime}\right)  \tag{1.67}\\
& =0,  \tag{1.68}\\
{\left[a_{k}^{\dagger}, a_{k^{\prime}}^{\dagger}\right] } & \sim\left[\mathcal{F}_{+}\left[i \omega_{k} \phi\right], \mathcal{F}_{+}[\pi]\right]+\left[\mathcal{F}_{+}[\pi], \mathcal{F}_{+}\left[i \omega_{k^{\prime}} \phi\right]\right]  \tag{1.69}\\
& =-\omega_{k} \delta\left(k+k^{\prime}\right)+\omega_{k^{\prime}} \delta\left(k+k^{\prime}\right)  \tag{1.70}\\
& =0 \tag{1.71}
\end{align*}
$$

So, finally,

$$
\begin{align*}
{\left[a_{k}, a_{k^{\prime}}^{\dagger}\right] } & =\frac{\delta\left(k-k^{\prime}\right)}{2 \omega_{k} c_{1} c_{2}}  \tag{1.72}\\
{\left[a_{k}, a_{k^{\prime}}\right] } & =\left[a_{k}^{\dagger}, a_{k^{\prime}}^{\dagger}\right]=0 \tag{1.73}
\end{align*}
$$

### 1.2.1 Free Fields

To continue our discussion, we first notice

$$
\begin{equation*}
\frac{\partial \phi}{\partial x}=\partial_{x} \phi=\mathcal{K}_{+}\left[i k c_{1} a\right]+\mathcal{K}_{-}\left[-i k c_{2} a^{\dagger}\right] \tag{1.74}
\end{equation*}
$$

Now assume a general form of a free (namely, no interactions) Hamiltonian density, with real coefficients $h_{1,2,3}$ to be given for a physical system:

$$
\begin{equation*}
\mathcal{H}(x)=h_{1} \pi^{2}+h_{2}\left(\partial_{x} \phi\right)^{2}+h_{3} m^{2} \phi^{2} \tag{1.75}
\end{equation*}
$$

where $m$ is a real parameter to be determined that is introduced just to account for the dimension of $h_{3}$. Upon expanding and collecting terms of $a$ and $a^{\dagger}$, the Hamiltonian is then

$$
\begin{align*}
H & =\int \mathrm{d} x \mathcal{H}(x)  \tag{1.76}\\
& =A+B+C \tag{1.77}
\end{align*}
$$

where

$$
\begin{align*}
& A \sim \mathcal{K}_{+} \mathcal{K}_{+},  \tag{1.78}\\
& B \sim \mathcal{K}_{-} \mathcal{K}_{-}, \text {and }  \tag{1.79}\\
& C \sim \mathcal{K}_{+} \mathcal{K}_{-}+\mathcal{K}_{-} \mathcal{K}_{+} \tag{1.80}
\end{align*}
$$

Let's calculate the $A$-term first:
$A=\int \mathrm{d} x\left\{h_{1} \mathcal{K}_{+}\left[-i \omega c_{1} a\right] \mathcal{K}_{+}[\cdots]+h_{2} \mathcal{K}_{+}\left[i k c_{1} a\right] \mathcal{K}_{+}[\cdots]+h_{3} m^{2} \mathcal{K}_{+}\left[c_{1} a\right] \mathcal{K}_{+}[\cdots]\right\}$,
where $[\cdots]$ indicates the repetitive terms that are similar up to swapping. Note that (Exercise 1. 3 c )

$$
\begin{equation*}
\int \mathrm{d} x \tilde{s}_{+}(x) \tilde{t}_{+}(x)=\int \mathrm{d} k s(k) t(-k) \tag{1.82}
\end{equation*}
$$

for arbitrary functions $s(k)$ and $t(k)$. We have

$$
\begin{align*}
A= & \int \mathrm{d} k\left[h_{1}\left(-i \omega_{k} c_{1, k}\right)\left(-i \omega_{-k} c_{1,-k}\right) a_{k} a_{-k}\right. \\
& +h_{2}\left(i k c_{1, k}\right)\left(-i k c_{1,-k}\right) a_{k} a_{-k} \\
& \left.+h_{3} m^{2} c_{1, k} c_{1,-k} a_{k} a_{-k}\right]  \tag{1.83}\\
= & \int \mathrm{d} k\left(-h_{1} \omega^{2}+h_{2} k^{2}+h_{3} m^{2}\right) c_{1, k} c_{1,-k} a_{k} a_{-k} \tag{1.84}
\end{align*}
$$

[^0]where, in the last step, we have used the assumption Eq. 1.57.
Similarly, we have
\[

$$
\begin{equation*}
B=\int \mathrm{d} k\left(-h_{1} \omega^{2}+h_{2} k^{2}+h_{3} m^{2}\right) c_{2, k} c_{2,-k} a_{k}^{\dagger} a_{-k}^{\dagger} \tag{1.85}
\end{equation*}
$$

\]

Finally, for $C$-term,

$$
\begin{align*}
C=\int \mathrm{d} x\{ & h_{1}\left(\mathcal{K}_{+}\left[-i \omega c_{1} a\right] \mathcal{K}_{-}\left[i \omega c_{2} a^{\dagger}\right]+\mathcal{K}_{-}[\cdots] \mathcal{K}_{+}[\cdots]\right) \\
& +h_{2}\left(\mathcal{K}_{+}\left[i k c_{1} a\right] \mathcal{K}_{-}\left[-i k c_{2} a^{\dagger}\right]+\mathcal{K}_{-}[\cdots] \mathcal{K}_{+}[\cdots]\right) \\
& \left.+h_{3} m^{2}\left(\mathcal{K}_{+}\left[c_{1} a\right] \mathcal{K}_{-}\left[c_{2} a^{\dagger}\right]+\mathcal{K}_{-}[\cdots] \mathcal{K}_{+}[\cdots]\right)\right\} \tag{1.86}
\end{align*}
$$

Note that (Exercise 1. 3c), similar to Eq. 1.82,

$$
\begin{equation*}
\int \mathrm{d} x \tilde{s}_{+}(x) \tilde{t}_{-}(x)=\int \mathrm{d} k s(k) t(k) \tag{1.87}
\end{equation*}
$$

And therefore,

$$
\begin{align*}
C & =\int \mathrm{d} k\left[h_{1} \omega^{2} c_{1} c_{2}\left(a a^{\dagger}+a^{\dagger} a\right)+h_{2} k^{2} c_{1} c_{2}\left(a a^{\dagger}+a^{\dagger} a\right)+h_{3} m^{2} c_{1} c_{2}\left(a a^{\dagger}+a^{\dagger} a\right)\right]  \tag{1.88}\\
& =\int \mathrm{d} k\left(h_{1} \omega^{2}+h_{2} k^{2}+h_{3} m^{2}\right) c_{1} c_{2}\left(a a^{\dagger}+a^{\dagger} a\right) \tag{1.89}
\end{align*}
$$

From Eqs. 1.84, 1.85, and 1.89, we see that once the dispersion relation is satisfied:

$$
\begin{equation*}
\omega^{2}=\frac{h_{2}}{h_{1}} k^{2}+\frac{h_{3}}{h_{1}} m^{2} \tag{1.90}
\end{equation*}
$$

which is consistent with Eq. 1.57, both $A$-and $B$-terms vanish, leading to

$$
\begin{equation*}
H=C=\int \mathrm{d} k 2 h_{1} \omega^{2} c_{1} c_{2}\left(a^{\dagger} a+a a^{\dagger}\right) \tag{1.91}
\end{equation*}
$$

As we shall see in later chapters, for a free scalar field,

$$
\begin{equation*}
h_{1}=h_{2}=h_{3}=\frac{1}{2} \tag{1.92}
\end{equation*}
$$

with $m$ being the mass of the field particle. Then, following Eq. 1.90, $k$ and $\omega$ are interpreted as the momentum and energy of the scalar field,

$$
\begin{equation*}
\omega=\sqrt{k^{2}+m^{2}} \tag{1.93}
\end{equation*}
$$

By choosing

$$
\begin{equation*}
c_{1}=c_{2}=\frac{1}{\sqrt{2 \omega}} \tag{1.94}
\end{equation*}
$$

we have

$$
\begin{align*}
{\left[a_{k}, a_{k^{\prime}}^{\dagger}\right] } & =\delta\left(k-k^{\prime}\right)  \tag{1.95}\\
H & =\int \mathrm{d} k \frac{\omega}{2}\left(a^{\dagger} a+a a^{\dagger}\right) \tag{1.96}
\end{align*}
$$

with which it is straightforward to recover the same set of algebra as in the case of the harmonic oscillator.

### 1.2.2 Time Dependence

The assumed form of the free Hamiltonian, Eqs. 1.75 and 1.92 , ensures that the $a(t)$ and $a^{\dagger}(t)$ evolve with a phase factor $e^{\mp i \omega t}$ (cf. Eqs. 1.9, 1.12, 1.13, and 1.14); therefore, for the general case, it is customary to factor out these phases in the mode expansion (Eqs. 1.46 and 1.47, with the choice of $c_{1,2}$ from Eq. 1.94) so that $a$ and $a^{\dagger}$ only retain possible non-trivial time dependence:

$$
\begin{align*}
\phi(x, t) & =\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} k \frac{1}{\sqrt{2 \omega}}\left(a e^{i k x-i \omega t}+a^{\dagger} e^{-i k x+i \omega t}\right)  \tag{1.97}\\
\pi(x, t) & =\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} k \frac{1}{\sqrt{2 \omega}}\left(-i \omega a e^{i k x-i \omega t}+i \omega a^{\dagger} e^{-i k x+i \omega t}\right) \tag{1.98}
\end{align*}
$$

Or, in our shorthand notation:

$$
\begin{align*}
\phi & =\mathcal{K}_{+}[c a]+\mathcal{K}_{-}\left[c^{*} a^{\dagger}\right]  \tag{1.99}\\
\pi & =\mathcal{K}_{+}[-i \omega c a]+\mathcal{K}_{-}\left[i \omega c^{*} a^{\dagger}\right]  \tag{1.100}\\
\left(\partial_{x} \phi\right. & \left.=\mathcal{K}_{+}[i k c a]+\mathcal{K}_{-}\left[-i k c^{*} a^{\dagger}\right],\right) \tag{1.101}
\end{align*}
$$

where $c=\frac{1}{\sqrt{2 \omega}} e^{-i \omega t}$.
We notice that

$$
\begin{equation*}
\pi=\partial_{t} \phi \quad \Leftrightarrow \quad \partial_{t} a=0, \partial_{t} a^{\dagger}=0 \tag{1.102}
\end{equation*}
$$

The first is satisfied for Hamiltonians like Eq. 1.75 and the latter means that $a$ and $a^{\dagger}$ are time-independent. In this case, Eqs. 1.56 and 1.58 become (note the inverse phases $\left.e^{ \pm i \omega t}\right)$

$$
\begin{align*}
a & =\frac{\mathcal{F}_{-}\left[e^{i \omega t}\left(\omega \phi+i \partial_{t} \phi\right)\right]}{\sqrt{2 \omega}},  \tag{1.103}\\
a^{\dagger} & =\frac{\mathcal{F}_{+}\left[e^{-i \omega t}\left(\omega \phi-i \partial_{t} \phi\right)\right]}{\sqrt{2 \omega}} . \tag{1.104}
\end{align*}
$$

As a consistency check, differentiate Eq. 1.103 with respect to time,

$$
\begin{equation*}
\partial_{t} a=\frac{\mathcal{F}_{-}\left[i e^{i \omega t}\left(\omega^{2}+\partial_{t}^{2}\right) \phi\right]}{\sqrt{2 \omega}} \tag{1.105}
\end{equation*}
$$

With the free field (i.e., on shell) dispersion relation and using the Fourier transform of $\partial_{x}$ (cf. epigraph), we have

$$
\begin{equation*}
\omega^{2}=k^{2}+m^{2} \rightarrow-\partial_{x}^{2}+m^{2} \tag{1.106}
\end{equation*}
$$

namely,

$$
\begin{equation*}
\partial_{t} a=\frac{\mathcal{F}_{-}\left[i e^{i \omega t}\left(\partial_{t}^{2}-\partial_{x}^{2}+m^{2}\right) \phi\right]}{\sqrt{2 \omega}} \tag{1.107}
\end{equation*}
$$

And similarly,

$$
\begin{equation*}
\partial_{t} a^{\dagger}=\frac{\mathcal{F}_{+}\left[-i e^{-i \omega t}\left(\partial_{t}^{2}-\partial_{x}^{2}+m^{2}\right) \phi\right]}{\sqrt{2 \omega}} \tag{1.108}
\end{equation*}
$$

$\partial_{t} a^{(\dagger)}=0$ holds for $t \rightarrow \pm \infty$ (asymptotic states) where $\left(\partial_{t}^{2}-\partial_{x}^{2}+m^{2}\right) \phi=0$. In general, it should be understood as $\partial_{t} a^{(\dagger)}$ behaves like $\delta$-functions at some time, non-vanishing only where $\partial_{t}^{2}-\partial_{x}^{2}+m^{2}$ does not destroy the yield.

## Exercise 1

## 1. Fourier Transform—Part 1

(a) The Dirac delta function can be defined as

$$
\delta(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \mathrm{d} k e^{ \pm i k x}
$$

Using this definition, show that

$$
\int_{-\infty}^{\infty} \mathrm{d} x \delta(x)=1
$$

(Hint: use the Gaussian integral $\int_{-\infty}^{\infty} \mathrm{d} x e^{-x^{2}}=\sqrt{\pi}$.)
(b) Consider the following general forms of the Fourier transform and its inverse:

$$
\begin{aligned}
& \tilde{f}(k)=\mathcal{F}[f]=A \int \mathrm{~d} x f(x) e^{i C k x} \\
& f(x)=\mathcal{F}^{-1}[\tilde{f}]=B \int \mathrm{~d} k \tilde{f}(k) e^{-i C k x}
\end{aligned}
$$

Show that

$$
A B=\frac{|C|}{2 \pi}
$$

(c) Momentum-space operators are the Fourier-transformed ones from position space. Recall the definition of the Fourier transform:

$$
\tilde{f}(p)=\mathcal{F}[f]=\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} x f(x) e^{-i p x}
$$

Show that: had we mapped the $x$-space operators to the $p$-space operators with the following Fourier transform:

$$
\mathcal{F}[f]=A \int \mathrm{~d} x f(x) e^{i p x}
$$

where $A$ is an arbitrary constant, the $x$-space momentum operator would have been

$$
\hat{p}=i \frac{\mathrm{~d}}{\mathrm{~d} x}
$$

and therefore we would have the commutation relation $[\hat{x}, \hat{p}]=-i$ instead.

## 2. Quantum Harmonic Oscillator

(a) Only consider the Hamiltonian, Eq. 1.26,

$$
\hat{H}=h_{1} \hat{p}^{2}+h_{2} \hat{q}^{2}
$$

with the dispersion relation, Eq. 1.31,

$$
h_{1} \lambda^{2}+h_{2}=0 .
$$

Calculate $C$ and $D$ (defined in Eqs. 1.32 and 1.35, respectively),

$$
\begin{aligned}
\hat{H} & =C\left(\hat{a}^{\dagger} \hat{a}+\hat{a} \hat{a}^{\dagger}\right), \\
{\left[\hat{H}, \hat{a}^{\dagger}\right] } & \equiv D \hat{a}^{\dagger}, \quad D>0 .
\end{aligned}
$$

(b) Prove Eq. 1.40,

$$
\lambda=-i m \omega
$$

and Eq. 1.41 ,

$$
D=\omega
$$

for the case of the harmonic oscillator.
(c) Finally, prove Eq. 1.42,

$$
C=\frac{\omega}{2},
$$

and fully recover Eqs. 1.5 and 1.6,

$$
\begin{aligned}
\hat{a} & =\frac{1}{\sqrt{2}}\left(\sqrt{m \omega} \hat{x}+\frac{i}{\sqrt{m \omega}} \hat{p}\right) \\
\hat{a}^{\dagger} & =\frac{1}{\sqrt{2}}\left(\sqrt{m \omega} \hat{x}-\frac{i}{\sqrt{m \omega}} \hat{p}\right)
\end{aligned}
$$

(up to some phase factors) by requiring $\left[\hat{a}, \hat{a}^{\dagger}\right]=1$.
(d) The transforms, Eqs. 1.21 and 1.22 ,

$$
\begin{aligned}
& \hat{q}=c_{1} \hat{a}+c_{2} \hat{b} \\
& \hat{p}=\lambda\left(c_{1} \hat{a}-c_{2} \hat{b}\right),
\end{aligned}
$$

are general in the sense that $\hat{a}$ and $\hat{a}^{\dagger}$ are fully independent. Now, show that if $\left[\hat{\mathcal{O}}, \hat{a}^{\dagger}\right]=\omega \hat{a}^{\dagger}$, then $[\hat{\mathcal{O}}, \hat{a}]=-\omega \hat{a}$, for an arbitrary Hermitian operator $\hat{\mathcal{O}}$.

## 3. Fourier Transform—Part 2

$\eta$ and $\tau$ are arbitrary signs, show that
(a)

$$
\begin{aligned}
\mathcal{F}_{\eta}\left[s \tilde{t}_{\tau}\right] & =\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} x e^{\eta i k x} s(k) \frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} k^{\prime} e^{\tau i k^{\prime} x} t\left(k^{\prime}\right) \\
& =\left\{\begin{array}{l}
s(k) t(-k), \eta=\tau \\
s(k) t(k), \eta \neq \tau
\end{array}\right.
\end{aligned}
$$

(b)

$$
\begin{aligned}
& {\left[\tilde{f}_{\eta}(k), \tilde{g}_{\tau}\left(k^{\prime}\right)\right]=\left[\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} x e^{\eta i k x} f(x), \frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} y e^{\tau i k^{\prime} y} g(y)\right]} \\
& =\left\{\begin{array}{cc}
r \delta\left(k+k^{\prime}\right), & \eta=\tau \\
r \delta\left(k-k^{\prime}\right), & \eta \neq \tau
\end{array}\right.
\end{aligned}
$$

given that $[f(x), g(y)]=r \delta(x-y)$.
(c)

$$
\begin{aligned}
& \int \mathrm{d} x \tilde{s}_{\eta}(x) \tilde{t}_{\tau}(x)=\int \mathrm{d} x \frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} k e^{\eta i k x} s(k) \frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} k^{\prime} e^{\tau i k^{\prime} x} t\left(k^{\prime}\right) \\
& =\left\{\begin{array}{l}
\int \mathrm{d} k s(k) t(-k)=\int \mathrm{d} k s(-k) t(k), \eta=\tau \\
\int \mathrm{d} k s(k) t(k), \eta \neq \tau
\end{array}\right.
\end{aligned}
$$

N.B.: Effectively, we obtain the convolution theorem for the case of $\eta=\tau$.

### 1.3 LSZ Reduction

In a field theory, the interpretation of $|0\rangle$ is different from the harmonic oscillator case; it is now the vacuum state, and it should be stable:

$$
\begin{equation*}
a|0\rangle=0 \tag{1.109}
\end{equation*}
$$

The vacuum state is conventionally normalised as

$$
\begin{equation*}
\langle 0 \mid 0\rangle=1 \tag{1.110}
\end{equation*}
$$

There is a technical difference between the zero-point energy in Eq. 1.20 and its correspondence following Eqs. 1.95 and 1.96 the vacuum energy:

$$
\begin{align*}
\langle 0| H|0\rangle & =\int \mathrm{d} k \omega\langle 0| a^{\dagger} a+\frac{1}{2} \delta(0)|0\rangle  \tag{1.111}\\
& =\left(\int \mathrm{d} k \frac{\omega}{2}\right) \delta(0)\langle 0 \mid 0\rangle \tag{1.112}
\end{align*}
$$

where both the integral over $k$ and $\delta(0)\left(\sim \int \mathrm{d} x 1\right)$ are infinite as they are the sum of all energy times the whole spacial volume. This infinity needs to be subtracted by rearranging the order of $a$ and $a^{\dagger}$ and hence the introduction of normal ordering:

$$
\begin{equation*}
: a^{\dagger} a:=a^{\dagger} a, \quad: a a^{\dagger}:=a^{\dagger} a \tag{1.113}
\end{equation*}
$$

namely, all annihilation operators are moved to the right of the creation operators. This is our first experience dealing with infinite vacuum fluctuations to obtain physically meaningful results. The renormalised Hamiltonian (Exercise 2.14) and vacuum energy are

$$
\begin{align*}
& : H:=\int \mathrm{d} k \omega a^{\dagger} a  \tag{1.114}\\
& \langle 0|: H:|0\rangle=0 \tag{1.115}
\end{align*}
$$

and furthermore,

$$
\begin{equation*}
: H: a_{k}^{\dagger}|0\rangle=\omega_{k} a_{k}^{\dagger}|0\rangle \tag{1.116}
\end{equation*}
$$

so $a_{k}^{\dagger}|0\rangle$ is the eigenstate of the Hamiltonian created by $a_{k}^{\dagger}$ with momentum $k$. Symbolically, we write

$$
\begin{equation*}
|k\rangle \equiv Z_{k} a_{k}^{\dagger}|0\rangle \tag{1.117}
\end{equation*}
$$

where $Z_{k}$ is the normalisation convention (Exercise 2.2.2):

$$
\begin{align*}
\left\langle k \mid k^{\prime}\right\rangle & =Z_{k}^{*} Z_{k^{\prime}}\langle 0| a_{k} a_{k^{\prime}}^{\dagger}|0\rangle  \tag{1.118}\\
& =\left|Z_{k}\right|^{2} \delta\left(k-k^{\prime}\right) \tag{1.119}
\end{align*}
$$

While $[\phi, \pi]$ is consistently defined in the literature, $\left[a, a^{\dagger}\right]$ varies depending on the choices of $c_{1,2}$ in Eq. 1.72 (including the choice of the Fourier transform prefactor). This is another factor that implicitly affects the normalisation of $|k\rangle$.

Now, consider the asymptotic states,

$$
\begin{align*}
& \left.\mid k_{1} \text { in }\right\rangle \equiv a_{1}^{\dagger}(-\infty)|0\rangle  \tag{1.120}\\
& \left.\mid k_{2} \text { out }\right\rangle \equiv a_{2}^{\dagger}(+\infty)|0\rangle \tag{1.121}
\end{align*}
$$

where the subscripts ${ }_{1,2}$ of $a^{\dagger}$ are shorthand notation for $k_{1,2}$, and we have set $Z_{k}=1$ for convenience. We are interested in the transition amplitude between them,

$$
\begin{align*}
S_{21} & \left.\equiv\left\langle k_{2} \text { out }\right| k_{1} \text { in }\right\rangle  \tag{1.122}\\
& =\left\langle k_{2} \text { out }\right| a_{1}^{\dagger}(-\infty)|0\rangle \tag{1.123}
\end{align*}
$$

Because

$$
\begin{align*}
\left.\left\langle k_{2} \text { out }\right| k_{1} \text { out }\right\rangle & =\delta\left(k_{1}-k_{2}\right)  \tag{1.124}\\
& =\left\langle k_{2} \text { out }\right| a_{1}^{\dagger}(+\infty)|0\rangle \tag{1.125}
\end{align*}
$$

we have

$$
\begin{align*}
& \left.S_{21}-\left\langle k_{2} \text { out }\right| k_{1} \text { out }\right\rangle \\
= & -\left\langle k_{2} \text { out }\right| a_{1}^{\dagger}(+\infty)-a_{1}^{\dagger}(-\infty)|0\rangle  \tag{1.126}\\
= & \left.-\left\langle k_{2} \text { out }\right| a_{1}^{\dagger}(t)\left|{ }_{-\infty}^{+\infty}\right| 0\right\rangle  \tag{1.127}\\
= & -\left\langle k_{2} \text { out }\right| \underbrace{\int_{-\infty}^{+\infty} \mathrm{d} t_{1} \partial_{t_{1}} a_{1}^{\dagger}\left(t_{1}\right)}_{I_{1}^{\dagger}}|0\rangle  \tag{1.128}\\
\equiv & -\langle 0| a_{2}(+\infty) I_{1}^{\dagger}|0\rangle \tag{1.129}
\end{align*}
$$

At this point, one would tend to repeat the same trick and add a term $\left\langle k_{2}\right.$ in $| I_{1}^{\dagger}|0\rangle$ to generate a similar difference $\int_{-\infty}^{+\infty} \mathrm{d} t_{2} \partial_{t_{2}} a_{2}\left(t_{2}\right)$, but this extra term is non-trivial. Instead, we can use the following term that vanishes because $a|0\rangle=0$ :

$$
\begin{equation*}
\langle 0| I_{1}^{\dagger} a_{2}(-\infty)|0\rangle=0 \tag{1.130}
\end{equation*}
$$

So,

$$
\begin{align*}
& \left.S_{21}-\left\langle k_{2} \text { out }\right| k_{1} \text { out }\right\rangle \\
= & -\langle 0| a_{2}(+\infty) I_{1}^{\dagger}|0\rangle+\langle 0| I_{1}^{\dagger} a_{2}(-\infty)|0\rangle  \tag{1.131}\\
= & -\langle 0| \int \mathrm{d} t_{1}[\underbrace{a_{2}(+\infty) \partial_{t_{1}} a_{1}^{\dagger}\left(t_{1}\right)}_{\equiv s\left(t_{1},+\infty\right)}-\underbrace{\partial_{t_{1}} a_{1}^{\dagger}\left(t_{1}\right) a_{2}(-\infty)}_{\equiv s\left(t_{1},-\infty\right)}]|0\rangle  \tag{1.132}\\
= & -\langle 0| \int \mathrm{d} t_{1} \int_{-\infty}^{+\infty} \mathrm{d} t_{2} \partial_{t_{2}} s\left(t_{1}, t_{2}\right)|0\rangle . \tag{1.133}
\end{align*}
$$

We need to find a function $s\left(t_{1}, t_{2}\right)$ that satisfies the above boundary conditions at $t_{2}= \pm \infty$, hence the introduction of time ordering:

$$
\begin{align*}
T\left[A\left(t_{1}\right) B\left(t_{2}\right)\right] & =A\left(t_{1}\right) B\left(t_{2}\right) \theta\left(t_{1}-t_{2}\right)+B\left(t_{2}\right) A\left(t_{1}\right) \theta\left(t_{2}-t_{1}\right)  \tag{1.134}\\
& =\left\{\begin{array}{cc}
A\left(t_{1}\right) B\left(t_{2}\right), & t_{1}>t_{2} \\
B\left(t_{2}\right) A\left(t_{1}\right), & t_{2}>t_{1}
\end{array}\right. \tag{1.135}
\end{align*}
$$

namely, fields at an earlier time are moved to the right of later ones. Here we have used the Heaviside step function (Exercise 2.3),

$$
\theta(x)=\left\{\begin{array}{l}
1, x>0  \tag{1.136}\\
0, x<0
\end{array}, \quad \frac{\mathrm{~d} \theta}{\mathrm{~d} x}=\delta(x)\right.
$$

We have

$$
\begin{equation*}
s\left(t_{1}, t_{2}\right)=T\left[\partial_{t_{1}} a_{1}^{\dagger}\left(t_{1}\right) a_{2}\left(t_{2}\right)\right] \tag{1.137}
\end{equation*}
$$

So,

$$
\begin{align*}
& \left.S_{21}-\left\langle k_{2} \text { out }\right| k_{1} \text { out }\right\rangle \\
= & -\langle 0| \int \mathrm{d} t_{1} \mathrm{~d} t_{2} T\left[\partial_{t_{1}} a_{1}^{\dagger}\left(t_{1}\right) \partial_{t_{2}} a_{2}\left(t_{2}\right)\right]|0\rangle . \tag{1.138}
\end{align*}
$$

Recall, from Eqs. 1.107 and 1.108,

$$
\begin{align*}
\partial_{t_{1}} a_{1}^{\dagger}\left(t_{1}\right) & =\frac{1}{\sqrt{2 \omega_{1}}} \mathcal{F}_{+}\left[-i e^{-i \omega_{1} t_{1}}\left(\partial_{t_{1}}^{2}-\partial_{x_{1}}^{2}+m^{2}\right) \phi_{1}\right]  \tag{1.139}\\
\partial_{t_{2}} a_{2}\left(t_{2}\right) & =\frac{1}{\sqrt{2 \omega_{2}}} \mathcal{F}_{-}\left[i e^{i \omega_{2} t_{2}}\left(\partial_{t_{2}}^{2}-\partial_{x_{2}}^{2}+m^{2}\right) \phi_{2}\right] \tag{1.140}
\end{align*}
$$

Putting things together 2, we have

$$
\begin{align*}
S_{21}= & \delta\left(k_{1}-k_{2}\right)+\frac{i}{\sqrt{2 \omega_{1}}} \frac{i}{\sqrt{2 \omega_{2}}} \frac{1}{2 \pi} \int \mathrm{~d} t_{1} \mathrm{~d} x_{1} \mathrm{~d} t_{2} \mathrm{~d} x_{2} e^{i k_{1} x_{1}-i \omega_{1} t_{1}} e^{-i k_{2} x_{2}+i \omega_{2} t_{2}} \\
& \times\left(\partial_{t_{1}}^{2}-\partial_{x_{1}}^{2}+m^{2}\right)\left(\partial_{t_{2}}^{2}-\partial_{x_{2}}^{2}+m^{2}\right)\langle 0| T\left[\phi\left(x_{1}, t_{1}\right) \phi\left(x_{2}, t_{2}\right)\right]|0\rangle \tag{1.141}
\end{align*}
$$

which is known as the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula - the transition amplitude is reduced to the vacuum expectation value of the time-ordered product, which is also called the two-point Green's function. Its structure is readily identifiable: the integral is a Fourier transform over the spacetime coordinates of all involved fields, and the in-and out-states are distinguished by the $\pm$ signs in the phase factors. The prefactors come from our convention of the $|k\rangle$ normalisation.

With the free fields, the two-point Green's function, $G_{2}$, is, in fact, the Feynman propagator, $\Delta_{\mathrm{F}}$ (Exercise 2.4):

$$
\begin{align*}
& G_{2}\left(x_{1}-x_{2}, t_{1}-t_{2}\right) \equiv\langle 0| T\left[\phi\left(x_{1}, t_{1}\right) \phi\left(x_{2}, t_{2}\right)\right]|0\rangle  \tag{1.142}\\
& =i \int \frac{\mathrm{~d} k}{2 \pi} \frac{\mathrm{~d} \omega}{2 \pi} \frac{e^{i k\left(x_{1}-x_{2}\right)-i \omega\left(t_{1}-t_{2}\right)}}{\omega^{2}-k^{2}-m^{2}+i \epsilon} \equiv \Delta_{\mathrm{F}}\left(x_{1}-x_{2}, t_{1}-t_{2}\right) \tag{1.143}
\end{align*}
$$

with $\epsilon \rightarrow 0^{+}$. Applying $\partial_{t_{i}}^{2}$ and $\partial_{x_{i}}^{2}$ to $G_{2}$ will bring down $-\omega^{2}$ and $-k^{2}$ from the phase, and, therefore, either of the operators in Eq. 1.141, $\partial_{t_{i}}^{2}-\partial_{x_{i}}^{2}+m^{2} \sim$ $\omega^{2}-k^{2}-m^{2}$ (off-shell), cancels the denominator in the integral:

$$
\begin{equation*}
\left(\partial_{t_{i}}^{2}-\partial_{x_{i}}^{2}+m^{2}\right) G_{2}\left(x_{1}-x_{2}, t_{1}-t_{2}\right)=-i \delta\left(x_{1}-x_{2}\right) \delta\left(t_{1}-t_{2}\right) \tag{1.144}
\end{equation*}
$$

namely, $G_{2}$ is the Green's function of the equation

$$
\begin{equation*}
\left(\partial_{t}^{2}-\partial_{x}^{2}+m^{2}\right) \phi(x, t)=0 \tag{1.145}
\end{equation*}
$$

[^1]and hence the name. The remaining $\partial_{t_{i}}^{2}-\partial_{x_{i}}^{2}+m^{2} \sim \omega_{k}^{2}-k^{2}-m^{2}$ after integration by parts kills the integral due to the (on-shell) dispersion relation. Then the whole equation is reduced to $S_{21}=\delta\left(k_{1}-k_{2}\right)$, which makes sense since we are dealing with a free field.

As is with the time evolution of $a$ and $a^{\dagger}$ (Eqs. 1.107 and 1.108), the LSZ reduction is non-vanishing where $\partial_{t}^{2}-\partial_{x}^{2}+m^{2}$ does not destroy the field, that is, when interactions are turned on. The LSZ reduction is the cornerstone connecting experimental observables and the underlying perturbative field theories, where free fields are used to describe interactions.

### 1.3.1 Multi-Particle States

Non-interacting multi-particle states can be created by multiple creation operators (Exercise 2.5):

$$
\begin{equation*}
: H: a_{k_{2}}^{\dagger} a_{k_{1}}^{\dagger}|0\rangle=\left(\omega_{k_{1}}+\omega_{k_{2}}\right) a_{k_{2}}^{\dagger} a_{k_{1}}^{\dagger}|0\rangle \tag{1.146}
\end{equation*}
$$

Symbolically, we write

$$
\begin{equation*}
\left|k_{1} k_{2}\right\rangle \equiv Z_{k_{1} k_{2}} a_{k_{2}}^{\dagger} a_{k_{1}}^{\dagger}|0\rangle \tag{1.147}
\end{equation*}
$$

Repeating the procedure in Sec. 1.3, we can calculate the transition amplitude between the asymptotic states $\mid k_{1} k_{2} \cdots$ in $\rangle$ and $\mid p_{1} p_{2} \cdots$ out $\rangle$ :

$$
\begin{align*}
S_{\mathrm{fi}} & \left.\equiv\left\langle p_{1} p_{2} \cdots \text { out }\right| k_{1} k_{2} \cdots \text { in }\right\rangle  \tag{1.148}\\
& \sim \delta_{\mathrm{fi}}+\mathcal{F}\left[\prod_{i}^{n}\left(\partial_{t_{i}}^{2}-\partial_{x_{i}}^{2}+m^{2}\right)\langle 0| T\left[\phi\left(x_{1}, t_{1}\right) \phi\left(x_{2}, t_{2}\right) \cdots \phi\left(x_{n}, t_{n}\right)\right]|0\rangle\right] \tag{1.149}
\end{align*}
$$

where $\mathcal{F}$ is a shorthand notation for the multi-dimensional Fourier transform (absorbing all prefactors) overall all involved space-time, and the time-ordered product is the $n$-point Green's function, which can be broken down into combinations of Feynman propagators through Wick's theorem:

$$
\begin{equation*}
\langle 0| T\left[\phi\left(x_{1}, t_{1}\right) \phi\left(x_{2}, t_{2}\right) \cdots \phi\left(x_{n}, t_{n}\right)\right]|0\rangle=\Delta_{\mathrm{F}}^{12} \Delta_{\mathrm{F}}^{34} \cdots+\text { permutations } \tag{1.150}
\end{equation*}
$$

where $\Delta_{\mathrm{F}}^{i j} \equiv \Delta_{\mathrm{F}}\left(x_{i}-x_{j}, t_{i}-t_{j}\right)$. Note that, for an odd $n$, there is always a remaining field that does not form a propagator, and as a result, the Green's function vanishes due to the trailing $\langle 0| \phi|0\rangle$.

Before we conclude this chapter, let's look at a four-point Green's function,

$$
\begin{equation*}
\langle 0| T\left[\phi_{1} \phi_{2} \phi_{3} \phi_{4}\right]|0\rangle=\Delta_{\mathrm{F}}^{12} \Delta_{\mathrm{F}}^{34}+\Delta_{\mathrm{F}}^{13} \Delta_{\mathrm{F}}^{24}+\Delta_{\mathrm{F}}^{14} \Delta_{\mathrm{F}}^{23} \tag{1.151}
\end{equation*}
$$

This forms the basic idea of representing a transition amplitude $\left(S_{\mathrm{fi}}\right)$ as the sum of Feynman diagrams (e.g., $\Delta_{\mathrm{F}}^{12} \Delta_{\mathrm{F}}^{34}$ ) consisting of lines for propagators (e.g., $\Delta_{\mathrm{F}}^{12}$ ).


## Exercise 2

## 1. Normal Ordering

Show that normal ordering does not alter the commutator relations:

$$
\begin{aligned}
{[: H:, a] } & =-\omega a \\
{\left[: H:, a^{\dagger}\right] } & =\omega a^{\dagger}
\end{aligned}
$$

2. The Vacuum-Part 1
(a) Show that

$$
\langle 0| a_{k} a_{k^{\prime}}^{\dagger}|0\rangle=\delta\left(k-k^{\prime}\right) .
$$

(b) What is $\left\langle k \mid k^{\prime}\right\rangle$ if we use the following mode expansion,

$$
\begin{aligned}
& \phi(x, t)=\int \frac{\mathrm{d} k}{2 \pi} \frac{1}{2 \omega}\left(a e^{i k x-i \omega t}+a^{\dagger} e^{-i k x+i \omega t}\right) \\
& \pi(x, t)=\int \frac{\mathrm{d} k}{2 \pi} \frac{1}{2 \omega}\left(-i \omega a e^{i k x-i \omega t}+i \omega a^{\dagger} e^{-i k x+i \omega t}\right)
\end{aligned}
$$

instead of Eqs. 1.97 and 1.98 ,

$$
\begin{aligned}
& \phi(x, t)=\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} k \frac{1}{\sqrt{2 \omega}}\left(a e^{i k x-i \omega t}+a^{\dagger} e^{-i k x+i \omega t}\right) \\
& \pi(x, t)=\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} k \frac{1}{\sqrt{2 \omega}}\left(-i \omega a e^{i k x-i \omega t}+i \omega a^{\dagger} e^{-i k x+i \omega t}\right)
\end{aligned}
$$

while keeping the commutation relation as in Eq. 1.44

$$
[\phi(x, t), \pi(y, t)]=i \delta(x-y)
$$

(let's choose $Z_{k}=1$ )?

## 3. Time Ordering

Prove that for arbitrary operators, $A(t)$ and $B(t)$,

$$
\begin{aligned}
& \partial_{t_{1}} T\left[A\left(t_{1}\right) B\left(t_{2}\right)\right] \\
= & T\left[\partial_{t 1} A\left(t_{1}\right) B\left(t_{2}\right)\right]+\left[A\left(t_{1}\right), B\left(t_{2}\right)\right] \delta\left(t_{1}-t_{2}\right) .
\end{aligned}
$$

## 4. Feynman Propagator

Our usual mode expansion is Eq. 1.97,

$$
\phi(x, t)=\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} k \frac{1}{\sqrt{2 \omega}}\left(a e^{i k x-i \omega t}+a^{\dagger} e^{-i k x+i \omega t}\right)
$$

or in the shorthand notation,

$$
\phi(x, t)=\mathcal{K}_{+}[c a]+\mathcal{K}_{-}\left[c^{*} a^{\dagger}\right], \quad c=\frac{1}{\sqrt{2 \omega}} e^{-i \omega t}
$$

(a) Show that

$$
\langle 0| \phi|0\rangle=0
$$

(b) Show that

$$
\langle 0| \phi\left(x_{1}, t_{1}\right) \phi\left(x_{2}, t_{2}\right)|0\rangle=\int \frac{\mathrm{d} k}{2 \pi} \frac{1}{2 \omega_{k}} e^{i k\left(x_{1}-x_{2}\right)-i \omega_{k}\left(t_{1}-t_{2}\right)}
$$

(c) Use the identity

$$
\theta(t)=\frac{i}{2 \pi} \int_{-\infty}^{+\infty} \mathrm{d} \omega \frac{e^{-i \omega t}}{\omega+i \epsilon}, \quad \epsilon \rightarrow 0^{+}
$$

to show

$$
\begin{aligned}
\langle 0| \phi_{1} \phi_{2}|0\rangle \theta(t) & =\int \frac{\mathrm{d} k}{2 \pi} \frac{1}{2 \omega_{k}} e^{i k x-i \omega_{k} t} \theta(t) \\
& =\int \frac{\mathrm{d} k}{2 \pi} \frac{1}{2 \omega_{k}} \frac{i}{2 \pi} \int_{-\infty}^{+\infty} \mathrm{d} \omega\left(-\frac{1}{\omega+\omega_{k}-i \epsilon}\right) e^{-i k x+i \omega t}
\end{aligned}
$$

where $x=x_{1}-x_{2}, t=t_{1}-t_{2}$.
(d) Finally, show that

$$
\Delta_{\mathrm{F}}(x, t) \equiv\langle 0| T \phi_{1} \phi_{2}|0\rangle=i \int \frac{\mathrm{~d} k \mathrm{~d} \omega}{(2 \pi)^{2}} \frac{e^{i k x-i \omega t}}{\omega^{2}-\omega_{k}^{2}+i \epsilon}
$$

and show that

$$
\Delta_{\mathrm{F}}(x, t)=\Delta_{\mathrm{F}}(-x,-t)
$$

5. The Vacuum-Part 2

Prove Eq. 1.146,

$$
: H: a_{k_{2}}^{\dagger} a_{k_{1}}^{\dagger}|0\rangle=\left(\omega_{k_{1}}+\omega_{k_{2}}\right) a_{k_{2}}^{\dagger} a_{k_{1}}^{\dagger}|0\rangle
$$

## Chapter 2

## Quantum Fields

$$
\begin{aligned}
& \text { Taylor expand } f(x) \text { at } a, \\
& \qquad \begin{array}{ll} 
& f(a+\epsilon)=\sum_{n=0}^{\infty} \frac{d^{n} f(a)}{d x^{n}} \frac{\epsilon^{n}}{n!} \\
= & \sum_{n=0}^{\infty} \frac{\epsilon^{n}}{n!}\left(\frac{d}{d x}\right)^{n} f(a) \\
= & e^{\epsilon \frac{d}{d x}} f(a) .
\end{array}
\end{aligned}
$$

Still remember

$$
\hat{p}_{x}=-i \frac{d}{d x},
$$

so that,

$$
f(x)=e^{i(x-a) \hat{p}_{x}} f(a) .
$$

### 2.1 Lagrange-Hamilton Formalism

We use the real scalar field, $\phi\left(x^{\mu}\right)$, to demonstrate the Lagrange-Hamilton formalism. We start by rewriting the Lagrangian $L(t)$ as a Lagrangian density $\mathcal{L}$,

$$
\begin{equation*}
L=\int \mathrm{d}^{3} \vec{x} \mathcal{L} \tag{2.1}
\end{equation*}
$$

The dimensionless (in natural units) action (for relativistic notations, cf. Appendix A),

$$
\begin{equation*}
S=\int \mathrm{d} t L=\int \mathrm{d}^{4} x \mathcal{L}\left[\phi, \partial_{\mu} \phi\right] \tag{2.2}
\end{equation*}
$$

is a functional of the field $\phi$ and its four spacetime derivatives $\partial_{\mu} \phi$. In field theory, the degrees of freedom are $\phi$ and $\partial_{\mu} \phi$, and space and time are just labels.

Consider independent changes, $\delta \phi$ and $\delta\left(\partial_{\mu} \phi\right)$, that lead to a change of $\mathcal{L}$ :

$$
\begin{equation*}
\delta \mathcal{L}=\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi+\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)} \delta\left(\partial_{\mu} \phi\right) . \tag{2.3}
\end{equation*}
$$

[^2]The resulting change in the action is

$$
\begin{align*}
\delta S & =\int \mathrm{d}^{4} x(\mathcal{L}+\delta \mathcal{L})-\int \mathrm{d}^{4} x \mathcal{L}  \tag{2.4}\\
& =\int \mathrm{d}^{4} x\left[\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi+\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)} \delta\left(\partial_{\mu} \phi\right)\right] \tag{2.5}
\end{align*}
$$

(integrate by parts for the second term)

$$
\begin{align*}
& =\int \mathrm{d}^{4} x\left\{\frac{\partial \mathcal{L}}{\partial \phi} \partial \phi-\left[\partial_{\mu} \frac{\partial \mathcal{L}}{\partial(\mu \phi)}\right] \delta \phi\right\}  \tag{2.6}\\
& =\int \mathrm{d}^{4} x\left[\frac{\partial \mathcal{L}}{\partial \phi}-\partial_{\mu} \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)}\right] \delta \phi \tag{2.7}
\end{align*}
$$

The principle of least action requires that $\delta S=0$ for the arbitrary $\delta \phi$. Therefore,

$$
\begin{equation*}
\frac{\partial \mathcal{L}}{\partial \phi}=\partial_{\mu} \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)} \tag{2.8}
\end{equation*}
$$

This is the Euler-Lagrange equation.
Now consider a translation in spacetime by an infinitesimal amount, $\epsilon^{\mu}$,

$$
\begin{equation*}
x^{\mu} \rightarrow x^{\mu}+\epsilon^{\mu} \tag{2.9}
\end{equation*}
$$

The measure changes correspondingly,

$$
\begin{align*}
\mathrm{d} x^{\mu} & \rightarrow \mathrm{d} x^{\mu}+\mathrm{d} \epsilon^{\mu}  \tag{2.10}\\
& \text { (no Einstein summation convention on repeated } \mu \text { ) } \\
& =\mathrm{d} x^{\mu}\left(1+\frac{\mathrm{d} \epsilon^{\mu}}{\mathrm{d} x^{\mu}}\right)  \tag{2.11}\\
& \text { (resuming Einstein summation convention) } \\
\mathrm{d}^{4} x & \rightarrow \mathrm{~d}^{4} x\left(1+\partial_{\mu} \epsilon^{\mu}\right)+o\left(\epsilon^{2}\right) . \tag{2.12}
\end{align*}
$$

The change in $\mathcal{L}$ is

$$
\begin{align*}
\delta \mathcal{L} & =\frac{\partial \mathcal{L}}{\partial \phi} \partial_{\mu} \phi \epsilon^{\mu}+\frac{\partial \mathcal{L}}{\partial\left(\partial_{\nu} \phi\right)} \partial_{\mu}\left(\partial_{\nu} \phi\right) \epsilon^{\mu}  \tag{2.13}\\
& \text { (apply E-L to the first term, and swap the order of } \partial_{\mu, \nu} \text { on } \phi \text { in the second term) } \\
& =\left[\partial_{\nu} \frac{\partial \mathcal{L}}{\partial\left(\partial_{\nu} \phi\right)}\right] \partial_{\mu} \phi \epsilon^{\mu}+\frac{\partial \mathcal{L}}{\partial\left(\partial_{\nu} \phi\right)} \partial_{\nu} \partial_{\mu} \phi \epsilon^{\mu}  \tag{2.14}\\
& =\partial_{\nu}\left[\frac{\partial \mathcal{L}}{\partial\left(\partial_{\nu} \phi\right)} \partial_{\mu} \phi\right] \epsilon^{\mu} \tag{2.15}
\end{align*}
$$

The change of action is, omitting $o\left(\epsilon^{2}\right)$-terms,

$$
\begin{align*}
\delta S & =\int \mathrm{d}^{4} x\left(1+\partial_{\mu} \epsilon^{\mu}\right)(\mathcal{L}+\delta \mathcal{L})-\int \mathrm{d}^{4} x \mathcal{L}  \tag{2.16}\\
& =\int \mathrm{d}^{4} x\left(\delta \mathcal{L}+\partial_{\mu} \epsilon^{\mu} \mathcal{L}\right)  \tag{2.17}\\
& \text { (put in } \delta \mathcal{L} \text { and integrate by parts for the second term) } \\
& =\int \mathrm{d}^{4} x\left\{\partial_{\nu}\left[\frac{\partial \mathcal{L}}{\partial\left(\partial_{\nu} \phi\right)} \partial_{\mu} \phi\right] \epsilon^{\mu}-\epsilon^{\mu} \partial_{\mu} \mathcal{L}\right\}  \tag{2.18}\\
& \left(\epsilon^{\mu} \partial_{\mu} \mathcal{L}=\partial_{\nu}\left(g_{\mu}^{\nu} \mathcal{L}\right) \epsilon^{\mu}\right) \\
& =\int \mathrm{d}^{4} x \partial_{\nu}\left[\frac{\partial \mathcal{L}}{\partial\left(\partial_{\nu} \phi\right)} \partial_{\mu} \phi-g_{\mu}^{\nu} \mathcal{L}\right] \epsilon^{\mu} \tag{2.19}
\end{align*}
$$

which should vanish for arbitrary $\epsilon^{\mu}$. Therefore, we have

$$
\begin{equation*}
\partial_{\nu} T_{\mu}^{\nu}=0 \tag{2.20}
\end{equation*}
$$

with the energy-momentum tensor,

$$
\begin{equation*}
T^{\nu \mu} \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_{\nu} \phi\right)} \partial^{\mu} \phi-g^{\nu \mu} \mathcal{L} \tag{2.21}
\end{equation*}
$$

whose components are

$$
\begin{align*}
T^{00} & =\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi}-\mathcal{L}  \tag{2.22}\\
T^{0 \mu} & =\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \partial^{\mu} \phi, \quad \mu=1,2,3 \tag{2.23}
\end{align*}
$$

In the previous Section 1.2, we mentioned the canonical momentum, $\pi$, and Hamiltonian density, $\mathcal{H}$, as given. Now, in the Lagrange-Hamilton formalism, they are defined as follows:

$$
\begin{align*}
\pi & =\frac{\partial \mathcal{L}}{\partial \dot{\phi}}  \tag{2.24}\\
\mathcal{H} & =\pi \dot{\phi}-\mathcal{L} \tag{2.25}
\end{align*}
$$

Therefore, we have

$$
\begin{align*}
& T^{00}=\pi \dot{\phi}-\mathcal{L}=\mathcal{H}  \tag{2.26}\\
& T^{0 \mu}=\pi \partial^{\mu} \phi, \quad \mu=1,2,3 \tag{2.27}
\end{align*}
$$

We can define the energy-momentum four-vector,

$$
\begin{equation*}
P^{\mu} \equiv \int \mathrm{d}^{3} \vec{x} T^{0 \mu} \tag{2.28}
\end{equation*}
$$

As before, the Hamiltonian is

$$
\begin{equation*}
H=\int \mathrm{d}^{3} \vec{x} \mathcal{H}=\int \mathrm{d}^{3} \vec{x} T^{00}=P^{0} \tag{2.29}
\end{equation*}
$$

and the momentum of component $\mu$ is (Exercise 3. 1 a)

$$
\begin{equation*}
P^{\mu}=\int \mathrm{d}^{3} \vec{x} T^{0 \mu}=\int \mathrm{d}^{3} \vec{x} \pi \partial^{\mu} \phi, \quad \mu=1,2,3 \tag{2.30}
\end{equation*}
$$

Now, we can derive the more general Heisenberg's equation of motion (Exercise $3 . \mathrm{Lb}$,

$$
\begin{equation*}
i\left[P^{\mu}, \phi\right]=\partial^{\mu} \phi \tag{2.31}
\end{equation*}
$$

We will see other similar "EOMs" in later sections. Restoring the ^-notation for the moment and rearrange the terms, we have

$$
\begin{equation*}
\hat{p}^{\mu} \hat{\phi}=\left[\hat{\phi}, \hat{P}^{\mu}\right] \tag{2.32}
\end{equation*}
$$

where $\hat{p}^{\mu}$ generates spacetime translation and $\hat{P}^{\mu}$ is the amount of the energymomentum of the field.

### 2.2 Real Scalars

We have the free scalar field Lagrangian,

$$
\begin{equation*}
\mathcal{L}[\phi]=\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi-\frac{1}{2} m^{2} \phi^{2}, \tag{2.33}
\end{equation*}
$$

with

$$
\begin{equation*}
\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)}=\partial^{\mu} \phi, \quad \frac{\partial \mathcal{L}}{\partial \phi}=-m^{2} \phi . \tag{2.34}
\end{equation*}
$$

The field equation of motion is given by the Euler-Lagrange equation,

$$
\begin{equation*}
\left(\partial_{\mu} \partial^{\mu}+m^{2}\right) \phi=0 . \tag{2.35}
\end{equation*}
$$

This is the Klein-Gordon equation. With $\hat{p} \sim i \partial$, we have $\hat{p}^{2}=-\partial^{2}$, and therefore,

$$
\begin{equation*}
\left(\hat{p}^{2}-m^{2}\right) \phi=0 . \tag{2.36}
\end{equation*}
$$

We have the canonical momentum,

$$
\begin{equation*}
\pi=\frac{\partial \mathcal{L}}{\partial \dot{\phi}}=\dot{\phi} \tag{2.37}
\end{equation*}
$$

implying time-independent $a$ and $a^{\dagger}$, as discussed in Sec. 1.2.2. The equal-time canonical commutators are

$$
\begin{align*}
& {[\phi(\vec{x}, t), \pi(\vec{y}, t)]=i \delta^{3}(\vec{x}-\vec{y}),}  \tag{2.38}\\
& {[\phi(\vec{x}, t), \phi(\vec{y}, t)]=[\pi(\vec{x}, t), \pi(\vec{y}, t)]=0 .} \tag{2.39}
\end{align*}
$$

The mode expansion is

$$
\begin{align*}
\phi & =\mathcal{K}_{+}^{3}[c a]+\mathcal{K}_{-}^{3}\left[c^{*} a^{\dagger}\right],  \tag{2.40}\\
\pi & =\mathcal{K}_{+}^{3}[-i \omega c a]+\mathcal{K}_{-}^{3}\left[i \omega c^{*} a^{\dagger}\right],  \tag{2.41}\\
\left(\partial_{i} \phi\right. & \left.=\mathcal{K}_{+}^{3}\left[i k_{i} c a\right]+\mathcal{K}_{-}^{3}\left[-i k_{i} c^{*} a^{\dagger}\right],\right) \tag{2.42}
\end{align*}
$$

where $c=\frac{1}{\sqrt{2 \omega}} e^{-i \omega t}$.
The Hamiltonian density is

$$
\begin{align*}
\mathcal{H}=\pi \dot{\phi}-\mathcal{L} & =\pi^{2}-\frac{1}{2}\left[\pi^{2}-(\nabla \phi)^{2}\right]+\frac{1}{2} m^{2} \phi^{2}  \tag{2.43}\\
& =\frac{1}{2} \pi^{2}+\frac{1}{2}(\nabla \phi)^{2}+\frac{1}{2} m^{2} \phi^{2}, \tag{2.44}
\end{align*}
$$

which has the assumed form of Eq. 1.75. Correspondingly, we have the dispersion relation,

$$
\begin{equation*}
\omega=\sqrt{\vec{k}^{2}+m^{2}} \tag{2.45}
\end{equation*}
$$

We see that the mode expansion, Eq. 2.40, when restricted to this on-shell dispersion relation, automatically satisfy the Klein-Gordon equation.

Furthermore, we have

$$
\begin{align*}
& {\left[a_{\vec{k}}, a_{\vec{k}^{\prime}}^{\dagger}\right]=\delta^{3}\left(\vec{k}-\vec{k}^{\prime}\right)}  \tag{2.46}\\
& {\left[a_{\vec{k}}, a_{\vec{k}^{\prime}}\right]=\left[a_{\vec{k}}^{\dagger}, a_{\vec{k}^{\prime}}^{\dagger}\right]=0}  \tag{2.47}\\
& a_{\vec{k}}|0\rangle=0  \tag{2.48}\\
& a_{\vec{k}}^{\dagger}|0\rangle=|\vec{k}\rangle \text { (up to a normalisation constant), }  \tag{2.49}\\
& : H:=\int \mathrm{d}^{3} \vec{k} \omega a^{\dagger} a  \tag{2.50}\\
& \langle 0| T \phi(x) \phi(y)|0\rangle=\Delta_{\mathrm{F}}(x-y)=i \int \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} \frac{e^{-i k(x-y)}}{k^{2}-m^{2}+i \epsilon} \tag{2.51}
\end{align*}
$$

Note that, while the expressions of $\left[a, a^{\dagger}\right]$ and $H$ depend on the conventions for writing down the mode expansion, the Feynman propagator $\Delta_{\mathrm{F}}$ does not.

### 2.3 Complex Scalars

Scalar fields are useful models to illustrate the mathematical structure of quantum fields; new properties arise with new construction. In the following, we will show that by uniting two independent real fields into a complex field, we have the first sight of particle-antiparticle dualism.

Consider two independent real scalar fields, $\phi_{1}$ and $\phi_{2}$, with the same mass. Their associated quantities, i.e., those discussed in the preceding section, are all labelled with subscripts $i=1,2$. Each paif of their associated operators commutes,

$$
\begin{equation*}
\left[\hat{\mathcal{O}}_{1}, \hat{\mathcal{O}}_{2}\right]=0 \tag{2.52}
\end{equation*}
$$

The two fields now form a non-interacting system, whose Lagrangian is

$$
\begin{equation*}
\mathcal{L}\left[\phi_{1}, \phi_{2}, \partial_{\mu} \phi_{1}, \partial_{\mu} \phi_{2}\right]=\mathcal{L}_{1}+\mathcal{L}_{2} \tag{2.53}
\end{equation*}
$$

Now, define a combined field,

$$
\begin{equation*}
\phi=\frac{\phi_{1}-i \phi_{2}}{\sqrt{2}} \tag{2.54}
\end{equation*}
$$

By taking the Hermitian conjugate, we have

$$
\begin{equation*}
\phi^{\dagger}=\frac{\phi_{1}+i \phi_{2}}{\sqrt{2}} \tag{2.55}
\end{equation*}
$$

which is considered independent of $\phi$ in terms of its contribution to the Lagrangian, which has become

$$
\begin{equation*}
\mathcal{L}\left[\phi_{1}, \phi_{2}, \partial_{\mu} \phi_{1}, \partial_{\mu} \phi_{2}\right]=\mathcal{L}\left[\phi, \phi^{\dagger}, \partial_{\mu} \phi, \partial_{\mu} \phi^{\dagger}\right]=\partial_{\mu} \phi \partial^{\mu} \phi^{\dagger}-m^{2} \phi \phi^{\dagger} \tag{2.56}
\end{equation*}
$$

The Euler-Lagrange equations can be derived from it:

$$
\begin{align*}
\left(\partial^{2}+m^{2}\right) \phi & =0  \tag{2.57}\\
\phi^{\dagger}\left(\overleftarrow{\partial}^{2}+m^{2}\right) & =0 \tag{2.58}
\end{align*}
$$

which do not look too interesting, but nevertheless, let's move on.

The canonical momentum of the $\phi$ field is

$$
\begin{equation*}
\pi=\frac{\partial \mathcal{L}}{\partial \dot{\phi}}=\dot{\phi}^{\dagger} \tag{2.59}
\end{equation*}
$$

which is no longer $\dot{\phi}$; And that of $\phi^{\dagger}$ is

$$
\begin{equation*}
\widetilde{\pi}=\frac{\partial \mathcal{L}}{\partial \dot{\phi}^{\dagger}}=\dot{\phi}=\pi^{\dagger} \tag{2.60}
\end{equation*}
$$

We can check that

$$
\begin{equation*}
[\phi(\vec{x}, t), \pi(\vec{y}, t)]=\left[\phi^{\dagger}(\vec{x}, t), \pi^{\dagger}(\vec{y}, t)\right]=i \delta^{3}(\vec{x}-\vec{y}) . \tag{2.61}
\end{equation*}
$$

For completeness, the remaining commutators are

$$
\begin{align*}
& {\left[\phi, \phi^{\dagger}\right]=\left[\pi, \pi^{\dagger}\right] } \\
= & {[\phi, \phi]=[\pi, \pi]=\left[\phi, \pi^{\dagger}\right] } \\
= & {\left[\phi^{\dagger}, \phi^{\dagger}\right]=\left[\pi^{\dagger}, \pi^{\dagger}\right]=\left[\phi^{\dagger}, \pi\right]=0 } \tag{2.62}
\end{align*}
$$

Because $\phi^{\dagger} \neq \phi$, we need to have an additional degree of freedom in the mode expansion (as usual, we have $c=\frac{1}{\sqrt{2 \omega}} e^{-i \omega t}$ ):

$$
\begin{equation*}
\phi=\mathcal{K}_{+}^{3}[c a]+\mathcal{K}_{-}^{3}\left[c^{*} b^{\dagger}\right] \tag{2.63}
\end{equation*}
$$

where the definitions of $a$ and $b^{\dagger}$ follow Eq. 2.54 with the underlying operators associated with $\phi_{1,2}$ :

$$
\begin{align*}
& a=\frac{a_{1}-i a_{2}}{\sqrt{2}}  \tag{2.64}\\
& b^{\dagger}=\frac{a_{1}^{\dagger}-i a_{2}^{\dagger}}{\sqrt{2}} \tag{2.65}
\end{align*}
$$

which have inherited the time-independence from $a_{1,2}^{(\dagger)}$. We can construct the additional expansion:

$$
\begin{align*}
\phi^{\dagger} & =\mathcal{K}_{+}^{3}[c b]+\mathcal{K}_{-}^{3}\left[c^{*} a^{\dagger}\right]  \tag{2.66}\\
\pi=\dot{\phi}^{\dagger} & =\mathcal{K}_{+}^{3}[-i \omega c b]+\mathcal{K}_{-}^{3}\left[i \omega c^{*} a^{\dagger}\right] \tag{2.67}
\end{align*}
$$

The full list of annilation and creation operator commutation relation is

$$
\begin{align*}
& {\left[a_{\vec{k}}, a_{\vec{k}^{\prime}}^{\dagger}\right]=\left[b_{\vec{k}}, b_{\vec{k}^{\prime}}^{\dagger}\right]=\delta^{3}\left(\vec{k}-\vec{k}^{\prime}\right) }  \tag{2.68}\\
& {\left[a_{\vec{k}}, b_{\vec{k}^{\prime}}^{\dagger}\right]=\left[a_{\vec{k}}^{\dagger}, b_{\vec{k}^{\prime}}\right] } \\
= & {\left[a_{\vec{k}}, a_{\vec{k}^{\prime}}\right]=\left[b_{\vec{k}}, b_{\vec{k}^{\prime}}\right]=\left[a_{\vec{k}}, b_{\vec{k}^{\prime}}\right] } \\
= & {\left[a_{\vec{k}}^{\dagger}, a_{\vec{k}^{\prime}}^{\dagger}\right]=\left[b_{\vec{k}}^{\dagger}, b_{\vec{k}^{\prime}}^{\dagger}\right]=\left[a_{\vec{k}}^{\dagger}, b_{\vec{k}^{\prime}}^{\dagger}\right]=0 } \tag{2.69}
\end{align*}
$$

Now we can discuss some physics. First, let's examine the vacuum:

$$
\begin{align*}
a_{\vec{k}}|0\rangle & =b_{\vec{k}}|0\rangle  \tag{2.70}\\
|\vec{k}\rangle_{a} & \equiv a_{\vec{k}}^{\dagger}|0\rangle
\end{align*}=\frac{|\vec{k}\rangle_{1}+i|\vec{k}\rangle_{2}}{\sqrt{2}}, ~ \begin{array}{|l|l|} 
 \tag{2.71}\\
|\vec{k}\rangle_{b} & \equiv b_{\vec{k}}^{\dagger}|0\rangle \tag{2.72}
\end{array}
$$

The states created by the two modes, $a^{\dagger}$ and $b^{\dagger}$ (Exercise 3.2a), are orthogonal:

$$
\begin{equation*}
{ }_{b}\left\langle\vec{k} \mid \vec{k}^{\prime}\right\rangle_{a}=0 . \tag{2.73}
\end{equation*}
$$

Since $\mathcal{H}=\mathcal{H}\left[\phi, \pi, \phi^{\dagger}, \pi^{\dagger}\right]$, we have

$$
\begin{align*}
\mathcal{H} & =\pi \dot{\phi}+\pi^{\dagger} \dot{\phi}^{\dagger}-\mathcal{L}  \tag{2.74}\\
& =2 \dot{\phi} \dot{\phi}^{\dagger}-\left(\dot{\phi} \dot{\phi}^{\dagger}-\nabla \phi \nabla \phi^{\dagger}-m^{2} \phi \phi^{\dagger}\right)  \tag{2.75}\\
& =\dot{\phi} \dot{\phi}^{\dagger}+\nabla \phi \nabla \phi^{\dagger}+m^{2} \phi \phi^{\dagger} . \tag{2.76}
\end{align*}
$$

The Hamiltonian is

$$
\begin{align*}
H & =\int \mathrm{d}^{3} \vec{k} \frac{\omega}{2}\left(a_{1}^{\dagger} a_{1}+a_{1} a_{1}^{\dagger}+a_{2}^{\dagger} a_{2}+a_{2} a_{2}^{\dagger}\right)  \tag{2.77}\\
& =\int \mathrm{d}^{3} \vec{k} \omega\left(a^{\dagger} a+b b^{\dagger}\right),  \tag{2.78}\\
: H: & =\int \mathrm{d}^{3} \vec{k} \omega\left(a^{\dagger} a+b^{\dagger} b\right) . \tag{2.79}
\end{align*}
$$

Second, the spacetime picture: The two-point Green's functions are (Exercise 3-2b)

$$
\begin{align*}
& \langle 0| T \phi(x) \phi(y)|0\rangle=\langle 0| T \phi^{\dagger}(x) \phi^{\dagger}(y)|0\rangle=0,  \tag{2.80}\\
& \langle 0| T \phi(x) \phi^{\dagger}(y)|0\rangle=\Delta_{\mathrm{F}}(x-y) . \tag{2.81}
\end{align*}
$$

From the non-vanishing one, we can see that both modes propagate separately from early to late time:

$$
\begin{align*}
& \langle 0| T \phi(x) \phi^{\dagger}(y)|0\rangle \\
& = \begin{cases}\langle 0| \phi(x) \phi^{\dagger}(y)|0\rangle \sim \mathcal{K}\left[\langle 0| a a^{\dagger}|0\rangle\right], & x^{0}>y^{0} \\
\langle 0| \phi^{\dagger}(y) \phi(x)|0\rangle \sim \mathcal{K}\left[\langle 0| b b^{\dagger}|0\rangle\right], & y^{0}>x^{0}\end{cases} \tag{2.82}
\end{align*}
$$

### 2.3.1 Noether's Theorem

Noether's theorem says that "a conserved current is associated with each generator of a continuous symmetry [Zee03]." A conserved current, $J^{\mu}$, has a vanishing total divergence,

$$
\begin{equation*}
\partial_{\mu} J^{\mu}=\frac{\partial J^{0}}{\partial t}+\nabla \cdot \vec{J}=0 \tag{2.83}
\end{equation*}
$$

Note that this is also known as the continuity equation. With a vanishing threecurrent at the spatial boundaries, we have a conserved charge:

$$
\begin{align*}
Q & \equiv \int \mathrm{~d}^{3} \vec{x} J^{0},  \tag{2.84}\\
\because 0=\int \mathrm{d}^{3} \vec{x} \partial_{\mu} J^{\mu} & =\frac{\mathrm{d}}{\mathrm{~d} t} \int \mathrm{~d}^{3} \vec{x} J^{0}+\int \mathrm{d}^{3} \vec{x} \nabla \cdot \vec{J}  \tag{2.85}\\
& =\frac{\mathrm{d} Q}{\mathrm{~d} t}+\underbrace{\int \mathrm{d} \vec{S} \cdot \vec{J}}_{=0 \text { at boundaries }} . \tag{2.86}
\end{align*}
$$

The conserved current and charge are called the Noether current and Noether charge of the symmetry, respectively.


The complex scalar fields we discussed in the preceding section are formed by two real scalar fields (Eqs. 2.54 and 2.55):

$$
\begin{equation*}
\phi=\frac{\phi_{1}-i \phi_{2}}{\sqrt{2}}, \quad \phi^{\dagger}=\frac{\phi_{1}+i \phi_{2}}{\sqrt{2}} \tag{2.87}
\end{equation*}
$$

A complex phase, $e^{ \pm i \alpha}(\alpha \in \mathbb{R})$, rotates $\phi$ and $\phi^{\dagger}$, while keeping $|\phi|^{2} \sim \phi_{1}^{2}+\phi_{2}^{2}$ invariant-a global $U(1)$ symmetry we shall revisit briefly in Sec. 2.5.1.

We can verify that the Lagrangian is unchanged by the transformation:

$$
\begin{equation*}
\phi \rightarrow e^{-i \alpha} \phi, \quad \phi^{\dagger} \rightarrow e^{i \alpha} \phi^{\dagger} \tag{2.88}
\end{equation*}
$$

For infinitesimal $\alpha$, they become

$$
\begin{equation*}
\phi \rightarrow \phi-i \alpha \phi, \quad \phi^{\dagger} \rightarrow \phi^{\dagger}+i \alpha \phi^{\dagger} \tag{2.89}
\end{equation*}
$$

and therefore the changes in the fileds and the Lagrangian are, respectively,

$$
\begin{align*}
\delta \phi & =-i \alpha \phi \quad, \delta \phi^{\dagger}=i \alpha \phi^{\dagger}  \tag{2.90}\\
0=\delta \mathcal{L} & =\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi+\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)} \delta \partial_{\mu} \phi+\delta \phi^{\dagger} \frac{\partial \mathcal{L}}{\partial \phi^{\dagger}}+\delta \partial_{\mu} \phi^{\dagger} \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi^{\dagger}\right)}  \tag{2.91}\\
& =\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi+\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)} \delta \partial_{\mu} \phi+\text { h.c. } \tag{2.92}
\end{align*}
$$

(apply E-L for the first term)
$=\left[\partial_{\mu} \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)}\right] \delta \phi+\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)}\left(\partial_{\mu} \delta \phi\right)+$ h.c.
$=\partial_{\mu}\left[\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)} \delta \phi\right]+$ h.c.

$$
\begin{equation*}
\equiv \partial_{\mu} J^{\mu} \tag{2.94}
\end{equation*}
$$

where

$$
\begin{equation*}
J^{\mu} \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \phi\right)} \delta \phi+\text { h.c. } \tag{2.96}
\end{equation*}
$$

is the Noether current. Therefore, the Noether charge is

$$
\begin{align*}
Q=\int \mathrm{d}^{3} \vec{x} J^{0} & =\int \mathrm{d}^{3} \vec{x} \frac{\partial \mathcal{L}}{\partial \dot{\phi}}(-i \alpha \phi)+\text { h.c. }  \tag{2.97}\\
& =-i \alpha \int \mathrm{~d}^{3} \vec{x}(\pi \phi-\text { h.c. }) \tag{2.98}
\end{align*}
$$

We have (Exercise 3.3)

$$
\begin{equation*}
[Q, \phi]=-\alpha \phi, \quad\left[Q, \phi^{\dagger}\right]=\alpha \phi^{\dagger} \tag{2.99}
\end{equation*}
$$

Or,

$$
\begin{equation*}
i[Q, \phi]=-i \alpha \phi=\delta \phi, \quad i\left[Q, \phi^{\dagger}\right]=i \alpha \phi^{\dagger}=\delta \phi^{\dagger} \tag{2.100}
\end{equation*}
$$

—another set of "EOMs" in addition to Eq. 2.31!

Let's continue the physics discussion of complex scalar fields. We find that (Exercise 3.20)

$$
\begin{gather*}
Q=\alpha \int \mathrm{d}^{3} \vec{k}\left(a^{\dagger} a-b b^{\dagger}\right)  \tag{2.101}\\
: Q:=\alpha \int \mathrm{d}^{3} \vec{k}\left(a^{\dagger} a-b^{\dagger} b\right) \tag{2.102}
\end{gather*}
$$

whose sign between the two modes is in contrast to the Hamiltonian case (Eq. 2.79). It leads to

$$
\begin{align*}
{\left[: Q:, a^{\dagger}\right] } & =\alpha a^{\dagger}  \tag{2.103}\\
{[: Q:, a] } & =-\alpha a  \tag{2.104}\\
{\left[: Q:, b^{\dagger}\right] } & =-\alpha b^{\dagger}  \tag{2.105}\\
{[: Q:, b] } & =\alpha b, \tag{2.106}
\end{align*}
$$

and, therefore,

$$
\begin{align*}
& : Q:|\vec{k}\rangle_{a}=\alpha|\vec{k}\rangle_{a}  \tag{2.107}\\
& : Q:|\vec{k}\rangle_{b}=-\alpha|\vec{k}\rangle_{b} \tag{2.108}
\end{align*}
$$

namely, the two modes created by $a^{\dagger}$ and $b^{\dagger}$ carry opposite charges!

## Exercise 3

## 1. Energy-Momentum Tensor

Consider only the spatial components (i.e., $\mu=1,2,3$ ).
(a) Use the mode expansion for real scalar fields, Eqs. 2.40-2.42,

$$
\begin{aligned}
\phi & =\mathcal{K}_{+}^{3}[c a]+\mathcal{K}_{-}^{3}\left[c^{*} a^{\dagger}\right] \\
\pi & =\mathcal{K}_{+}^{3}[-i \omega c a]+\mathcal{K}_{-}^{3}\left[i \omega c^{*} a^{\dagger}\right] \\
\left(\partial_{i} \phi\right. & \left.=\mathcal{K}_{+}^{3}\left[i k_{i} c a\right]+\mathcal{K}_{-}^{3}\left[-i k_{i} c^{*} a^{\dagger}\right],\right)
\end{aligned}
$$

where $c=\frac{1}{\sqrt{2 \omega}} e^{-i \omega t}$, to show

$$
P^{\mu}=\int \mathrm{d}^{3} \vec{k} \frac{k^{\mu}}{2}\left(a^{\dagger} a+a a^{\dagger}\right)
$$

(b) Prove Eq. 2.32,

$$
\hat{p}^{\mu} \hat{\phi}=\left[\hat{\phi}, \hat{P}^{\mu}\right]
$$

## 2. Complex Scalar Fields

(a) Verify Eq. 2.73,

$$
{ }_{b}\left\langle\vec{k} \mid \vec{k}^{\prime}\right\rangle_{a}=0
$$

and Eq. 2.78,

$$
\begin{aligned}
H & =\int \mathrm{d}^{3} \vec{k} \frac{\omega}{2}\left(a_{1}^{\dagger} a_{1}+a_{1} a_{1}^{\dagger}+a_{2}^{\dagger} a_{2}+a_{2} a_{2}^{\dagger}\right) \\
& =\int \mathrm{d}^{3} \vec{k} \omega\left(a^{\dagger} a+b b^{\dagger}\right)
\end{aligned}
$$

explicitly.
(b) Modify your answer of Exercise 2. 4 b to verify Eqs. 2.80 and 2.81,

$$
\begin{aligned}
& \langle 0| T \phi(x) \phi(y)|0\rangle=\langle 0| T \phi^{\dagger}(x) \phi^{\dagger}(y)|0\rangle=0 \\
& \langle 0| T \phi(x) \phi^{\dagger}(y)|0\rangle=\Delta_{\mathrm{F}}(x-y)
\end{aligned}
$$

(c) Prove Eq. 2.101,

$$
Q=\alpha \int \mathrm{d}^{3} \vec{k}\left(a^{\dagger} a-b b^{\dagger}\right)
$$

using the field expansion, Eqs. 2.63, 2.66 and 2.67,

$$
\begin{aligned}
\phi & =\mathcal{K}_{+}^{3}[c a]+\mathcal{K}_{-}^{3}\left[c^{*} b^{\dagger}\right] \\
\phi^{\dagger} & =\mathcal{K}_{+}^{3}[c b]+\mathcal{K}_{-}^{3}\left[c^{*} a^{\dagger}\right] \\
\pi=\dot{\phi}^{\dagger} & =\mathcal{K}_{+}^{3}[-i \omega c b]+\mathcal{K}_{-}^{3}\left[i \omega c^{*} a^{\dagger}\right]
\end{aligned}
$$

## 3. Noether's Theorem

(a) Derive Eq. 2.99,

$$
[Q, \phi]=-\alpha \phi, \quad\left[Q, \phi^{\dagger}\right]=\alpha \phi^{\dagger}
$$

(b) Based on Eqs. 2.98,

$$
Q=\int \mathrm{d}^{3} \vec{x} J^{0}=-i \alpha \int \mathrm{~d}^{3} \vec{x}(\pi \phi-\text { h.c. }),
$$

and 2.99 (see above), we define $Q_{0}$ and $\psi_{1,2}$ as follows:

$$
\begin{aligned}
Q & =\alpha Q_{0} \quad, \quad\left[Q_{0}, \phi\right]=-\phi \\
\psi_{1} & =e^{\alpha Q_{0}} \phi e^{-\alpha Q_{0}} \\
\psi_{2} & =e^{-\alpha} \phi
\end{aligned}
$$

Prove that $\psi_{1}=\psi_{2}$.
(Hint: Expand $\psi_{1,2}$ in Taylor series as a function of $\alpha$ around 0 , compare the coefficients order by order.)

### 2.4 Spinors

The free Dirac Lagrangian is

$$
\begin{equation*}
\mathcal{L}[\psi, \partial \psi, \bar{\psi}, \partial \bar{\psi}]=\bar{\psi}(i \gamma \partial-m) \psi, \quad \bar{\psi}=\psi^{\dagger} \gamma^{0} \tag{2.109}
\end{equation*}
$$

with

$$
\begin{align*}
& \frac{\partial \mathcal{L}}{\partial \psi}=-m \bar{\psi}, \quad \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \psi\right)}=\bar{\psi} i \gamma^{\mu}  \tag{2.110}\\
& \frac{\partial \mathcal{L}}{\partial \bar{\psi}}=(i \gamma \partial-m) \psi, \quad \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \bar{\psi}\right)}=0 \tag{2.111}
\end{align*}
$$

We have the Euler-Lagrange equations, which gives us the Dirac equations (Exercise 4. 1a),

$$
\begin{align*}
\bar{\psi}(i \gamma \overleftarrow{\partial}+m) & =0  \tag{2.112}\\
(i \gamma \partial-m) \psi & =0 \tag{2.113}
\end{align*}
$$

Applying $(i \gamma \partial+m)$ on the left of Eq. 2.113, we recover the Klein-Gordon equation 2 :

$$
\begin{align*}
0 & =(i \gamma \partial+m)(i \gamma \partial-m) \psi  \tag{2.114}\\
& =\left(-\gamma_{\mu} \partial^{\mu} \gamma_{\nu} \partial^{\nu}-m^{2}\right) \psi=-\left(\partial^{2}+m^{2}\right) \psi \tag{2.115}
\end{align*}
$$

The canonical momenta read

$$
\begin{equation*}
\pi=\frac{\partial \mathcal{L}}{\partial \dot{\psi}}=\bar{\psi} i \gamma^{0}=i \psi^{\dagger}, \quad \widetilde{\pi}=\frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}}=0 \tag{2.116}
\end{equation*}
$$

which leads us to the Hamiltonian density,

$$
\begin{align*}
\mathcal{H} & =\pi \dot{\psi}+\widetilde{\pi} \dot{\bar{\psi}}-\mathcal{L}  \tag{2.117}\\
& =i \psi^{\dagger} \dot{\psi}-\bar{\psi}(i \gamma \partial-m) \psi  \tag{2.118}\\
& =i \psi^{\dagger} \dot{\psi} \tag{2.119}
\end{align*}
$$

where the last step has applied the Euler-Lagrange (Dirac) equation.
Similar to the complex field with the global $U(1)$ symmetry $\psi \rightarrow e^{-i \alpha} \psi$ (Eq. 2.96), we have the Noether current,

$$
\begin{align*}
J^{\mu} & =\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \psi\right)} \delta \psi+\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \bar{\psi}\right)} \delta \bar{\psi}  \tag{2.120}\\
& =\alpha \bar{\psi} \gamma^{\mu} \psi \tag{2.121}
\end{align*}
$$

and the Neother charge,

$$
\begin{align*}
Q=\int \mathrm{d}^{3} \vec{x} J^{0} & =\alpha \int \mathrm{d}^{3} \vec{x} \bar{\psi} \gamma^{0} \psi  \tag{2.122}\\
& =\alpha \int \mathrm{d}^{3} \vec{x} \psi^{\dagger} \psi \tag{2.123}
\end{align*}
$$

[^3]
### 2.4.1 Dirac Equation

Let's pause and take a look at the solutions of the Dirac equation-Dirac spinors.
First of all, we have the anticommutation relation of the gamma matrix:

$$
\begin{equation*}
\left\{\gamma^{\mu}, \gamma^{\nu}\right\}=2 g^{\mu \nu} \tag{2.124}
\end{equation*}
$$

from which we have (Exercises 4.1 B and 4.1 c ),

$$
\begin{align*}
\left(\gamma^{0}\right)^{2} & =1  \tag{2.125}\\
(\gamma x)^{2} & =x^{2}  \tag{2.126}\\
\gamma x \gamma^{0} & =2 x^{0}-\gamma^{0} \gamma x \tag{2.127}
\end{align*}
$$

with an abbitrary 4 -vector $x$ 3. In the Dirac basis,

$$
\begin{align*}
& \left(\gamma^{\mu}\right)^{\dagger}=\gamma^{0} \gamma^{\mu} \gamma^{0}  \tag{2.128}\\
& \gamma^{0}=\operatorname{diag}(1,1,-1,-1)=\left(\begin{array}{ll}
I_{2} & 0 \\
0 & -I_{2}
\end{array}\right) \tag{2.129}
\end{align*}
$$

where $I_{2}$ is the $2 \times 2$ identity matrix.
Second, we notice that,

$$
\begin{align*}
& i \partial e^{i p x}=-p e^{i p x}  \tag{2.130}\\
& i \partial e^{-i p x}=p e^{-i p x} \tag{2.131}
\end{align*}
$$

So, we have two plane-wave solutions:

$$
\begin{gather*}
\psi=u(p) e^{-i p x}, \quad(\gamma p-m) u(p)=0  \tag{2.132}\\
\psi=v(p) e^{i p x}, \quad(\gamma p+m) v(p)=0 \tag{2.133}
\end{gather*}
$$

Consider the solution for $\vec{p}=0$, i.e., in the rest frame of the field particle. We have 4

$$
\begin{align*}
& p=(m, \overrightarrow{0})  \tag{2.134}\\
& \left(\gamma^{0}-1\right) u(0)=0  \tag{2.135}\\
& \left(\gamma^{0}+1\right) v(0)=0 \tag{2.136}
\end{align*}
$$

$$
\begin{aligned}
& { }^{3} \text { Equation } 2.126 \text { leads to the following useful expressions: } \\
& \qquad \begin{aligned}
&(\gamma p+m)(\gamma p-m)=(\gamma p-m)(\gamma p+m)=0 \\
& \Rightarrow \quad \gamma p(\gamma p-m)=-m(\gamma p-m), \quad \gamma p(\gamma p+m)=m(\gamma p+m)
\end{aligned}
\end{aligned}
$$

Namely, one can replace $\gamma p$ right in front of $\gamma p \pm m$ directly by $\pm m —$ Try the following:

$$
\begin{aligned}
& (\gamma p+m)(\gamma p+m)=2 m(\gamma p+m), \\
& (\gamma p-m)(\gamma p-m)=-2 m(\gamma p-m), \\
& (\gamma p+m) \gamma^{0}(\gamma p+m)=\left(2 p^{0}-\gamma^{0} \gamma p+m \gamma^{0}\right)(\gamma p+m)=2 p^{0}(\gamma p+m), \\
& (\gamma p-m) \gamma^{0}(\gamma p-m)=\left(2 p^{0}-\gamma^{0} \gamma p-m \gamma^{0}\right)(\gamma p-m)=2 p^{0}(\gamma p-m) .
\end{aligned}
$$

${ }^{4}$ Direct implications:

$$
\bar{u}(0)=u^{\dagger}(0), \quad \bar{v}(0)=-v^{\dagger}(0) .
$$

In the Dirac basis, we can choose the following orthogonal solutions ${ }^{\text {雨: }}$

$$
\begin{align*}
& u_{1}(0)=\left(\begin{array}{l}
1 \\
0 \\
0 \\
0
\end{array}\right), \quad u_{2}(0)=\left(\begin{array}{l}
0 \\
1 \\
0 \\
0
\end{array}\right)  \tag{2.137}\\
& v_{1}(0)=\left(\begin{array}{l}
0 \\
0 \\
1 \\
0
\end{array}\right), \quad v_{2}(0)=\left(\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right) \tag{2.138}
\end{align*}
$$

The two-fold solutions of $u$ and $v$ each are the spin degrees of freedom for a spin- $\frac{1}{2}$ particle. We can now construct the solutions for non-vanishing $\vec{p}$ :

$$
\begin{align*}
& u(p)=N(\gamma p+m) u(0)  \tag{2.139}\\
& v(p)=N(\gamma p-m) v(0) \tag{2.140}
\end{align*}
$$

where $N$ is a normalisation factor to be chosen later. To see that they are indeed solutions of the Dirac equation, we can put them back into the equation. For example,

$$
\begin{equation*}
(\gamma p-m) u(p)=N(\gamma p-m)(\gamma p+m) u(0)=0 \tag{2.141}
\end{equation*}
$$

The normalisation of these solutions are 6 (Exercises 4. 1d and 4. 18)

$$
\begin{align*}
u_{s}^{\dagger}(p) u_{s^{\prime}}(p) & =\delta_{s s^{\prime}} 2 E(m+E) N^{2}  \tag{2.142}\\
& =v_{s}^{\dagger}(p) v_{s^{\prime}}(p)  \tag{2.143}\\
\bar{u}_{s}(p) u_{s^{\prime}}(p) & =\delta_{s s^{\prime}} 2 m(m+E) N^{2}  \tag{2.144}\\
& =-\bar{v}_{s}(p) v_{s^{\prime}}(p) \tag{2.145}
\end{align*}
$$

It is very often that we need to do spin sums, which are given as follows

$$
\begin{align*}
& \sum_{s=1,2} u_{s}(p) \bar{u}_{s}(p)=N^{2}(m+E)(\gamma p+m)  \tag{2.146}\\
& \sum_{s=1,2} v_{s}(p) \bar{v}_{s}(p)=N^{2}(m+E)(\gamma p-m) \tag{2.147}
\end{align*}
$$

[^4]${ }^{6}$ The proof for the first case is sketched below. The steps involve Footnotes $3-5$.
\[

$$
\begin{aligned}
& u_{s}^{\dagger}(p) u_{s^{\prime}}(p)=u_{s}^{\dagger}(0)\left(\gamma^{\dagger} p+m\right) N^{2}(\gamma p+m) u_{s^{\prime}}(0)=\bar{u}_{s}(0)(\gamma p+m) \gamma^{0}(\gamma p+m) u_{s^{\prime}}(0) N^{2} \\
& =\bar{u}_{s}(0)(\gamma p+m) u_{s^{\prime}}(0) N^{2} 2 p^{0}=u_{s}^{\dagger}(0)(\gamma p+m) u_{s^{\prime}}(0) N^{2} 2 p^{0}=\delta_{s s^{\prime}}\left(m+p^{0}\right) N^{2} 2 p^{0} .
\end{aligned}
$$
\]

The "-" sign in Eq. 2.145 originates from the one in Footnote 3.
${ }^{7}$ The proof for the first case is sketched below:

$$
\begin{aligned}
& \sum_{s} u_{s}(p) \bar{u}_{s}(p)=N^{2}(\gamma p+m) \sum_{s} u_{s}(0) u_{s}^{\dagger}(0)\left(\gamma^{\dagger} p+m\right) \gamma^{0}=N^{2}(\gamma p+m) \frac{\gamma^{0}+1}{2}(\gamma p+m) \\
& =N^{2} \frac{2 p^{0}+2 m^{0}}{2}(\gamma p+m)=N^{2}\left(m+p^{0}\right)(\gamma p+m) .
\end{aligned}
$$

### 2.4.2 Quantising the Spinor Field

We are going to build our spinor field expansion on top of what we did for the complex scalar field, Eq. 2.63. Now, the differences are that

1. there are additional degrees of freedom for spins;
2. the field expansion needs to automatically satisfy the Dirac equation.

Therefore, we have the following expansion of the $\psi$ field, trying to maintain a parallelism with Eq. 2.63:

$$
\begin{equation*}
\psi=\mathcal{K}_{+}^{3}\left[c \sum_{s} a_{s} u_{s}\right]+\mathcal{K}_{-}^{3}\left[c^{*} \sum_{s} b_{s}^{\dagger} v_{s}\right] \tag{2.148}
\end{equation*}
$$

and then $\pi$ and $\dot{\psi}$ can be constructed out of it:

$$
\begin{align*}
\pi & =i \psi^{\dagger}  \tag{2.149}\\
& =\mathcal{K}_{+}^{3}\left[i c \sum_{s} b_{s} v_{s}^{\dagger}\right]+\mathcal{K}_{-}^{3}\left[i c^{*} \sum_{s} a_{s}^{\dagger} u_{s}^{\dagger}\right]  \tag{2.150}\\
\dot{\psi} & =\mathcal{K}_{+}^{3}\left[-i \omega c \sum_{s} a_{s} u_{s}\right]+\mathcal{K}_{-}^{3}\left[i \omega c^{*} \sum_{s} b_{s}^{\dagger} v_{s}\right] . \tag{2.151}
\end{align*}
$$

Note that the $\dagger$ 's on the spinors need to be consistent in all the terms. The last modification to make is to change commutators to anticommutators. The only non-vanishing anticommutators are

$$
\begin{equation*}
\left\{a_{s, \vec{k}}, a_{s^{\prime}, \overrightarrow{k^{\prime}}}^{\dagger}\right\}=\left\{b_{s, \vec{k}}, b_{s^{\prime}, \overrightarrow{k^{\prime}}}^{\dagger}\right\}=\delta_{s s^{\prime}} \delta^{3}\left(\vec{k}-\overrightarrow{k^{\prime}}\right) \tag{2.152}
\end{equation*}
$$

The approach we take here is somehow opposite to what we did for the scalar fields; now we are going to use the creation and annihilation anticommutators to determine the field anticommutator 8 :

$$
\begin{equation*}
\{\psi(\vec{x}, t), \pi(\vec{y}, t)\}=i \int \frac{\mathrm{~d}^{3} \vec{k}}{(2 \pi)^{3}} e^{i \vec{k} \cdot(\vec{x}-\vec{y})} N^{2}\left(m+\omega_{k}\right) \tag{2.153}
\end{equation*}
$$

[^5]If we choose the spinor normalisation as (otherwise we need to change our conventions for $c_{1,2}$ )

$$
\begin{equation*}
N=\frac{1}{\sqrt{m+\omega_{k}}} \tag{2.154}
\end{equation*}
$$

the integral becomes a $\delta$ function, and, finally,

$$
\begin{equation*}
\{\psi(\vec{x}, t), \pi(\vec{y}, t)\}=i \delta(\vec{x}-\vec{y}) \tag{2.155}
\end{equation*}
$$

which is similar to the scalar cases, Eqs. 2.38 and 2.61. With this normalisation (and our usual $c=\frac{1}{\sqrt{2 \omega}} e^{-i \omega t}$ ), we have

$$
\begin{align*}
u_{s}^{\dagger}(p) u_{s^{\prime}}(p) & =\delta_{s s^{\prime}} 2 E  \tag{2.156}\\
& =v_{s}^{\dagger}(p) v_{s^{\prime}}(p)  \tag{2.157}\\
\bar{u}_{s}(p) u_{s^{\prime}}(p) & =\delta_{s s^{\prime}} 2 m  \tag{2.158}\\
& =-\bar{v}_{s}(p) v_{s^{\prime}}(p)  \tag{2.159}\\
\sum_{s=1,2} u_{s}(p) \bar{u}_{s}(p) & =\gamma p+m  \tag{2.160}\\
\sum_{s=1,2} v_{s}(p) \bar{v}_{s}(p) & =\gamma p-m \tag{2.161}
\end{align*}
$$

Furthermore, we have 9

$$
\begin{equation*}
H=\int \mathrm{d}^{3} \vec{k} \omega \sum_{s}\left(a_{s}^{\dagger} a_{s}-b_{s} b_{s}^{\dagger}\right) \tag{2.162}
\end{equation*}
$$

and (Exercise 4.14),

$$
\begin{equation*}
Q=\alpha \int \mathrm{d}^{3} \vec{k} \sum_{s}\left(a_{s}^{\dagger} a_{s}+b_{s} b_{s}^{\dagger}\right) \tag{2.163}
\end{equation*}
$$

Note the different "-" signs compared to the complex scalar fields (Eqs. 2.78 and 2.101). Because of anticommutation, we now introduce a "-" sign when the normal ordering takes effect. Therefore,

$$
\begin{align*}
& : H:=\int \mathrm{d}^{3} \vec{k} \omega \sum_{s}\left(a^{\dagger} a+b^{\dagger} b\right)  \tag{2.164}\\
& : Q:=\alpha \int \mathrm{d}^{3} \vec{k} \sum_{s}\left(a^{\dagger} a-b^{\dagger} b\right) \tag{2.165}
\end{align*}
$$

which have the same forms as for the complex scalar (Eqs. 2.79 and 2.102).

$$
\begin{aligned}
& \hline{ }^{9} \text { The calculation of } H \text { is relatively straightforward: } \\
& H=\int \mathrm{d}^{3} \vec{x} i \psi^{\dagger} \dot{\psi} \\
& \text { (cross terms vanish because of orthogonality) } \\
&= i \int \mathrm{~d}^{3} \vec{x}\left(\mathcal{K}_{+}^{3}\left[c \sum_{s} b_{s} v_{s}^{\dagger}\right] \mathcal{K}_{-}^{3}\left[i \omega c^{*} \sum_{s} b_{s}^{\dagger} v_{s}\right]+\mathcal{K}_{-}^{3}\left[c^{*} \sum_{s} a_{s}^{\dagger} u_{s}^{\dagger}\right] \mathcal{K}_{+}^{3}\left[-i \omega c \sum_{s} a_{s} u_{s}\right]\right) \\
&= i \int \mathrm{~d}^{3} \vec{k}\left(\frac{i}{2} \sum_{s s^{\prime}} v_{s}^{\dagger} v_{s^{\prime}} b_{s} b_{s^{\prime}}^{\dagger}-\frac{i}{2} \sum_{s s^{\prime}} u_{s}^{\dagger} u_{s^{\prime}} a_{s}^{\dagger} a_{s^{\prime}}\right) .
\end{aligned}
$$

The time-ordered product and spinor Feynman propagator are, respectively,

$$
\begin{align*}
& T \psi_{\alpha}(x) \bar{\psi}_{\beta}(y) \equiv \psi_{\alpha}(x) \bar{\psi}_{\beta}(y) \theta\left(x^{0}-y^{0}\right)-\bar{\psi}_{\beta}(y) \psi_{\alpha}(x) \theta\left(y^{0}-x^{0}\right)  \tag{2.166}\\
& \langle 0| T \psi_{\alpha}(x) \bar{\psi}_{\beta}(y)|0\rangle=S_{\mathrm{F} \alpha \beta} \tag{2.167}
\end{align*}
$$

where $\alpha$ and $\beta$ are spinor indices (see Footnote 8) and

$$
\begin{equation*}
S_{\mathrm{F}} \equiv i \int \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} \frac{\gamma k+m}{k^{2}-m^{2}+i \epsilon} e^{-i k(x-y)}=i \int \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} \frac{e^{-i k(x-y)}}{\gamma k-m+i \epsilon} . \tag{2.168}
\end{equation*}
$$

It can be readily verified that $S_{\mathrm{F}}$ is indeed the Green's function of the Dirac equation:

$$
\begin{equation*}
(i \gamma \partial-m) S_{\mathrm{F}}=i \delta^{4}(x-y) \tag{2.169}
\end{equation*}
$$

### 2.5 Photons

The free Lagrangian of a photon field is

$$
\begin{equation*}
\mathcal{L}[A, \partial A]=-\frac{1}{4} F_{\mu \nu} F^{\mu \nu} \tag{2.170}
\end{equation*}
$$

with the electromagnetic tensor,

$$
\begin{equation*}
F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu} \tag{2.171}
\end{equation*}
$$

Note that,

$$
\begin{align*}
\frac{\partial F_{\mu \nu}}{\partial\left(\partial_{\alpha} A_{\beta}\right)} & =\frac{\partial}{\partial\left(\partial_{\alpha} A_{\mu}\right)}\left(\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}\right)  \tag{2.172}\\
& =\delta_{\mu}^{\alpha} \delta_{\nu}^{\beta}-\delta_{\nu}^{\alpha} \delta_{\mu}^{\beta} \tag{2.173}
\end{align*}
$$

We have

$$
\begin{align*}
\frac{\partial \mathcal{L}}{\partial\left(\partial_{\alpha} A_{\beta}\right)} & =-\frac{1}{4} \frac{\partial\left(F_{\mu \nu} F^{\mu \nu}\right)}{\partial F_{\mu \nu}} \frac{\partial F_{\mu \nu}}{\left(\partial \partial_{\alpha} A_{\beta}\right)}  \tag{2.174}\\
& =-\frac{1}{4} \cdot 2 F^{\mu \nu}\left(\delta_{\mu}^{\alpha} \delta_{\nu}^{\beta}-\delta_{\nu}^{\alpha} \delta_{\mu}^{\beta}\right)  \tag{2.175}\\
& =-\frac{1}{2}\left(F^{\alpha \beta}-F^{\beta \alpha}\right)  \tag{2.176}\\
& =-\frac{1}{2}\left(F^{\alpha \beta}+F^{\alpha \beta}\right)  \tag{2.177}\\
& =-F^{\alpha \beta} \tag{2.178}
\end{align*}
$$

This gives us the Euler-Lagrange equations,

$$
\begin{align*}
0=\frac{\partial \mathcal{L}}{\partial A_{\beta}} & =\partial_{\alpha} \frac{\partial \mathcal{L}}{\partial\left(\partial_{\alpha} A_{\beta}\right)}  \tag{2.179}\\
& =-\partial_{\alpha} F^{\alpha \beta}  \tag{2.180}\\
& =-\partial_{\alpha}\left(\partial^{\alpha} A^{\beta}-\partial^{\beta} A^{\alpha}\right)  \tag{2.181}\\
& =-\partial^{2} A^{\beta}+\partial^{\beta} \partial_{\alpha} A^{\alpha} \tag{2.182}
\end{align*}
$$

These four equations (with $\beta=0, \ldots, 3$ ) do not determine $A_{\beta}$ uniquely because a shift in $A_{\beta}$ as follows leaves the field tensor, $F_{\mu \nu}$, unchanged and therefore preserves the above Euler-Lagrange equations:

$$
\begin{equation*}
A_{\beta} \rightarrow A_{\beta}+\partial_{\beta} G \tag{2.183}
\end{equation*}
$$

where $G$ is an arbitrary scalar field. This redundant degree of freedom of the $A_{\beta^{-}}$ field needs to be gauged away. With the Lorenz gauge 10 condition,

$$
\begin{equation*}
\partial_{\mu} A^{\mu}=0 \tag{2.184}
\end{equation*}
$$

this becomes

$$
\begin{equation*}
\partial^{2} A^{\mu}=0 \tag{2.185}
\end{equation*}
$$

The gauge fixing can be elegantly imposed by adding a term to the Lagarangian:

$$
\begin{align*}
\mathcal{L} & =-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}+\mathcal{L}_{\mathrm{g}}  \tag{2.186}\\
\mathcal{L}_{\mathrm{g}} & \equiv-\frac{1}{2} \partial_{\mu} A^{\mu} \partial_{\nu} A^{\nu} \tag{2.187}
\end{align*}
$$

with

$$
\begin{align*}
\frac{\partial \mathcal{L}_{\mathrm{g}}}{\partial\left(\partial_{\alpha} A_{\beta}\right)} & =-\frac{1}{2} \cdot 2\left(\partial_{\mu} A^{\mu}\right) \underbrace{\frac{\partial\left(\partial_{\mu} A^{\mu}\right)}{\partial\left(\partial_{\alpha} A_{\beta}\right)}}_{=\underbrace{\delta_{\mu}^{\alpha} \delta^{\mu \beta}}_{=\delta^{\alpha \beta}}}  \tag{2.188}\\
& =-\delta^{\alpha \beta} \partial_{\mu} A^{\mu} . \tag{2.189}
\end{align*}
$$

So,

$$
\begin{equation*}
\frac{\partial \mathcal{L}}{\partial\left(\partial_{\alpha} A_{\beta}\right)}=-F^{\alpha \beta}-\delta^{\alpha \beta} \partial_{\mu} A^{\mu} \tag{2.190}
\end{equation*}
$$

Now, we have an Euler-Lagrange equation that is automatically gauge-fixed:

$$
\begin{align*}
0=\partial_{\alpha} \frac{\partial \mathcal{L}}{\partial\left(\partial_{\alpha} A_{\beta}\right)} & =-\partial_{\alpha} F^{\alpha \beta}-\partial^{\beta} \partial_{\mu} A^{\mu}  \tag{2.191}\\
& =-\partial_{\alpha}\left(\partial^{\alpha} A^{\beta}-\partial^{\beta} A^{\alpha}\right)-\partial^{\beta} \partial_{\mu} A^{\mu}  \tag{2.192}\\
& =-\partial^{2} A^{\beta}+\partial^{\beta} \partial_{\alpha} A^{\alpha}-\partial^{\beta} \partial_{\mu} A^{\mu}  \tag{2.193}\\
& =-\partial^{2} A^{\beta} \tag{2.194}
\end{align*}
$$

This gauge choice of the Lagrangian is called the Feynman gauge (although here effectively, it is doing what the Lorenz gauge does).

The existence of one choice means that there are many others - there is always more than one way to remove a degree of freedom, that is, to add an independent field equation such as Eq. 2.184. One way of generalising is to parameterise the gauge-fixing Lagrangian:

$$
\begin{equation*}
\mathcal{L}_{\mathrm{g}}(\xi) \equiv-\frac{1}{2 \xi} \partial_{\mu} A^{\mu} \partial_{\nu} A^{\nu} \tag{2.195}
\end{equation*}
$$

with the corresponding Euler-Lagrange equation (Exercise 4.2a):

$$
\begin{equation*}
\left[-\partial^{2} g_{\alpha}^{\mu}-\left(\frac{1}{\xi}-1\right) \partial^{\mu} \partial_{\alpha}\right] A^{\alpha}=0 \tag{2.196}
\end{equation*}
$$

The photon Feynman propagator is

$$
\begin{equation*}
\langle 0| T A^{\mu}(x) A^{\nu}(y)|0\rangle=\Pi_{\mathrm{F}}^{\mu \nu} \equiv-i \int \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} \frac{g^{\mu \nu}-(1-\xi) \frac{k^{\mu} k^{\nu}}{k^{2}}}{k^{2}+i \epsilon} \mathrm{e}^{-i k(x-y)} \tag{2.197}
\end{equation*}
$$

which is the Green's function of Eq. 2.196 (Exercise 4.2b):

$$
\begin{equation*}
\left[-\partial^{2} g_{\alpha}^{\mu}-\left(\frac{1}{\xi}-1\right) \partial^{\mu} \partial_{\alpha}\right] \Pi_{\mathrm{F}}^{\alpha \nu}=-i g^{\mu \nu} \delta^{4}(x-y) \tag{2.198}
\end{equation*}
$$

[^6]
### 2.5.1 Local Gauge Principle

In Sec. 2.3.1, we discussed an example of the global gauge invance: the Lagrangian is invariant under a phase rotation, Eq. 2.88, that is independent of space and time:

$$
\phi \rightarrow e^{-i \alpha} \phi, \quad \phi^{\dagger} \rightarrow e^{i \alpha} \phi^{\dagger}
$$

Now, let's consider the free (hence the subscript ${ }_{0}$ ) Lagrangian of a spinor field and a local phase change, which is a function of space and time:

$$
\begin{align*}
& \mathcal{L}_{0}=\bar{\psi}(i \gamma \partial-m) \psi  \tag{2.199}\\
& \psi \rightarrow e^{-i \alpha(x)} \psi, \quad \bar{\psi} \rightarrow e^{i \alpha(x)} \bar{\psi} \tag{2.200}
\end{align*}
$$

Since

$$
\begin{equation*}
\partial_{\mu} \psi=\left(-i \partial_{\mu} \alpha\right) e^{-i \alpha} \psi+e^{-i \alpha} \partial_{\mu} \psi \tag{2.201}
\end{equation*}
$$

we have a change in the Lagrangian,

$$
\begin{align*}
\delta \mathcal{L}_{0} & =e^{i \alpha} \bar{\psi} i \gamma^{\mu}\left(-i \partial_{\mu} \alpha\right) e^{-i \alpha} \psi  \tag{2.202}\\
& =\left(\partial_{\mu} \alpha\right) \bar{\psi} \gamma^{\mu} \psi  \tag{2.203}\\
& \equiv\left(\partial_{\mu} \alpha\right) J^{\mu} \tag{2.204}
\end{align*}
$$

where $J$ is the Noether current we obtained in Eq. 2.121 (note that we have taken $\alpha$ out of the definition here).

We shall correct this by ${ }^{11}$ replacing the ordinary derivative with the covariant derivative:

$$
\begin{equation*}
\partial_{\mu} \rightarrow \partial_{\mu}+i e A_{\mu} \tag{2.205}
\end{equation*}
$$

where $A_{\mu}$ is a vector field and $e$ some constant. Now we have an additional term in the Lagrangian:

$$
\begin{align*}
\mathcal{L}^{\prime} & =\bar{\psi} i \gamma^{\mu}\left(i e A_{\mu}\right) \psi  \tag{2.206}\\
& =-e A_{\mu} \bar{\psi} \gamma^{\mu} \psi  \tag{2.207}\\
& =-e A_{\mu} J^{\mu} \tag{2.208}
\end{align*}
$$

The new Lagrangian is (Exercise 4.3)

$$
\begin{align*}
\mathcal{L} & =\mathcal{L}_{0}+\mathcal{L}^{\prime}  \tag{2.209}\\
& =\bar{\psi}\left(i \gamma^{\mu} \partial_{\mu}-m\right) \psi-e A_{\mu} J^{\mu} \tag{2.210}
\end{align*}
$$

Then we require the $A_{\mu}$-field to change in the local gauge transform, Eq. 2.200, as follows:

$$
\begin{equation*}
A_{\mu} \rightarrow A_{\mu}+\delta A_{\mu} \tag{2.211}
\end{equation*}
$$

which should keep the new Lagrangian invariant.

$$
\begin{equation*}
\delta \mathcal{L}=\left(\partial_{\mu} \alpha\right) J^{\mu}-e\left(\delta A_{\mu}\right) J^{\mu}=0 \tag{2.212}
\end{equation*}
$$

The required change of the $A_{\mu}$ field is then

$$
\begin{equation*}
\delta A_{\mu}=\frac{1}{e} \partial_{\mu} \alpha \tag{2.213}
\end{equation*}
$$

[^7]namely,
\[

$$
\begin{equation*}
A_{\mu} \rightarrow A_{\mu}+\frac{1}{e} \partial_{\mu} \alpha \tag{2.214}
\end{equation*}
$$

\]

We identify the $A_{\mu}$-field as the photon field, with Eq. 2.214 a special case of Eq. 2.183; $F_{\mu \nu}$ is already compatible with the covariant derivative, namely, the modification, Eq. 2.205, leaves $F_{\mu \nu}$ invariant. By the same token of gauge invariance, a mass term $\sim \frac{1}{2} M^{2} A_{\mu} A^{\mu}$ is forbidden in the free photon Lagarangian, Eq. 2.170.

## Exercise 4

## 1. Spinors

(a) Derive Eq. 2.112,

$$
\bar{\psi}(i \gamma \overleftarrow{\partial}+m)=0
$$

from Eq. 2.113,

$$
(i \gamma \partial-m) \psi=0
$$

(b) Prove Eq. 2.126,

$$
(\gamma x)^{2}=x^{2}
$$

(c) Prove Eq. 2.127,

$$
\gamma x \gamma^{0}=2 x^{0}-\gamma^{0} \gamma x
$$

(d) The spinors, $\psi, u, v$, are $4 \times 1$ objects, namely vectors in the spinor space. What are the dimensions of the following quantities?
i. $\bar{\psi}$,
ii. $\gamma^{\mu}$,
iii. the Lagrangian, $\mathcal{L}$,
iv. the Hamiltonian density, $\mathcal{H}$,
v. $\bar{u} u$,
vi. $u \bar{u}$, and
vii. the current $J^{\mu}=\bar{\psi} \gamma^{\mu} \psi$.
(e) Prove
i.

$$
\sum_{s} u_{s}(0) u_{s}^{\dagger}(0)\left(\gamma^{\dagger} p+m\right) \gamma^{0}=\frac{\gamma^{0}+1}{2}(\gamma p+m)
$$

ii. Eqs. 2.143, 2.144, 2.145 and 2.147 ,

$$
\begin{aligned}
\delta_{s s^{\prime}} 2 E(m+E) N^{2} & =v_{s}^{\dagger}(p) v_{s^{\prime}}(p) \\
\bar{u}_{s}(p) u_{s^{\prime}}(p) & =\delta_{s s^{\prime}} 2 m(m+E) N^{2} \\
& =-\bar{v}_{s}(p) v_{s^{\prime}}(p) \\
\sum_{s=1,2} v_{s}(p) \bar{v}_{s}(p) & =N^{2}(m+E)(\gamma p-m)
\end{aligned}
$$

(f) Derive Eq. 2.163 ,

$$
Q=\alpha \int \mathrm{d}^{3} \vec{k} \sum_{s}\left(a_{s}^{\dagger} a_{s}+b_{s} b_{s}^{\dagger}\right) .
$$

## 2. Photons

(a) Derive Eq. 2.196 ,

$$
\left[-\partial^{2} g_{\alpha}^{\mu}-\left(\frac{1}{\xi}-1\right) \partial^{\mu} \partial_{\alpha}\right] A^{\alpha}=0
$$

(b) Verify Eq. 2.198,

$$
\left[-\partial^{2} g_{\alpha}^{\mu}-\left(\frac{1}{\xi}-1\right) \partial^{\mu} \partial_{\alpha}\right] \Pi_{\mathrm{F}}^{\alpha \nu}=-i g^{\mu \nu} \delta^{4}(x-y) .
$$

## 3. Local Gauge Principle

What is the Euler-Lagrange equation of the spinor field by the new Lagargian, Eq. 2.210,

$$
\mathcal{L}=\bar{\psi}\left(i \gamma^{\mu} \partial_{\mu}-m\right) \psi-e A_{\mu} J^{\mu} \quad ?
$$

## Appendix

## A Relativistic Notations

Here is a reminder of the signs in special relativity. We temporarily restore the ${ }^{\wedge}$-notation for operators in this subsection to avoid confusion. We use Greek letters, $\mu, \nu$, etc., for Lorentz indices, while Latin letters, $i, j$, etc., do not need to follow the index rule.

Metric:

$$
g^{\mu \nu}=g_{\mu \nu}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{A.1}\\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{array}\right)
$$

Note that, independent of the signature convention, we have

$$
\begin{equation*}
g^{\mu \alpha} g_{\alpha \nu}=g_{\nu}^{\mu}=\delta_{\nu}^{\mu} \tag{A.2}
\end{equation*}
$$

where $\delta_{\nu}^{\mu}=\delta^{\mu \nu}=\delta_{\mu \nu}$ is the Kronecker delta.
Contravariant vectors $A^{\mu}$ :

$$
\begin{align*}
x^{\mu} & =(t, \vec{x})=(t, x, y, z)  \tag{A.3}\\
p^{\mu} & =(E, \vec{p})=\left(E, p_{x}, p_{y}, p_{z}\right)  \tag{A.4}\\
\partial^{\mu} & =\frac{\partial}{\partial x_{\mu}}=\left(\frac{\partial}{\partial t},-\nabla\right)=\left(\frac{\partial}{\partial t},-\frac{\partial}{\partial x},-\frac{\partial}{\partial y},-\frac{\partial}{\partial z}\right)  \tag{A.5}\\
\hat{p}^{\mu} & =i \partial^{\mu}=\left(i \frac{\partial}{\partial t},-i \nabla\right)=\left(i \frac{\partial}{\partial t},-i \frac{\partial}{\partial x},-i \frac{\partial}{\partial y},-i \frac{\partial}{\partial z}\right) . \tag{A.6}
\end{align*}
$$

Covariant vectors $A_{\mu}=g_{\mu \nu} A^{\nu}$ :

$$
\begin{align*}
x_{\mu} & =(t,-\vec{x}),  \tag{A.7}\\
p_{\mu} & =(E,-\vec{p})  \tag{A.8}\\
\partial_{\mu} & =\frac{\partial}{\partial x^{\mu}}=\left(\frac{\partial}{\partial t}, \nabla\right),  \tag{A.9}\\
\hat{p}_{\mu} & =i \partial_{\mu}=\left(i \frac{\partial}{\partial t}, i \nabla\right) . \tag{A.10}
\end{align*}
$$

Note the freedom of rearranging the indices within the scalar product:

$$
\begin{equation*}
p x=p^{\mu} x_{\mu}=p_{\mu} x^{\mu}=E t-\vec{p} \cdot \vec{x} . \tag{A.11}
\end{equation*}
$$

Differentials:

$$
\begin{align*}
& \mathrm{d}^{4} x=\mathrm{d} t \mathrm{~d}^{3} \vec{x}=\mathrm{d} t \mathrm{~d} x \mathrm{~d} y \mathrm{~d} z  \tag{A.12}\\
& \mathrm{~d}^{4} p=\mathrm{d} E \mathrm{~d}^{3} \vec{p}=\mathrm{d} E \mathrm{~d} p_{x} \mathrm{~d} p_{y} \mathrm{~d} p_{z}  \tag{A.13}\\
& \mathrm{~d}^{4} k=\mathrm{d} \omega \mathrm{~d}^{3} \vec{k}=\mathrm{d} \omega \mathrm{~d} k_{x} \mathrm{~d} k_{y} \mathrm{~d} k_{z} \tag{A.14}
\end{align*}
$$

The calculation in the previous chapter can be straightforwardly extended to the $1+3$ spacetime. In particular, equations with the shorthand notations for Fourier transforms can be mostly carried forward without modification-additional spatial dimensions need to be added:

$$
\begin{align*}
\tilde{f}(E) & =\mathcal{T}_{\eta}[f]=\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} t f(t) e^{\eta i E t}  \tag{A.15}\\
\tilde{f}(\vec{p}) & =\mathcal{F}_{\eta}^{3}[f]=\frac{1}{(\sqrt{2 \pi})^{3}} \int \mathrm{~d}^{3} \vec{x} f(\vec{x}) e^{\eta i \vec{p} \cdot \vec{x}}  \tag{A.16}\\
\tilde{f}(p) & =\mathcal{F}_{\eta}^{4}[f]=\mathcal{T}_{\eta} \mathcal{F}_{-\eta}^{3}[f]  \tag{A.17}\\
& =\frac{1}{(\sqrt{2 \pi})^{4}} \int \mathrm{~d}^{4} x f(x) e^{\eta i(E t-\vec{p} \cdot \vec{x})}=\frac{1}{(\sqrt{2 \pi})^{4}} \int \mathrm{~d}^{4} x f(x) e^{\eta i p x} \tag{A.18}
\end{align*}
$$

Now, consider the Fourier transform of $\hat{p}^{\mu}$ :

$$
\begin{align*}
\mathcal{F}_{+}^{4}\left[\hat{p}^{\mu} f\right] & =\frac{1}{(\sqrt{2 \pi})^{4}} \int \mathrm{~d}^{4} x e^{i p x}\left(i \partial^{\mu} f\right)  \tag{A.19}\\
& \text { (integration by parts) } \\
& =\frac{-i}{(\sqrt{2 \pi})^{4}} \int \mathrm{~d}^{4} x \underbrace{\left(\partial^{\mu} e^{i p x}\right)}_{\frac{\partial}{\partial x_{\mu}} e^{i \mu^{\mu} x_{\mu}}=i p^{\mu} e^{i p^{\mu} x_{\mu}}} f  \tag{A.20}\\
& =\frac{1}{(\sqrt{2 \pi})^{4}} \int \mathrm{~d}^{4} x e^{i p x} p^{\mu} f  \tag{A.21}\\
& =p^{\mu} \mathcal{F}_{+}^{4}[f] \tag{A.22}
\end{align*}
$$

So, $\mathcal{F}_{+}^{4}\left(=\mathcal{T}_{+} \mathcal{F}_{-}^{3}\right)$ is the Fourier transform from the position space to the momentum space.

## Bibliography

[AH13] Ian J.R. Aitchison and Anthony J.G. Hey. Gauge Theories in Particle Physics: A Practical Introduction, volume 1: From Relativistic Quantum Mechanics to QED. CRC Press, 4. edition, 2013.
[AJ02] David Atkinson and Porter Wear Johnson. Quantum Field Theory - A Selfcontained Course, volume 2. Rinton Press, 2002.
[Eng] Christoph Englert. Quantum Field Theory. https://conference. ippp.dur.ac.uk/event/1181/attachments/5050/6482/QFTnotes.pdf, accessed 2023-12-24.
[Gel19] François Gelis. Quantum Field Theory-From Basics to Modern Topics. Cambridge University Press, 2019.
[IZ80] Claude Itzykson and Jean-Bernard Zuber. Quantum Field Theory. International Series In Pure and Applied Physics. McGraw-Hill, New York, Dover edition, 1980.
[Sch14] Matthew D. Schwartz. Quantum Field Theory and the Standard Model. Cambridge University Press, 2014.
[Ton] David Tong. Quantum Field Theory. https://www.damtp.cam.ac.uk/ user/tong/qft/qft.pdf, accessed 2023-12-24.
[Zee03] Anthony Zee. Quantum Field Theory in a Nutshell. Princeton University Press, 2003.

## Solutions to Exercises

## Exercise 1

## 1. Fourier Transform-Part 1

(a) Just need to prove the $+i k x$ case.

$$
\begin{aligned}
\delta(x) & =\frac{1}{2 \pi} \int_{-\infty}^{\infty} \mathrm{d} k e^{i k x} \\
& =\frac{1}{2 \pi} \lim _{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \mathrm{d} k e^{-\varepsilon k^{2}+i k x} \\
& =\frac{1}{2 \pi} \lim _{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \mathrm{d} k e^{-\varepsilon\left(k-\frac{i x}{2 \varepsilon}\right)^{2}-\frac{x^{2}}{4 \varepsilon}} \\
& =\frac{1}{2 \pi} \lim _{\varepsilon \rightarrow 0} e^{-\frac{x^{2}}{4 \varepsilon}} \sqrt{\frac{\pi}{\varepsilon}} \\
\int_{-\infty}^{\infty} \mathrm{d} x \delta(x) & =\frac{1}{2 \pi} \lim _{\varepsilon \rightarrow 0} \sqrt{\frac{\pi}{\varepsilon}} \int_{-\infty}^{\infty} \mathrm{d} x e^{-\frac{x^{2}}{4 \varepsilon}} \\
& =\frac{1}{2 \pi} \lim _{\varepsilon \rightarrow 0} \sqrt{\frac{\pi}{\varepsilon}} 2 \sqrt{\varepsilon \pi} \\
& =1 .
\end{aligned}
$$

(b)

$$
\begin{aligned}
f(x) & =\mathcal{F}^{-1}[\mathcal{F}[f]]=A B \int \mathrm{~d} k \mathrm{~d} y f(y) e^{i C k y-i C k x}=\int \mathrm{d} y f(y) \delta(y-x) \\
\delta(y-x) & =A B \int \mathrm{~d} k e^{i C k(y-x)} \\
& =\frac{A B}{|C|} \int \mathrm{d} k^{\prime} e^{i k^{\prime}(y-x)} .
\end{aligned}
$$

Matching the last step to the form of the $\delta$-function.
(c) (direct calculation as in the epigraph.)

## 2. Quantum Harmonic Oscillator

(a)

$$
\left[\hat{H}, \hat{a}^{\dagger}\right]=C\left[\hat{a}^{\dagger} \hat{a}+\hat{a} \hat{a}^{\dagger}, \hat{a}^{\dagger}\right]=2 C\left[\hat{a}, \hat{a}^{\dagger}\right] \hat{a}^{\dagger}
$$

With the D-R

$$
\begin{aligned}
C & =\left(h_{1} \lambda^{2}+h_{2}\right) c_{1} c_{2} \\
& =-2 h_{1} \lambda^{2} c_{1} c_{2}=2 h_{2} c_{1} c_{2} \\
D & =2 C\left[\hat{a}, \hat{a}^{\dagger}\right] \\
& =2 i h_{1} \lambda=-\frac{2 i h_{2}}{\lambda}
\end{aligned}
$$

(b) With the harmonic oscillator Hamiltonian, $h_{1}=\frac{1}{2 m}, h_{2}=\frac{1}{2} m \omega^{2}$,

$$
\begin{aligned}
C & =m \omega^{2} c_{1} c_{2} \\
\lambda^{2} & =-\frac{h_{2}}{h_{1}} \\
& =-m^{2} \omega^{2} \\
\lambda & =\left\{\begin{array}{c}
i m \omega \Rightarrow D=-\omega,\left[\hat{a}, \hat{a}^{\dagger}\right]=-\frac{1}{2 m \omega c_{1} c_{2}} \\
-i m \omega \Rightarrow D=\omega,[\hat{a}, \\
\left.\hat{a}^{\dagger}\right]=\frac{1}{2 m \omega c_{1} c_{2}}
\end{array}\right.
\end{aligned}
$$

So far, the treatment for $a$ and $a^{\dagger}$ is symmetric, hence the two options in the sign of $D$. We can take the + solution.
(c) Furthermore,

$$
\begin{aligned}
{\left[\hat{a}, \hat{a}^{\dagger}\right]=1 } & \Rightarrow c_{1} c_{2}=\frac{1}{2 m \omega} \\
& \Rightarrow C=\frac{\omega}{2}
\end{aligned}
$$

At this point, because $c_{1}^{*}=c_{2}, c_{1,2}$ can be determined up to a phase $e^{ \pm i \alpha}$. Choose real values,

$$
c_{1}=c_{2}=\frac{1}{\sqrt{2 m \omega}}
$$

and the harmonic oscillator case is fully recovered.
(d) For arbitrary operators $A$ and $B,(A B)^{\dagger}=B^{\dagger} A^{\dagger}$.

$$
\begin{aligned}
H^{\dagger} & =H \\
\left(\left[H, a^{\dagger}\right]\right)^{\dagger} & =\left(H a^{\dagger}-a^{\dagger} H\right)^{\dagger} \\
& =a H-H a \\
& =-[H, a] \\
\left(\omega a^{\dagger}\right)^{\dagger} & =\omega a \\
\therefore \quad-[H, a] & =\omega a \\
\text { i.e. }[H, a] & =-\omega a
\end{aligned}
$$

## 3. Fourier Transform—Part 2

(a)

$$
\begin{aligned}
\mathcal{F}_{\eta}\left[s \tilde{t}_{\tau}\right] & =\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} x e^{\eta i k x} s(k) \frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} k^{\prime} e^{\tau i k^{\prime} x} t\left(k^{\prime}\right) \\
& =\frac{1}{2 \pi} \int \mathrm{~d} x \mathrm{~d} k^{\prime} s(k) t\left(k^{\prime}\right) e^{\tau i\left(\frac{\eta}{\tau} k+k^{\prime}\right) x} \\
& =\int \mathrm{d} k^{\prime} s(k) t\left(k^{\prime}\right) \delta\left(\frac{\eta}{\tau} k+k^{\prime}\right) \\
& =s(k) t\left(-\frac{\eta}{\tau} k\right) \\
& =\left\{\begin{array}{l}
s(k) t(-k), \eta=\tau \\
s(k) t(k), \eta \neq \tau
\end{array}\right.
\end{aligned}
$$

(b)

$$
\begin{aligned}
& {[f(x), g(y)]=r \delta(x-y),} \\
& {\left[\tilde{f}_{\eta}(k), \tilde{g}_{\tau}\left(k^{\prime}\right)\right]=\left[\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} x e^{\eta i k x} f(x), \frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} y e^{\tau i k^{\prime} y} g(y)\right]} \\
& =\frac{1}{2 \pi} \int \mathrm{~d} x \mathrm{~d} y e^{\eta i k x+\tau i k^{\prime} y}[f(x), g(y)] \\
& =\frac{r}{2 \pi} \int \mathrm{~d} x \mathrm{~d} y e^{\eta i k x+\tau^{\prime} k^{\prime} y} \delta(x-y) \\
& =\frac{r}{2 \pi} \int \mathrm{~d} x e^{\tau i\left(\frac{\eta}{\tau} k+k^{\prime}\right) x} \\
& =r \delta\left(\frac{\eta}{\tau} k+k^{\prime}\right) \\
& = \begin{cases}r \delta\left(k+k^{\prime}\right), & \eta=\tau \\
r \delta\left(k-k^{\prime}\right), & \eta \neq \tau .\end{cases}
\end{aligned}
$$

(c)

$$
\begin{aligned}
& \int \mathrm{d} x \tilde{s}_{\eta}(x) \tilde{t}_{\tau}(x)=\int \mathrm{d} x \frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} k e^{\eta i k x} s(k) \frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} k^{\prime} e^{\tau i k^{\prime} x} t\left(k^{\prime}\right) \\
& =\frac{1}{2 \pi} \int \mathrm{~d} x \mathrm{~d} k \mathrm{~d} k^{\prime} e^{\tau i\left(\frac{\eta}{\tau} k+k^{\prime}\right) x} s(k) t\left(k^{\prime}\right) \\
& =\int \mathrm{d} k \mathrm{~d} k^{\prime} s(k) t\left(k^{\prime}\right) \delta\left(\frac{\eta}{\tau} k+k^{\prime}\right) \\
& =\int \mathrm{d} k s(k) t\left(-\frac{\eta}{\tau} k\right) \\
& =\left\{\begin{array}{l}
\int \mathrm{d} k s(k) t(-k), \eta=\tau \\
\int \mathrm{d} k s(k) t(k), \eta \neq \tau
\end{array}\right.
\end{aligned}
$$

## Exercise 2

1. Normal Ordering
(directly following the definitions)

## 2. The Vacuum-Part 1

(a)

$$
\begin{aligned}
& \langle 0| a_{k} a_{k^{\prime}}^{\dagger}|0\rangle \\
= & \langle 0| a_{k^{\prime}}^{\dagger} a_{k}+\left[a_{k}, a_{k^{\prime}}^{\dagger}\right]|0\rangle \\
= & {\left[a_{k}, a_{k^{\prime}}^{\dagger}\right]\langle 0 \mid 0\rangle } \\
= & \delta\left(k-k^{\prime}\right)
\end{aligned}
$$

(b) (Ignore the $Z_{k^{\prime}}$-dependence as $Z_{k}=1$.) The scaling between $\left[a_{k}, a_{k^{\prime}}^{\dagger}\right]$ and $c_{1,2}$ is

$$
\left[a_{k}, a_{k^{\prime}}^{\dagger}\right]=\frac{\delta\left(k-k^{\prime}\right)}{2 \omega_{k} c_{1} c_{2}}
$$

With

$$
\begin{aligned}
\phi(x, t) & =\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} k \frac{1}{\sqrt{2 \omega}}\left(a e^{i k x-i \omega t}+a^{\dagger} e^{-i k x+i \omega t}\right) \\
\pi(x, t) & =\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} k \frac{1}{\sqrt{2 \omega}}\left(-i \omega a e^{i k x-i \omega t}+i \omega a^{\dagger} e^{-i k x+i \omega t}\right)
\end{aligned}
$$

we had

$$
\left\langle k \mid k^{\prime}\right\rangle=\left[a_{k}, a_{k^{\prime}}^{\dagger}\right]=\delta\left(k-k^{\prime}\right)
$$

Now with

$$
\begin{aligned}
& \phi(x, t)=\int \frac{\mathrm{d} k}{2 \pi} \frac{1}{2 \omega}\left(a e^{i k x-i \omega t}+a^{\dagger} e^{-i k x+i \omega t}\right) \\
& \pi(x, t)=\int \frac{\mathrm{d} k}{2 \pi} \frac{1}{2 \omega}\left(-i \omega a e^{i k x-i \omega t}+i \omega a^{\dagger} e^{-i k x+i \omega t}\right)
\end{aligned}
$$

we need to scale down $c_{1,2}$ each by a factor of $\sqrt{2 \pi 2 \omega}$ and therefore, now we have

$$
\left\langle k \mid k^{\prime}\right\rangle=\left[a_{k}, a_{k^{\prime}}^{\dagger}\right]=2 \pi 2 \omega_{k} \delta\left(k-k^{\prime}\right)
$$

## 3. Time Ordering

$$
\begin{aligned}
T\left[A\left(t_{1}\right) B\left(t_{2}\right)\right] & =A\left(t_{1}\right) B\left(t_{2}\right) \theta\left(t_{1}-t_{2}\right)+B\left(t_{2}\right) A\left(t_{1}\right) \theta\left(t_{2}-t_{1}\right), \\
\partial_{t_{1}} T\left[A\left(t_{1}\right) B\left(t_{2}\right)\right] & =\left[\partial_{1} A\left(t_{1}\right)\right] B\left(t_{2}\right) \theta\left(t_{1}-t_{2}\right)+A\left(t_{1}\right) B\left(t_{2}\right) \delta\left(t_{1}-t_{2}\right) \\
& +B\left(t_{2}\right)\left[\partial_{t 1} A\left(t_{1}\right)\right] \theta\left(t_{2}-t_{1}\right)-B\left(t_{2}\right) A\left(t_{1}\right) \delta\left(t_{2}-t_{1}\right) \\
& =T\left[\partial_{t 1} A\left(t_{1}\right) B\left(t_{2}\right)\right]+\left[A\left(t_{1}\right), B\left(t_{2}\right)\right] \delta\left(t_{1}-t_{2}\right) .
\end{aligned}
$$

## 4. Feynman Propagator

(a)

$$
\langle 0| \phi|0\rangle=\mathcal{K}_{+}[c\langle 0| a|0\rangle]+\mathcal{K}_{-}\left[c^{*}\langle 0| a^{\dagger}|0\rangle\right]=0
$$

as $a|0\rangle=0$ and $\langle 0| a^{\dagger}=0$.
(b) In calculating $\langle 0| a^{(\dagger)} a^{(\dagger)}|0\rangle$, only the $a a^{\dagger}$ term survives.

$$
\begin{aligned}
& \langle 0| \phi\left(x_{1}, t_{1}\right) \phi\left(x_{2}, t_{2}\right)|0\rangle=\langle 0| \mathcal{K}_{+}\left[c_{1} a_{1}\right] \mathcal{K}_{-}\left[c_{2}^{\dagger} a_{2}^{\dagger}\right]|0\rangle \\
= & \frac{1}{2 \pi} \int \mathrm{~d} k_{1} \mathrm{~d} k_{2} \frac{1}{\sqrt{2 \omega_{1}}} \frac{1}{\sqrt{2 \omega_{2}}} e^{i k_{1} x_{1}-i \omega_{1} t_{1}} e^{-i k_{2} x_{2}+i \omega_{2} t_{2}} \underbrace{\langle 0| a_{1} a_{2}^{\dagger}|0\rangle}_{=\delta\left(k_{1}-k_{2}\right)} \\
= & \int \frac{\mathrm{d} k}{2 \pi} \frac{1}{2 \omega_{k}} e^{i k\left(x_{1}-x_{2}\right)-i \omega_{k}\left(t_{1}-t_{2}\right)} .
\end{aligned}
$$

(c)

$$
\begin{align*}
& \langle 0| \phi_{1} \phi_{2}|0\rangle \theta(t)=\int \frac{\mathrm{d} k}{2 \pi} \frac{1}{2 \omega_{k}} e^{i k x-i \omega_{k} t} \theta(t) \\
& =\int \frac{\mathrm{d} k}{2 \pi} \frac{1}{2 \omega_{k}} \frac{i}{2 \pi} \int \mathrm{~d} \omega \frac{e^{i k x-i\left(\omega+\omega_{k}\right) t}}{\omega+i \epsilon} \\
& \left(\omega_{0} \equiv \omega+\omega_{k}\right) \\
& =\int \frac{\mathrm{d} k}{2 \pi} \frac{1}{2 \omega_{k}} \frac{i}{2 \pi} \int \mathrm{~d} \omega_{0} \frac{e^{i k x-i \omega_{0} t}}{\omega_{0}-\omega_{k}+i \epsilon}  \tag{*~A}\\
& \left(k^{\prime} \equiv-k, \omega^{\prime} \equiv-\omega_{0}, \omega_{k^{\prime}}^{2}=k^{\prime 2}+m^{2}=\omega_{k}^{2}\right)
\end{align*}
$$

(the two flips of integral limits cancel each other)

$$
=\int \frac{-\mathrm{d} k^{\prime}}{2 \pi} \frac{1}{2 \omega_{k}} \frac{i}{2 \pi} \int\left(-\mathrm{d} \omega^{\prime}\right) \frac{e^{-i k^{\prime} x+i \omega^{\prime} t}}{-\omega^{\prime}-\omega_{k}+i \epsilon}
$$

$$
=\int \frac{\mathrm{d} k^{\prime}}{2 \pi} \frac{1}{2 \omega_{k}} \frac{i}{2 \pi} \int \mathrm{~d} \omega^{\prime}\left(-\frac{1}{\omega^{\prime}+\omega_{k}-i \epsilon}\right) e^{-i k^{\prime} x+i \omega^{\prime} t}
$$

(restoring dummy variables)

$$
\begin{equation*}
=\int \frac{\mathrm{d} k}{2 \pi} \frac{1}{2 \omega_{k}} \frac{i}{2 \pi} \int \mathrm{~d} \omega\left(-\frac{1}{\omega+\omega_{k}-i \epsilon}\right) e^{-i k x+i \omega t} \tag{*B}
\end{equation*}
$$

(d) Still using $x=x_{1}-x_{2}, t=t_{1}-t_{2}$, and following ( ${ }^{*} \mathrm{~A}$ ) in the previous question but now flip the signs of $x$ and $t$,

$$
\begin{gather*}
\langle 0| \phi_{2} \phi_{1}|0\rangle \theta\left(t_{2}-t_{1}\right)=\int \frac{\mathrm{d} k}{2 \pi} \frac{1}{2 \omega_{k}} e^{-i k x+i \omega_{k} t} \theta(-t), \\
\quad=\int \frac{\mathrm{d} k}{2 \pi} \frac{1}{2 \omega_{k}} \frac{i}{2 \pi} \int \mathrm{~d} \omega \frac{1}{\omega-\omega_{k}+i \epsilon} e^{-i k x+i \omega t} \tag{}
\end{gather*}
$$

Adding (*B) and (*)

$$
\begin{aligned}
& \langle 0| T \phi_{1} \phi_{2}|0\rangle=\int \frac{\mathrm{d} k}{2 \pi} \frac{1}{2 \omega_{k}} \frac{i}{2 \pi} \int \mathrm{~d} \omega \frac{2\left(\omega_{k}-i \epsilon\right)}{\omega^{2}-\left(\omega_{k}-i \epsilon\right)^{2}} e^{-i k x+i \omega t} \\
& \left(\varepsilon \equiv 2 \omega_{k} \epsilon\right) \\
& =i \int \frac{\mathrm{~d} k \mathrm{~d} \omega}{(2 \pi)^{2}} \frac{e^{-i k x+i \omega t}}{\omega^{2}-\omega_{k}^{2}+i \varepsilon} \\
& =i \int \frac{\mathrm{~d} k \mathrm{~d} \omega}{(2 \pi)^{2}} \frac{e^{i k x-i \omega t}}{\omega^{2}-\omega_{k}^{2}+i \varepsilon}
\end{aligned}
$$

where the last step has flip the signs of $k$ and $\omega$ and therefore proved $\Delta_{\mathrm{F}}(x, t)=\Delta_{\mathrm{F}}(-x,-t)$.

## 5. The Vacuum-Part 2

$$
\begin{aligned}
& : H: a_{2}^{\dagger} a_{1}^{\dagger}|0\rangle \\
= & \left(a_{2}^{\dagger}: H:+\omega_{k_{2}} a_{2}^{\dagger}\right) a_{1}^{\dagger}|0\rangle \\
= & a_{2}^{\dagger}: H: a_{1}^{\dagger}|0\rangle+\omega_{2} a_{2}^{\dagger} a_{1}^{\dagger}|0\rangle \\
= & a_{2}^{\dagger} \omega_{1} a_{1}^{\dagger}|0\rangle+\omega_{2} a_{2}^{\dagger} a_{1}^{\dagger}|0\rangle \\
= & \left(\omega_{1}+\omega_{2}\right) a_{2}^{\dagger} a_{1}^{\dagger}|0\rangle .
\end{aligned}
$$

## Exercise 3

## 1. Energy-Momentum Tensor

(a) $(i=x, y, z$ is not Lorentz index.)

$$
\begin{aligned}
& \pi \partial_{i} \phi=\mathcal{K}_{+}[-i \omega c a] \mathcal{K}_{+}\left[i k_{i} c a\right] \\
&+\mathcal{K}_{+}[-i \omega c a] \mathcal{K}_{-}\left[-i k_{i} c^{*} a^{\dagger}\right] \\
&+\mathcal{K}_{-}\left[i \omega c^{*} a^{\dagger}\right] \mathcal{K}_{+}\left[i k_{i} c a\right] \\
&+\mathcal{K}_{-}\left[i \omega c^{*} a^{\dagger}\right] \mathcal{K}_{-}\left[-i k_{i} c^{*} a^{\dagger}\right] \\
& \int \mathrm{d}^{3} \vec{x} \pi \partial_{i} \phi=\int \mathrm{d}^{3} \vec{k} {\left[-i \omega_{k} c_{k} a_{k} i\left(-k_{i}\right) c_{-k} a_{-k}\right.} \\
&+(-i) \omega_{k} c_{k} a_{k}(-i) k_{i} c_{k}^{*} a_{k}^{\dagger} \\
&+i \omega_{k} c_{k}^{*} a_{k}^{\dagger} i k_{i} c_{k} a_{k} \\
&\left.+i \omega_{k} c_{k}^{*} a_{k}^{\dagger}(-i)\left(-k_{i}\right) c_{-k}^{*} a_{-k}^{\dagger}\right]
\end{aligned}
$$

The first and fourth integrands are odd in $k_{i}$ and so the integrals go to 0 when integrating from $-\infty$ to $\infty$. With the remaining terms, we have

$$
\begin{aligned}
\int \mathrm{d}^{3} \vec{x} \pi \partial_{i} \phi & =-\int \mathrm{d}^{3} \vec{k} \omega k_{i} c c^{*}\left(a^{\dagger} a+a a^{\dagger}\right) \\
& =-\int \mathrm{d}^{3} \vec{k} \frac{k_{i}}{2}\left(a^{\dagger} a+a a^{\dagger}\right)
\end{aligned}
$$

Finally, for $\mu=1,2,3$,

$$
\begin{aligned}
P^{\mu} & =\int \mathrm{d}^{3} \vec{x} \pi \partial^{\mu} \phi \\
& =-\int \mathrm{d}^{3} \vec{x} \pi \partial_{i} \phi \\
& =\int \mathrm{d}^{3} \vec{k} \frac{k_{i}}{2}\left(a^{\dagger} a+a a^{\dagger}\right) \\
& =\int \mathrm{d}^{3} \vec{k} \frac{k^{\mu}}{2}\left(a^{\dagger} a+a a^{\dagger}\right)
\end{aligned}
$$

(b) For $\mu=0$,

$$
\begin{aligned}
{\left[P^{0}, \phi(y)\right] } & =[H, \phi(y)] \\
& =-i \dot{\phi}(y) \\
& =-i \partial^{0} \phi(y)
\end{aligned}
$$

For $\mu=1,2,3$,

$$
\begin{aligned}
{\left[P^{\mu}, \phi(y)\right] } & =\int \mathrm{d}^{3} \vec{x}\left[\pi \partial^{\mu} \phi, \phi(y)\right] \\
& \left(\left[\partial_{i} \phi(x), \phi(y)\right] \sim \int \mathrm{d}^{3} \vec{k} k_{i}=0\right) \\
& =\int \mathrm{d}^{3} \vec{x}[\pi, \phi(y)] \partial^{\mu} \phi \\
& =-\int \mathrm{d}^{3} \vec{x} i \delta^{(3)}(x-y) \partial^{\mu} \phi \\
& =-i \partial^{\mu} \phi(y)
\end{aligned}
$$

## 2. Complex Scalar Fields

(a) Both parts are routine - just note that,

$$
\begin{aligned}
& \quad\langle\vec{k} \mid \vec{k}\rangle_{a}=0, \quad\langle\vec{k} \mid \vec{k}\rangle_{a}={ }_{b}\langle\vec{k} \mid \vec{k}\rangle_{b} . \\
& a=\frac{a_{1}-i a_{2}}{\sqrt{2}}, \quad a^{\dagger}=\frac{a_{1}^{\dagger}+i a_{2}^{\dagger}}{\sqrt{2}}, \quad b=\frac{a_{1}+i a_{2}}{\sqrt{2}}, \quad b^{\dagger}=\frac{a_{1}^{\dagger}-i a_{2}^{\dagger}}{\sqrt{2}}, \\
& a^{\dagger} a=\frac{a_{1}^{\dagger} a_{1}+a_{2}^{\dagger} a_{2}-i a_{1}^{\dagger} a_{2}+i a_{2}^{\dagger} a_{1}}{2}, \quad b b^{\dagger}=\frac{a_{1} a_{1}^{\dagger}+a_{2} a_{2}^{\dagger}-i a_{1} a_{2}^{\dagger}+i a_{2} a_{1}^{\dagger}}{2}, \\
& \therefore a^{\dagger} a+b b^{\dagger} \\
& \left(\left[a_{1}^{(\dagger)}, a_{2}^{(\dagger)}\right]=0\right) \\
& =\frac{a_{1}^{\dagger} a_{1}+a_{1} a_{1}^{\dagger}}{2}+\frac{a_{2}^{\dagger} a_{2}+a_{2} a_{2}^{\dagger}}{2} .
\end{aligned}
$$

(b) For complex scalars,

$$
\begin{aligned}
\langle 0| T \phi \phi|0\rangle & \sim \mathcal{K}\left[\langle 0|\left(a+b^{\dagger}\right)\left(a+b^{\dagger}\right)|0\rangle\right], \\
\langle 0| T \phi^{\dagger} \phi^{\dagger}|0\rangle & \sim \mathcal{K}\left[\langle 0|\left(b+a^{\dagger}\right)\left(b+a^{\dagger}\right)|0\rangle\right], \\
\langle 0| \phi^{\dagger} \phi|0\rangle & \sim \mathcal{K}\left[\langle 0|\left(b+a^{\dagger}\right)\left(a+b^{\dagger}\right)|0\rangle\right], \\
\langle 0| \phi \phi^{\dagger}|0\rangle & \sim \mathcal{K}\left[\langle 0|\left(a+b^{\dagger}\right)\left(b+a^{\dagger}\right)|0\rangle\right],
\end{aligned}
$$

where only $a a^{\dagger}$ and $b b^{\dagger}$ survive, eacho contributing a $\delta$ just like the case in real scalar field. So, $\langle 0| T \phi \phi^{\dagger}|0\rangle$ equals to the real scalar case $\langle 0| T \phi \phi|0\rangle=\Delta_{\mathrm{F}}$.
(c) Field expansion:

$$
\begin{aligned}
\phi & =\mathcal{K}_{+}^{3}[c a]+\mathcal{K}_{-}^{3}\left[c^{*} b^{\dagger}\right] \\
\pi & =\dot{\phi}^{\dagger}=\mathcal{K}_{+}^{3}[-i \omega c b]+\mathcal{K}_{-}^{3}\left[i \omega c^{*} a^{\dagger}\right]
\end{aligned}
$$

So,

$$
\begin{aligned}
\int \mathrm{d}^{3} \vec{x} \pi \phi=\int \mathrm{d}^{3} \vec{x} & \left(\mathcal{K}_{+}^{3}[-i \omega c b] \mathcal{K}_{+}^{3}[c a]\right. \\
& +\mathcal{K}_{+}^{3}[-i \omega c b] \mathcal{K}_{-}^{3}\left[c^{*} b^{\dagger}\right] \\
& +\mathcal{K}_{-}^{3}\left[i \omega c^{*} a^{\dagger}\right] \mathcal{K}_{+}^{3}[c a] \\
& \left.+\mathcal{K}_{-}^{3}\left[i \omega c^{*} a^{\dagger}\right] \mathcal{K}_{-}^{3}\left[c^{*} b^{\dagger}\right]\right) \\
=\int \mathrm{d}^{3} \vec{k}( & -i \omega c c b_{\vec{k}} a_{-\vec{k}} \\
& -i \omega c c^{*} b b^{\dagger} \\
& +i \omega c c^{*} a^{\dagger} a \\
& \left.+i \omega c^{*} c^{*} a_{\vec{k}}^{\dagger} b_{-\vec{k}}^{\dagger}\right) \\
\text { h.c. }=\int \mathrm{d}^{3} \vec{k} & \left(i \omega c^{*} c^{*} a_{-\vec{k}}^{\dagger} b_{\vec{k}}^{\dagger}\right. \\
& +i \omega c c^{*} b b^{\dagger} \\
& -i \omega c c^{*} a^{\dagger} a \\
& \left.-i \omega c c b_{-\vec{k}} a_{\vec{k}}\right)
\end{aligned}
$$

Because

$$
\int \mathrm{d}^{3} \vec{k} a_{\vec{k}} b_{-\vec{k}}=\int \mathrm{d}^{3} \vec{k} a_{-\vec{k}} b_{\vec{k}}
$$

the first and fourth terms in the integral cancel when we evaluate

$$
\begin{aligned}
\int \mathrm{d}^{3} \vec{x} \pi \phi-\text { h.c. } & =\int \mathrm{d}^{3} \vec{k}\left(-2 i \omega c c^{*} b b^{\dagger}+2 i \omega c c^{*} a^{\dagger} a\right) \\
& =\int \mathrm{d}^{3} \vec{k} 2 i \omega c c^{*}\left(a^{\dagger} a-b b^{\dagger}\right) \\
& =i \int \mathrm{~d}^{3} \vec{k}\left(a^{\dagger} a-b b^{\dagger}\right)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
Q & =\alpha \int \mathrm{d}^{3} \vec{k}\left(a^{\dagger} a-b b^{\dagger}\right) \\
: Q: & =\alpha \int \mathrm{d}^{3} \vec{k}\left(a^{\dagger} a-b^{\dagger} b\right)
\end{aligned}
$$

## 3. Noether's Theorem

(a)

$$
\begin{aligned}
{[Q, \phi] } & =-i \alpha \int \mathrm{~d}^{3} \vec{x}[\pi \phi, \phi]=-i \alpha \int \mathrm{~d}^{3} \vec{x}[\pi, \phi] \phi=-\alpha \phi \\
{\left[Q, \phi^{\dagger}\right] } & =-i \alpha \int \mathrm{~d}^{3} \vec{x}\left[-\phi^{\dagger} \pi^{\dagger}, \phi^{\dagger}\right]=-i \alpha \int \mathrm{~d}^{3} \vec{x} \phi^{\dagger}\left[\phi^{\dagger}, \pi^{\dagger}\right]=\alpha \phi^{\dagger}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{\mathrm{d} \psi_{1}}{\mathrm{~d} \alpha} & =Q_{0} \psi_{1}-\psi_{1} Q_{0} & & \left.\frac{\mathrm{~d} \psi_{1}}{\mathrm{~d} \alpha}\right|_{\alpha=0}=\left[Q_{0}, \phi\right]=-\phi ; \\
& =\left[Q_{0}, \psi_{1}\right], & & \left.\frac{\mathrm{d}^{2} \psi_{1}}{\mathrm{~d} \alpha^{2}}\right|_{\alpha=0}=\left[Q_{0},-\phi\right]=\phi ; \\
\frac{\mathrm{d}^{2} \psi_{1}}{\mathrm{~d} \alpha^{2}} & =\left[Q_{0}, \frac{\mathrm{~d} \psi_{1}}{\mathrm{~d} \alpha}\right] & & \\
& =\left[Q_{0},\left[Q_{0}, \psi_{1}\right]\right], & \ldots & \\
\frac{\mathrm{d} \psi_{2}}{\mathrm{~d} \alpha} & =-\psi_{2}, & & \left.\frac{\mathrm{~d} \psi_{2}}{\mathrm{~d} \alpha}\right|_{\alpha=0}=-\phi ; \\
\frac{\mathrm{d}^{2} \psi_{2}}{\mathrm{~d} \alpha^{2}} & =-\frac{\mathrm{d} \psi_{2}}{\mathrm{~d} \alpha}, & & \left.\frac{\mathrm{~d}^{2} \psi_{2}}{\mathrm{~d} \alpha^{2}}\right|_{\alpha=0}=\phi ; \\
\therefore \psi_{1} & =\psi_{2} . & \ldots &
\end{aligned}
$$

## Exercise 4

## 1. Spinor Fields

(a)

$$
\begin{aligned}
(i \gamma \partial-m) \psi & =0 \\
0=[(i \gamma \partial-m) \psi]^{\dagger} & =\psi^{\dagger}\left(-i \gamma^{\dagger} \overleftarrow{\partial}-m\right) \\
& \left(\gamma^{0} \gamma^{0}=1, \gamma^{\dagger}=\gamma^{0} \gamma \gamma^{0}\right) \\
& =\psi^{\dagger} \gamma^{0} \gamma^{0}\left(-i \gamma^{0} \gamma \gamma^{0} \overleftarrow{\partial}-m\right) \\
& =\bar{\psi}\left(-i \gamma \gamma^{0} \overleftarrow{\partial}-\gamma^{0} m\right) \\
& =\bar{\psi}(-i \gamma \overleftarrow{\partial}-m) \gamma^{0}, \\
\therefore \bar{\psi}(i \gamma \overleftarrow{\partial}+m) & =0 .
\end{aligned}
$$

(b) Swap indices and average, then use

$$
\gamma^{\mu} \gamma^{\nu}+\gamma^{\nu} \gamma^{\mu}=2 g^{\mu \nu}
$$

(c) Use

$$
\gamma^{\mu} \gamma^{0}=2 g^{\mu 0}-\gamma^{0} \gamma^{\mu}
$$

(d) i. $\bar{\psi}: 1 \times 4$
ii. $\gamma^{\mu}: 4 \times 1$
iii. the Lagrangian, $\mathcal{L}: 1 \times 1$
iv. the Hamiltonian density, $\mathcal{H}: 1 \times 1$
v. $\bar{u} u: 1 \times 1$
vi. $u \bar{u}: 4 \times 4$
vii. the current $J^{\mu}=\bar{\psi} \gamma^{\mu} \psi: 1 \times 1$
(e) i.

$$
\begin{aligned}
& \sum_{s} u_{s}(0) u_{s}^{\dagger}(0)\left(\gamma^{\dagger} p+m\right) \gamma^{0} \\
= & \sum_{s} u_{s}(0) u_{s}^{\dagger}(0) \gamma^{0} \gamma^{0}\left(\gamma^{\dagger} p+m\right) \gamma^{0} \\
& \left(\sum_{s} u_{s}(0) u_{s}^{\dagger}(0)=\frac{\gamma^{0}+1}{2}, \gamma^{\mu \dagger}=\gamma^{0} \gamma^{\mu} \gamma^{0}, \gamma^{0} \gamma^{0}=1\right) \\
= & \frac{\gamma^{0}+1}{2}(\gamma p+m)
\end{aligned}
$$

ii. Follow the examples in the footnotes:

$$
\begin{aligned}
& v_{s}^{\dagger}(p) v_{s^{\prime}}(p) \\
& =v_{s}^{\dagger}(0)\left(\gamma^{\dagger} p-m\right) N^{2}(\gamma p-m) v_{s^{\prime}}(0) \\
& =\bar{v}_{s}(0)(\gamma p-m) \gamma^{0}(\gamma p-m) v_{s^{\prime}}(0) N^{2} \\
& =\bar{v}_{s}(0)(\gamma p-m) v_{s^{\prime}}(0) N^{2} 2 p^{0} \\
& =-v_{s}^{\dagger}(0)(\gamma p-m) v_{s^{\prime}}(0) N^{2} 2 p^{0} \\
& =-\delta_{s s^{\prime}}\left(-p^{0}-m\right) N^{2} 2 p^{0} \\
& =\delta_{s s^{\prime}}\left(m+p^{0}\right) N^{2} 2 p^{0} . \\
& \\
& \bar{u}_{s}(p) u_{s^{\prime}}(p) \\
& =u_{s}^{\dagger}(0)\left(\gamma^{\dagger} p+m\right) \gamma^{0} N^{2}(\gamma p+m) u_{s^{\prime}}(0) \\
& =\bar{u}_{s}(0)(\gamma p+m)(\gamma p+m) u_{s^{\prime}}(0) N^{2} \\
& =\bar{u}_{s}(0)(\gamma p+m) u_{s^{\prime}}(0) N^{2} 2 m \\
& =u_{s}^{\dagger}(0)(\gamma p+m) u_{s^{\prime}}(0) N^{2} 2 m \\
& =\delta_{s s^{\prime}}\left(m+p^{0}\right) N^{2} 2 m . \\
& \bar{v}_{s}(p) v_{s^{\prime}}(p) \\
& =v_{s}^{\dagger}(0)\left(\gamma^{\dagger} p-m\right) \gamma^{0} N^{2}(\gamma p-m) v_{s^{\prime}}(0) \\
& =\bar{v}_{s}(0)(\gamma p-m)(\gamma p-m) v_{s^{\prime}}(0) N^{2} \\
& =-\bar{v}_{s}(0)(\gamma p-m) v_{s^{\prime}}(0) N^{2} 2 m \\
& =v_{s}^{\dagger}(0)(\gamma p-m) v_{s^{\prime}}(0) N^{2} 2 m \\
& =\delta_{s s^{\prime}}\left(-p^{0}-m\right) N^{2} 2 m \\
& =-\delta_{s s^{\prime}}\left(m+p^{0}\right) N^{2} 2 m . \\
& =v_{s}(p) \bar{v}_{s}(p) \\
& =N^{2}(\gamma p-m) \sum v_{s}(0) v_{s}^{\dagger}(0)\left(\gamma^{\dagger} p-m\right) \gamma^{0} \\
& =N^{2}(\gamma p-m) \frac{\gamma^{0}-1}{2}(\gamma p-m) \\
& =N^{2} 2 p^{0}-\left(-2 m^{0}\right) \\
& =N^{2}\left(r p+p^{0}\right)(\gamma p-m) .
\end{aligned}
$$

(f)

$$
\begin{aligned}
& Q=\alpha \int \mathrm{d}^{3} \vec{x} \psi^{\dagger} \psi \\
= & \alpha \int \mathrm{d}^{3} \vec{x}\left(\mathcal{K}_{+}^{3}\left[c \sum_{s} b_{s} v_{s}^{\dagger}\right] \mathcal{K}_{-}^{3}\left[c^{*} \sum_{s} b_{s}^{\dagger} v_{s}\right]\right. \\
& \left.+\mathcal{K}_{-}^{3}\left[c^{*} \sum_{s} a_{s}^{\dagger} u_{s}^{\dagger}\right] \mathcal{K}_{+}^{3}\left[c \sum_{s} a_{s} u_{s}\right]\right) \\
= & \alpha \int \mathrm{d}^{3} \vec{k}\left(\frac{1}{2 \omega} \sum_{s s^{\prime}} v_{s}^{\dagger} v_{s^{\prime}} b_{s} b_{s^{\prime}}^{\dagger}+\frac{1}{2 \omega} \sum_{s s^{\prime}} u_{s}^{\dagger} u_{s^{\prime}} a_{s}^{\dagger} a_{s^{\prime}}\right) \\
= & \alpha \int \mathrm{d}^{3} \vec{k} \sum_{s}\left(a_{s}^{\dagger} a_{s}+b_{s} b_{s}^{\dagger}\right) .
\end{aligned}
$$

## 2. Photons

(a)

$$
\begin{aligned}
& \frac{\partial \mathcal{L}}{\partial\left(\partial_{\alpha} A_{\beta}\right)}=-F^{\alpha \beta}-\frac{1}{\xi} \delta^{\alpha \beta} \partial_{\mu} A^{\mu} \\
& 0=\partial_{\alpha} \frac{\partial \mathcal{L}}{\partial\left(\partial_{\alpha} A_{\beta}\right)}=-\partial_{\alpha} F^{\alpha \beta}-\frac{1}{\xi} \partial^{\beta} \partial_{\mu} A^{\mu} \\
&=-\partial_{\alpha}\left(\partial^{\alpha} A^{\beta}-\partial^{\beta} A^{\alpha}\right)-\frac{1}{\xi} \partial^{\beta} \partial_{\mu} A^{\mu} \\
&=-\partial^{2} A^{\beta}+\partial^{\beta} \partial_{\alpha} A^{\alpha}-\frac{1}{\xi} \partial^{\beta} \partial_{\mu} A^{\mu} \\
&=-\partial^{2} A^{\beta}-\left(\frac{1}{\xi}-1\right) \partial^{\beta} \partial_{\alpha} A^{\alpha} \\
&\left(A^{\beta}=g_{\alpha}^{\beta} A^{\alpha}\right) \\
&=\left[-\partial^{2} g_{\alpha}^{\beta}-\left(\frac{1}{\xi}-1\right) \partial^{\beta} \partial_{\alpha}\right] A^{\alpha} .
\end{aligned}
$$

(b)

$$
\begin{aligned}
& D_{\alpha}^{\mu} \equiv-\partial^{2} g_{\alpha}^{\mu}-\left(\frac{1}{\xi}-1\right) \partial^{\mu} \partial_{\alpha} \\
& D_{\alpha}^{\mu} e^{-i k(x-y)} \\
& =\left[k^{2} g_{\alpha}^{\mu}+\left(\frac{1}{\xi}-1\right) k^{\mu} k_{\alpha}\right] e^{-i k(x-y)}, \\
& k^{2}\left[g_{\alpha}^{\mu}+\left(\frac{1}{\xi}-1\right) \frac{k^{\mu} k_{\alpha}}{k^{2}}\right] \cdot \frac{g^{\alpha \nu}-(1-\xi) \frac{k^{\alpha} k^{\nu}}{k^{2}}}{k^{2}} \\
& =g^{\mu \nu}+\left(\frac{1}{\xi}-1\right) \frac{k^{\mu} k^{\nu}}{k^{2}}-(1-\xi) \frac{k^{\mu} k^{v}}{k^{2}}-\left(\frac{1}{\xi}-1\right)(1-\xi) \frac{k^{\mu} k^{\nu}}{k^{2}} \\
& =g^{\mu \nu}+\underbrace{\left[\frac{1}{\xi}-1-1+\xi-\left(\frac{1}{\xi}-1-1+\xi\right)\right]}_{=0} \frac{k^{\mu} k^{\nu}}{k^{2}} \\
& =g^{\mu \nu} \\
& D_{\alpha}^{\mu} \Pi_{\mathrm{F}}^{\alpha \nu}=-i g^{\mu \nu} \delta^{4}(x-y)
\end{aligned}
$$

## 3. Local Gauge Principle

$$
\begin{aligned}
\mathcal{L} & =\bar{\psi}(i \gamma \partial-m) \psi-e A \bar{\psi} \gamma \psi, \\
\frac{\partial \mathcal{L}}{\partial \bar{\psi}} & =(i \gamma \partial-m) \psi-e A \gamma \psi=0,
\end{aligned}
$$

$$
\text { E-L: }(i \gamma \partial-m) \psi=e A \gamma \psi .
$$


[^0]:    ${ }^{1}$ You might have already guessed that this is some sort of field's mass. However, this information is irrelevant for the calculation; we can proceed with the analysis without explicitly considering its nature.

[^1]:    ${ }^{2}$ Caveat on pulling $T$ through $\partial_{t}$, cf. e.g. Schwartz (2014) p. 72 Sch14], Aitchison and Hey (2013) p. 210 AH13], and Itzykson and Zuber (1980) Section 6.1.4 (p. 284, Dover edition) [IZ80].

[^2]:    ${ }^{1}$ If the term "density" is clear in the context and its omission does not lead to confusion, we will drop it for brevity.

[^3]:    ${ }^{2}$ One can prove the second line with Eq. 2.126.

[^4]:    ${ }^{5}$ For an arbitrary $4 \times 4$ matrix $M, u_{s}^{\dagger}(0) M u_{s^{\prime}}(0)=M_{s s^{\prime}}, v_{s}^{\dagger}(0) M v_{s^{\prime}}(0)=M_{2+s, 2+s^{\prime}}$. This is particularly useful when $M$ is the $\gamma$-matrix: $u_{s}^{\dagger}(0) \gamma x u_{s^{\prime}}(0)=u_{s}^{\dagger}(0) \gamma^{0} x_{0} u_{s^{\prime}}(0)=\delta_{s s^{\prime}} x_{0}$, $v_{s}^{\dagger}(0) \gamma x v_{s^{\prime}}(0)=v_{s}^{\dagger}(0) \gamma^{0} x_{0} v_{s^{\prime}}(0)=-\delta_{s, s^{\prime}} x_{0}$, with an arbitrary 4 -vector $x$. Further more,

    $$
    \sum_{s=1,2} u_{s}(0) u_{s}^{\dagger}(0)=\left(\begin{array}{cc}
    I_{2} & 0 \\
    0 & 0
    \end{array}\right)=\frac{\gamma^{0}+1}{2}, \quad \sum_{s=1,2} v_{s}(0) v_{s}^{\dagger}(0)=\left(\begin{array}{cc}
    0 & 0 \\
    0 & I_{2}
    \end{array}\right)=\frac{1-\gamma^{0}}{2} .
    $$

[^5]:    ${ }^{8}$ It should be understood that here $\psi$ and $\pi$ are broken down into their components, that is, $\psi$ and $\pi$ carry some spinor indices $\alpha$ and $\beta$, respectively, which then propagate throughout.

    $$
    \begin{aligned}
    & \{\psi(\vec{x}, t), \pi(\vec{y}, t)\} \\
    & =\left\{\mathcal{K}_{+}^{3}\left[c \sum_{s} a_{s} u_{s}\right], \mathcal{K}_{-}^{3}\left[i c^{*} \sum_{s} a_{s}^{\dagger} u_{s}^{\dagger}\right]\right\}+\left\{\mathcal{K}_{-}^{3}\left[c^{*} \sum_{s} b_{s}^{\dagger} v_{s}\right], \mathcal{K}_{+}^{3}\left[i c \sum_{s} b_{s} v_{s}^{\dagger}\right]\right\} \\
    & =\frac{i}{(2 \pi)^{3}} \int \mathrm{~d}^{3} \vec{k} \mathrm{~d}^{3} \vec{k}^{\prime} \frac{1}{\sqrt{2 \omega_{k}}} \frac{1}{\sqrt{2 \omega_{k^{\prime}}}} e^{i \vec{k} \cdot \vec{x}-i \omega_{k} t} e^{-i \vec{k}^{\prime} \cdot \vec{y}+i \omega_{k^{\prime}} t} \\
    & \times \underbrace{\sum_{N^{2}\left(m+w_{k}\right)(\gamma k+m) \gamma^{0}} u_{s} u_{s^{\prime}}^{\dagger}}_{\left(\text {if } s=s^{\prime}\right)} \underbrace{\left\{a_{s}, a_{s^{\prime}}^{\dagger}\right\}}_{\delta_{s, s^{\prime}} \delta^{3}\left(\vec{k}-\vec{k}^{\prime}\right)}+\int \cdots \\
    & =\frac{i}{(2 \pi)^{3}} \int \mathrm{~d}^{3} \vec{k} \frac{1}{2 \omega_{k}} e^{i \vec{k} \cdot(\vec{x}-\vec{y})} N^{2}\left(m+\omega_{k}\right)(\gamma k+m) \gamma^{0}+\int \ldots \\
    & =\frac{i}{(2 \pi)^{3}} \int \mathrm{~d}^{3} \vec{k} \frac{N^{2}\left(m+\omega_{k}\right)}{2 \omega_{k}}\left[e^{i \vec{k} \cdot(\vec{x}-\vec{y})}(\gamma k+m)+e^{-i \vec{k} \cdot(\vec{x}-\vec{y})}(\gamma k-m)\right] \gamma^{0} \\
    & (\gamma k \pm m=\underbrace{\gamma^{0} \omega_{k}}_{\text {even }}-\underbrace{\gamma^{i} k^{i}}_{\text {odd }} \pm \underbrace{m}_{\text {even }}) \\
    & =\frac{i}{(2 \pi)^{3}} \int \mathrm{~d}^{3} \vec{k} \frac{N^{2}\left(m+\omega_{k}\right)}{2 \omega_{k}}\left[e^{i \vec{k} \cdot(\vec{x}-\vec{y})}+e^{-i \vec{k} \cdot(\vec{x}-\vec{y})}\right] \omega_{k} \text {. }
    \end{aligned}
    $$

[^6]:    ${ }^{10}$ Not Lorentz!

[^7]:    ${ }^{11}$ Also known as minimal substitution.

