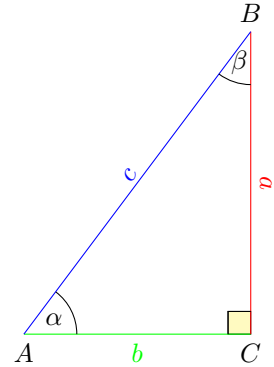


1 Introduction: Right-Angled Triangles

The most common definitions of trigonometric functions is done via a right-angled triangle (from which the functions take their name). Consider the triangle ABC . We take the angle α to be the angle of interest. In that case the side a is called the opposite, b is the hypotenuse, and c is the adjacent. With these three numbers we can define six ratios, or three pairs of reciprocals. These ratios are independent of the actual size of the triangle but solely depend on the angles. In a right-angled triangle one angle is always known (the right angle) and the other two are linked by $\alpha + \beta = 90^\circ$ and as a result we only need to specify one angle, let's say α . In this case the six ratios become a function of α : as we change the angle the ratios will change in a defined manner. We now give these ratios the names *sine*, *cosine*, *tangent*, *cosecant*, *secant*, and *cotangent* and define them as:

$$\begin{aligned} \sin \alpha &= \frac{a}{c}; & \cos \alpha &= \frac{b}{c}; & \tan \alpha &= \frac{a}{b}; \\ \csc \alpha &= \frac{c}{a}; & \sec \alpha &= \frac{c}{b}; & \cot \alpha &= \frac{b}{a}. \end{aligned}$$



Note that the functions in the second row are the reciprocals of the function immediately above. With these functions we can perform many operations on right-angled triangle, the most prominent function is in a process called triangulation which is commonly used to survey landscapes. It relied on knowing two sides of a right-angled triangle and one angle. With this information, using the appropriate trigonometric function, the third side or distance can be determined. However, currently we are limited to values of α between 0 and 90° .

2 The Unit Circle

Let's examine the definition of the functions above, take the sine as an example. In the case of $c = 1$ we find directly that $\sin \alpha = a$, i.e. the sine of an angle α is just the length of the opposite. If we fix the point A of the triangle in space and the length of the hypotenuse c we can change the angle α to an arbitrary value and complete the triangle such that we have a right-angle at C . This allows us to extend the definition of the sine function. The point B of the triangle will move around A on a fixed distance of 1, forming a circle around A . Now, define A to be at the origin of a Cartesian coordinate system, then the circle described by B will be a circle of radius 1 around the origin and the hypotenuse, always of length 1, will be a line connecting any point on the circle to the origin, like a spoke in a wheel.

By completing the right-angled triangle with the x -axis of the coordinate system, the length of the opposite will be the value of $\sin \alpha$. This corresponds to the y -coordinate of the point B . With this we have defined the function $\sin \alpha$ for any angle.

We can use the technique above for any of the functions by choosing the correct side of a right-angled triangle to be of unit length. In the case of the cosine it is equally the hypotenuse that needs to be of length 1 and then the x -coordinate of the point B is the cosine of the angle α .

For the tangent we require the adjacent to be of unit length. For this purpose we draw a vertical line intersecting the x -axis at 1. This means that this line is a tangent to the circle (hence the name for the corresponding function). Now, the y -coordinate of the intersection of the

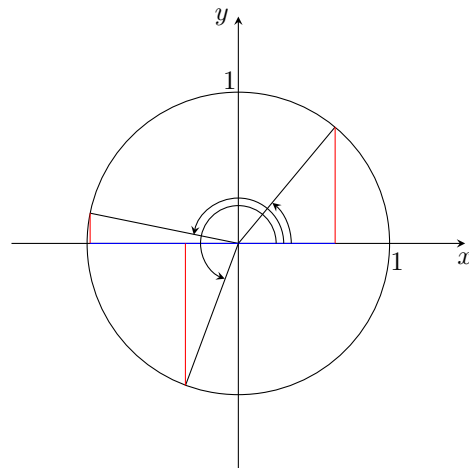


Figure 1: Visualisation of the sine and cosine for three different angles on the unit circle. The functional values correspond to the length of the red and blue segments, respectively.

extended hypotenuse of the triangle with this tangent gives the value of $\tan \alpha$. This shows that $\tan 90^\circ$ is undefined as the hypotenuse becomes parallel to the tangent and we cannot construct a triangle. For angles larger than 90° the tangent intersecting the x -axis at -1 has to be used to determine the functional value.

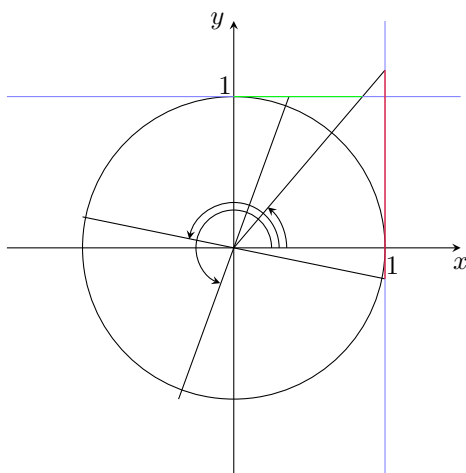


Figure 2: Visualisation of the tangent and cotangent for three different angles on the unit circle. The functional values correspond to the length of the red and green segments, respectively. The blue lines are the tangents of the circle. Tangent and cotangent are omitted for the larger angles.

that,

$$\cos \alpha = \frac{z + z^*}{2} = \Re(z); \quad \sin \alpha = \frac{z - z^*}{2i} = \Im(z).$$

This, itself, is not particularly useful. However, we will now postulate that $\cos \alpha + i \sin \alpha = e^{i\alpha}$, i.e. we can express any complex number as a complex exponential. This is Euler's Formula, one of the most useful and beautiful equations in modern Mathematics. Then,

$$\cos \alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}; \quad \sin \alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}; \quad \tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{e^{i\alpha} - e^{-i\alpha}}{e^{i\alpha} + e^{-i\alpha}}.$$

While this may on first examination not be particularly useful, it allows us to derive nearly all trigonometric identities without memorising them, as the rules on how to manipulate exponentials are particularly simple. We will make use of these when defining the derivatives of trigonometric functions.

4 Analytics of Trigonometric Functions

With the definitions of the trigonometric functions so far, we can now plot the graphs for real numbers, x , or angles, α . We will from now on only use angles in radians (x). The conversion to degrees is straightforward by introducing factors of $180^\circ/\pi$.

Due to the definition of the functions on the unit circle and the repetition of angles in multiples of 2π the functions will be of periodic nature. Using the unit circle definition, we see that for the sine we start off at a functional value of 0 which will be passed through at every multiple of π with the graph repeating itself every 2π . Similar for the cosine, however the passing through 0 will no occur at every odd multiple of $\pi/2$. In fact, cosine and sine are identical bar a shift along the x -axis of $\pi/2$. This is called a *phase* and we can see that $\cos(x + \pi/2) = \sin(x)$.

Equivalent definitions can be found for the other three trigonometric functions.

The definition on the unit circle is superior to the right-angled triangle definition as it enables the definition of the functions for any argument. Moreover it highlights the connection between an angle in degrees and radians, as the length of the arc from the x -axis to the hypotenuse is just the value of the angle.

3 Trigonometric Functions and Complex Numbers

Complex numbers of the form $z = a + ib$, where i is the imaginary unit defined as $i^2 = -1$, can be visualised in the complex plane using an Argand diagram. In this case the coordinates $(a; b)$ specify the real and imaginary parts of the number z . Moreover, we can construct a *pointer* connecting the number z to the origin. This pointer will have a length r and an angle with the real axis, call it θ . If we require $r = 1$ the number will be part of the unit circle and its coordinates will be given as $(\cos \alpha; \sin \alpha)$. But these have to be equivalent to $(a; b)$ (it's the same number). And hence, $z = \cos \alpha + i \sin \alpha$. If we now define the complex conjugate of z as $z^* = \cos \alpha - i \sin \alpha$ we find

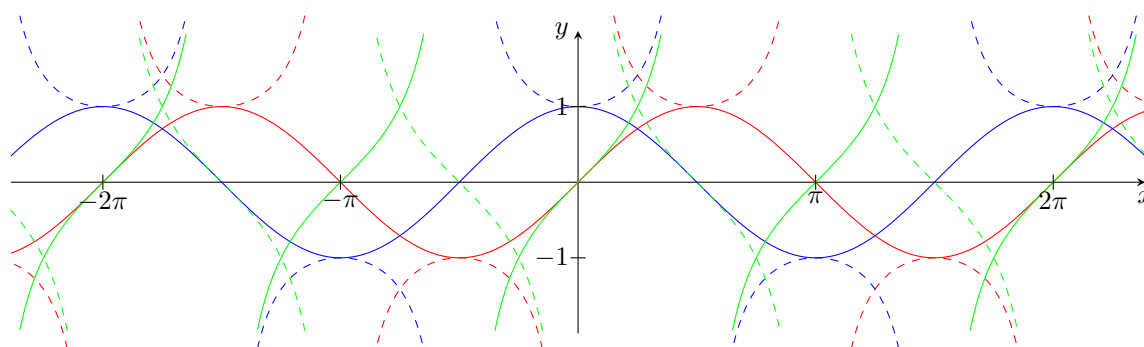


Figure 3: Functional graphs for $\sin(x)$ (red), $\cos(x)$ (blue), $\tan(x)$ (green), $\csc(x)$ (red dashed), $\sec(x)$ (blue dashed), and $\cot(x)$ (green dashed).

As all trigonometric functions are periodic, they have either no roots (cosecans and secans) or infinitely many (the others) with a given periodicity. As can be seen the tangent shares its roots with the sine function at multiples of π and the cotangent with the cosine at odd multiples of $\pi/2$. Furthermore, all trigonometric functions are odd with exception of the cosine and secant, which are even. Apart from the sine and cosine every function has poles, or values at which the function is not defined. For the tangent and secant these are at odd multiples of $\pi/2$, for the tangent and sine they are at multiples of π .

As sine, cosine,... are functions we can find their derivatives. Using differential quotients we find,

$$\frac{d}{dx} \sin x = \lim_{\delta x \rightarrow 0} \frac{\sin(x + \delta x) - \sin x}{\delta x}.$$

We now need to understand how to manipulate $\sin(x + \delta x)$. To this end take $\sin(x + y)$ and rewrite it using the exponential expressions above as,

$$\begin{aligned} \sin(x + y) &= \frac{e^{i(x+y)} - e^{-i(x+y)}}{2i} = \frac{e^{ix}e^{iy} - e^{-ix}e^{-iy}}{2i} \\ &= \frac{1}{2i} [(\cos x + i \sin x)(\cos y + i \sin y) - (\cos x - i \sin x)(\cos y - i \sin y)] \\ &= \frac{1}{2i} [\cos x \cos y + i \sin x \cos y + i \cos x \sin y - \sin x \sin y - \cos x \cos y + i \sin x \cos y + i \cos x \sin y + \sin x \sin y] \\ &= \frac{1}{2i} [2i \sin x \cos y + 2i \cos x \sin y] = \sin x \cos y + \cos x \sin y. \end{aligned}$$

Hence, for our differential,

$$\frac{d}{dx} \sin x = \lim_{\delta x \rightarrow 0} \frac{\sin x \cos \delta x + \cos x \sin \delta x - \sin x}{\delta x},$$

we can separate the fraction into three terms,

$$\frac{d}{dx} \sin x = \lim_{\delta x \rightarrow 0} \left[\frac{\sin x \cos \delta x}{\delta x} + \frac{\cos x \sin \delta x}{\delta x} - \frac{\sin x}{\delta x} \right],$$

where we can consider each term separately. First the middle term,

$$\lim_{\delta x \rightarrow 0} \frac{\cos x \sin \delta x}{\delta x} = \cos x \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x} = \cos x,$$

as we will show below. Then the first term,

$$\lim_{\delta x \rightarrow 0} \frac{\sin x \cos \delta x}{\delta x} = \sin x \lim_{\delta x \rightarrow 0} \frac{\cos \delta x}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{\sin x}{\delta x},$$

and putting it all together then gives,

$$\frac{d}{dx} \sin x = \lim_{\delta x \rightarrow 0} \left[\frac{\sin x}{\delta x} + \cos x - \frac{\sin x}{\delta x} \right] = \cos x.$$

We can find equivalent results for the other functions. We then find:

$$\sin x = -\frac{d}{dx} \cos x = -\frac{d^2}{dx^2} \sin x = \frac{d^3}{dx^3} \cos x = \frac{d^4}{dx^4} \sin x = \dots$$

Integration of the sine and cosine is then also rather simple. Integrating the other functions heavily relies on trigonometric identities. Below is a table of simple derivatives and integrals. Note that the constant of integration is omitted.

$\int f(x)dx$	$f(x)$	df/dx
$-\cos(x)$	$\sin(x)$	$\cos(x)$
$\sin(x)$	$\cos(x)$	$-\sin(x)$
$\ln \sec x $	$\tan(x)$	$\sec^2(x)$
$\ln \sec x + \tan x $	$\sec(x)$	$\tan(x) \sec(x)$
$-\ln \csc x + \cot x $	$\csc(x)$	$-\csc(x) \cot(x)$
$\ln \sin x $	$\cot(x)$	$-\csc^2(x)$

5 Series Definition

Having the derivatives of the trigonometric functions we can now find the Taylor expansions of these functions, for instance for the sine function as,

$$\begin{aligned} \sin x &= \sin 0 + \left. \frac{d \sin x}{dx} \right|_{x=0} x + \frac{1}{2!} \left. \frac{d^2 \sin x}{dx^2} \right|_{x=0} x^2 + \frac{1}{3!} \left. \frac{d^3 \sin x}{dx^3} \right|_{x=0} x^3 + \frac{1}{4} \left. \frac{d^4 \sin x}{dx^4} \right|_{x=0} x^4 + \dots \\ &= 0 + x \cos 0 - \frac{x^2}{2!} \sin 0 - \frac{x^3}{3!} \cos 0 + \frac{x^4}{4!} \sin 0 + \frac{x^5}{5!} \cos 0 + \dots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, \end{aligned}$$

which is an infinite series with some regularities: firstly, only the powers that are odd in x survive and secondly they have alternating sign. We can then set this series up as an infinite series:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

convince yourself that this is correct: the factor of $(-1)^n$ ensures that we have an alternating sign, starting with a + and the term $(2n+1)$ makes sure that we only have odd numbers. We can, in fact, use this endless series as a definition for the sine function and can find equivalent series for the other trigonometric functions. However, the definitions of the tangent, secant, cosecant, and cosecant, contain special numbers that we will not consider further here.

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

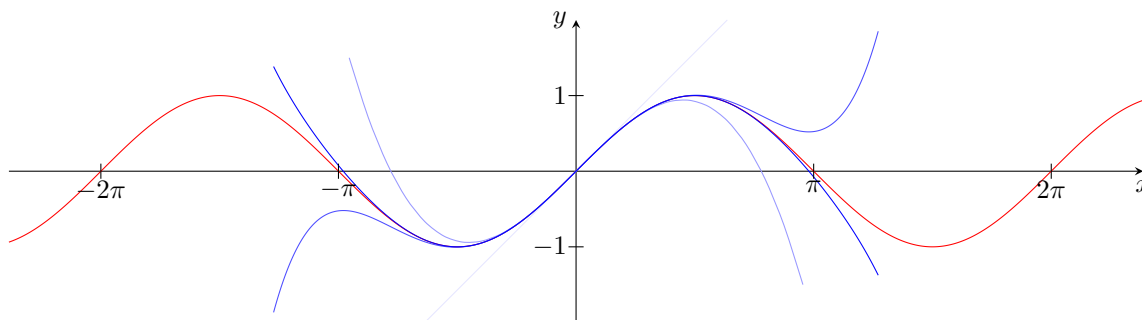


Figure 4: Series approximations to $\sin(x)$. With increasing number of terms in the series (here up to 4) the expansion becomes more accurate over a larger domain.

With these, we can prove Euler's Formula:

$$\begin{aligned} \cos x + i \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} i^{2n} \left[\frac{x^{2n}}{(2n)!} + i \frac{x^{2n+1}}{(2n+1)!} \right] \\ &= \sum_{n=0}^{\infty} \left[\frac{(ix)^{2n}}{(2n)!} + \frac{(ix)^{2n+1}}{(2n+1)!} \right] = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} = e^{ix}. \end{aligned}$$

6 Definition via Differential Equations

Sine and cosine are the functions that solve the differential equation,

$$y'' = -y,$$

as we can see by making using the trial function $y = e^{\lambda x}$ which results in,

$$\lambda^2 e^{\lambda x} = -e^{\lambda x},$$

giving the equation $\lambda^2 = -1$ which solves as $\lambda = \pm i$. This then gives the solution,

$$y(x) = Ae^{ix} + Be^{-ix}.$$

To obtain $\sin(x)$ we impose the boundary conditions $y(0) = 0$ and $y'(0) = 1$ which results in $A = -B = 2i$, i.e.

$$y(x) = 2i(e^{ix} - e^{-ix}) = \sin(x),$$

and for $\cos(x)$ we use the boundary condition that $y(0) = 1$ and $y'(0) = 0$, giving $A = B = 2$, i.e.

$$y(x) = 2(e^{ix} + e^{-ix}) = \cos(x).$$