Relativistic Quantum Mechanics

MPAGS

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Contents

Lecture 1

Special relativity and Lorentz invariance

1.1   Objectives of this course

The objectives of this lecture course are to:

- get familiar with the notation of relativistic equations;

- understand how to use perturbation theory to derive transition amplitudes;

- understand the properties of the Dirac and Klein-Gordon equations;

- and understand how to perform calculations based on Feynman rules (spin sums/trace techniques etc).

I am indebted to Michal Kreps, whose lectures this course is based on. Much of the content of the course can also be found in chapters 3, 4, 5 and 6 of

- *Quarks and Leptons: An Introductory Course in Modern Particle Physics*
  by Halzen and Martin.

1.2   Introduction

Why do we need Relativistic Quantum mechanics? In high energy physics we study processes that involve "very small things travelling very fast". For each aspect we have a well tested theory:

$$\text{"very small things"} \quad \Rightarrow \text{quantum mechanics}$$

$$\text{"travelling very fast"} \quad \Rightarrow \text{special relativity}$$

The unification of general relativity (GR) and quantum mechanics (QM) is beyond the scope of the MPAGS courses. There are numerous mathematical (quantum theories of gravity are non-renormalizable) and conceptual issues (to do with locailty and unitarity) with combining GR and QM. Attempts to unify GR and QM include string theory and loop quantum gravity.

1.3 Natural units

In particle physics the physical constants

$$c = 2.99792458 \times 10^8 \, \mathrm{m \, s^{-1}} \, ,$$
$$\hbar = 1.054571726(47) \times 10^{-34} \mathrm{J \, s} \, ,$$

appear in many places. It is convenient to choose a system of units where $\hbar = c = 1$, such that mass, momentum (mc) and energy (mc^2) have the same units. This allows us to be lazy when writing down our equations and we'll do this frequently throughout the course. If we want to, we can convert back to `SI` units using dimensional analysis.

1.4 Invariances in physical theories

Invariances play an important role in physical theories. In 1915 Emmy Noether demonstrated that every (differentiable) symmetry of the action of a physical system has a corresponding conservation law.
Examples are:

$$\text{invariance under translations} \quad \Rightarrow \quad \text{conservation of momentum}$$

$$\text{invariance under rotations} \quad \Rightarrow \quad \text{conservation of angular momentum}$$

$$\text{time invariance (same at } t \text{ and } t') \quad \Rightarrow \quad \text{conservation of energy}$$

1.5 Form invariance in classical mechanics

We will start by looking at what is known as form-invariance i.e. that the equations describing a physical system are the same for all observers.

$$\begin{array}{ccc} \text{law } i = jk & \rightarrow & i' = j'k' \\ \text{observer A} & & \text{observer } A' \end{array}$$

In classical mechanics we know that equations are invariant under translations and rotations of coordinate axis. However, these invariances are not obvious if we do not formulate our equations in a suitable language. In classical mechanics we use vectors.

If we consider two coordinate systems rotated with respect to each other. A vector $\vec{A}$ in system S will correspond to a vector $\vec{A}'$ in system S',

$$\vec{A} = (a^1, a^2, a^3) \, , \ \vec{A}' = (a'^1, a'^2, a'^3) \, . \tag{1.1}$$

The vectors $\vec{A}$ and $\vec{A}'$ correspond to the same physical object (they are the same vector) and have the same magnitude (length) in each of the coordinate systems

$$|\vec{A}|^2 = (a^1)^2 + (a^2)^2 + (a^3)^2 = (a'^1)^2 + (a'^2)^2 + (a'^3)^2 \, . \tag{1.2}$$

The quantity $|\vec{A}|^2$, *i.e.* the length of the vector, is an "invariant" quantity. We can write

$$\vec{A} = a^1\hat{e}_1 + a^2\hat{e}_2 + a^3\hat{e}_3 = a'^1\hat{e}'_1 + a'^2\hat{e}'_2 + a'^3\hat{e}'_3 \ , \tag{1.3}$$

where $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ and $\hat{e}'_1$, $\hat{e}'_2$, $\hat{e}'_3$ are basis vectors describing the coordinate system of S and S', respectively. The coordinates are related through a linear transformation (a rotation)

$$a^{i\prime} = \sum_{j=1}^{3} R^i_j a^j \ . \tag{1.4}$$

Lets apply this to Newtons law. In S

$$\vec{F} = m\vec{a} \quad \text{or} \quad F^i = ma^i \ , \tag{1.5}$$

where the index i refers to a component of $\vec{F}$. What form does this take in S'?

$$F^i = ma^i \tag{1.6}$$

$$\sum_{i=1}^{3} R^j_i F^i = m \sum_{i=1}^{3} R^j_i a^i \ ,$$

$$F'^j = ma'^j$$

The form invariance comes from having objects that transform in the same way on the left- and right-hand side of the equation.

1.6   Galilei transformations

In classical mechanics, if there are two systems of coordinates moving with a constant relative velocity v with respect to each other, we make a Galilei transformation

$$
\begin{aligned}
x'^1 &= x^1 - vt \\
x'^2 &= x^2 \\
x'^3 &= x^3 \\
t' &= t
\end{aligned}
\tag{1.7}
$$

Newtons law's are invariant under this type of transformation, but Maxwell's equations are not. Under a Galilei transformation,

$$
\frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\mathrm{d}x}{\mathrm{d}t} - v \quad \Rightarrow \quad c' = c - v
\tag{1.8}
$$

i.e. the speed of light is different for the different observers.

1.7   Special relativity

In 1905 Einstein postulated that:

- the laws of physics are invariant (the same) for all inertial systems (systems of coordinates moving with uniform velocity with respect to each other);

- the speed of light in the vacuum is the same for all observers.

This requires a much more general transformation as it is a time dependent invariance.

1.8   The Lorentz transformation

The appropriate transformation to leave c unchanged is the Lorentz transformation. The Lorentz transformation corresponding to a boost along x^1 is

$$
\begin{pmatrix} ct' \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} ,
\tag{1.9}
$$

where

$$
\beta = \frac{v}{c} \; , \; \gamma = \frac{1}{\sqrt{1-\beta^2}}
\tag{1.10}
$$

Here we have included ct as another component x^0 in a 4-vector

$$x^\mu = (x^0, x^1, x^2, x^3) \tag{1.11}$$

We conventionally use Greek indices to refer to components of a 4-vector (e.g. μ, ν etc) and roman indices (e.g. i, j, k etc) to refer to components of a spatial 3-vector.

Written in this way, it is obvious that Einsteins postulates lead to a mixing of space and time. In order to form invariant equations in special relativity we need to use 4-vectors. In 3D cartesian space we had 3-vectors ($\vec{x}$). In 4D Minkowski space we have space-time vectors (x^μ). The Lorentz transformation can be thought of like a (generalised) rotation in the Minkowski space. They are part of a larger group of transformations, which includes translations, called the Poincaré group. Writing equations in the form of 4-vectors and balancing indices guarantees form invariance (or *Lorentz covariance*) under Lorentz transformations. As the Lorentz transformation is linear, it can be written as

$$x'^\mu = \sum_{\nu=1}^{4} \Lambda^\mu_\nu x^\nu \tag{1.12}$$

At this point we will introduce *Einstein summation* notation (*contraction of indices*) and drop the explicit summation in the expression, *i.e.*

$$\sum_{\nu=1}^{4} \Lambda^\mu_\nu x^\nu \equiv \Lambda^\mu_\nu x^\nu , \tag{1.13}$$

where a repeated pair of upper and lower indices implies a sum over that index such that

$$\Lambda^\mu_\nu x^\nu = \Lambda^\mu_0 x^0 + \Lambda^\mu_1 x^1 + \Lambda^\mu_1 x^2 + \Lambda^\mu_1 x^3 . \tag{1.14}$$

As in the case of 3-vectors we can construct lengths that are invariant under Lorentz transformations, in this case the invariant length is

$$A^2 = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2 , \tag{1.15}$$

for example

$$p^2 = E^2/c^2 - |\vec{p}|^2 = m^2 c^2 , \ p^\mu = (E/c, \vec{p}) . \tag{1.16}$$

Put another way, the element in the 4D space

$$(\mathrm{d}s)^2 = (\mathrm{d}t)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2 , \tag{1.17}$$

is invariant under Lorentz transformations.

If we write this in terms of basis vectors

$$(A^\mu \hat{e}_\mu) \cdot (A^\nu \hat{e}_\nu) = A^\mu A^\nu \hat{e}_\mu \cdot \hat{e}_\nu , \tag{1.18}$$

we can see that the basis vectors for $\mu = \nu = 1, 2, 3$ are not orthogonal. Instead $\hat{e}_\mu \cdot \hat{e}_\mu = -1$.

At this point it is useful to introduce the (metric) tensor $g_{\mu\nu}$ and a new type of vector $x_\mu = (x^0, -\vec{x})$. We call this type of vector a *co-vector*. The metric tensor in Minkowski space is defined by

$$g_{00} = 1 \quad , \quad g_{ii} = -1 \text{ for } i = 1, 2, 3 \quad \text{and} \quad g_{\mu\nu} = 0 \text{ for } \mu \neq \nu \tag{1.19}$$

such that

$$(\mathrm{d}s)^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{1.20}$$

and similarly for $g^{\mu\nu}$ (which follows from $g_{\mu\nu}g^{\mu\nu} = \mathbb{1}$). The vectors and co-vectors are related through the metric tensor,

$$x_\mu = g_{\mu\nu}x^\nu \ , \tag{1.21}$$

which acts to lower and raise the indices. Anywhere we have a repeated upper and lower index we have a dot product and get a scalar quantity, *i.e.* the dot product of $A \cdot B$ can then be written as

$$A^\mu B_\mu = A_\mu B^\mu = g_{\mu\nu}A^\mu B^\nu \ . \tag{1.22}$$

Lorentz invariance implies

$$\begin{aligned} x'^\mu x'_\mu &= g_{\mu\nu}x'^\mu x'^\nu \\ &= x'_\mu \Lambda^\mu_\nu x^\nu \end{aligned} \tag{1.23}$$

$$\therefore x_\nu = x'_\mu \Lambda^\mu_\nu \ , \ x'_\mu = [\Lambda^{-1}]^\nu_\mu x_\nu \tag{1.24}$$

i.e. the co-vectors transform with the inverse of the Lorentz transformation.

We can also write down the derivatives

$$\partial^\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right) \quad , \quad \partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \tag{1.25}$$

The derivative ∂_μ is an important covector. Acting on a scalar field, ϕ,

$$\begin{aligned} \partial'_\mu \phi &= \frac{\partial\phi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial\phi}{\partial x^\nu} \\ &= [\Lambda^{-1}]^\nu_\mu \partial_\nu \phi \end{aligned} \tag{1.26}$$

We want the 4-vector to be basis independent,

$$A = A^\mu \vec{e}_\mu = A'^\nu \vec{e}'_\nu \ . \tag{1.27}$$

For this to hold,

$$A^\mu \vec{e}_\mu = \Lambda^\nu_\sigma A^\sigma M^\rho_\nu \vec{e}_\rho \qquad (1.28)$$
$$= \Lambda^\nu_\sigma M^\rho_\nu A^\sigma \vec{e}_\rho$$

where M is some transformation of the basis vector. This in turn implies

$$\Lambda^\nu_\sigma M^\rho_\nu = \delta^\rho_\sigma \qquad (1.29)$$

i.e. M is the inverse of the Lorentz transform, $M = [\Lambda^{-1}]$. The basis vectors of Minkowski space transform as

$$\vec{e}'_\mu = [\Lambda^{-1}]^\nu_\mu \vec{e}_\nu \ . \qquad (1.30)$$

We call A^μ a *contravariant* component and $\vec{e}_\mu$ a *covariant* basis vector. For the co-vectors, the components are *covariant* and the basis vectors *contravariant* ,

$$A'_\mu = [\Lambda^{-1}]^\nu_\mu A_\nu \qquad (1.31)$$
$$\vec{e}'^\mu = \Lambda^\mu_\nu \vec{e}^\nu$$

In summary, to guarantee Lorentz covariance of our equations we need to write equations that have the same type of object on both sides of the equation, e.g.

$$A^\mu B_\mu = k \qquad \text{scalar} \leftrightarrow \text{scalar}$$
$$A^\mu B_\mu C^\nu = kC^\nu \qquad \text{vector} \leftrightarrow \text{vector}$$
$$D^\mu T_{\mu\nu} = kC_\nu \qquad \text{vector} \leftrightarrow \text{vector}$$

Aside: the light-cone

When we build quantum mechanical theories, it is important to remember that information can not propagate faster than the speed of light. Events with space-time points at x_1^μ and x_2^μ are only causal if $(x_1 - x_2)^2 > 0$. This is referred to as *time-like* separation. Two events with $(x_1 - x_2)^2 < 0$ can not influence each other. The point x_1 is not inside the light-cone of x_2. This is referred to as *space-like* separation.

Lecture 2

Examples of Lorentz invariance – Maxwell's equations and the Klein-Gordon equation

2.1 Lorentz covariance of Maxwell's equations

Maxwell's equations, written in terms of an electric field $\vec{E}$ and a magnetic field $\vec{B}$ are

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \rho && \text{(Gauss' law) ,} \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} && \text{(Faraday-Lenz) ,} \\
\vec{\nabla} \cdot \vec{B} &= 0 && \text{(no magnetic charges) ,} \\
\vec{\nabla} \times \vec{B} &= \vec{j} + \frac{\partial \vec{E}}{\partial t} && \text{(Ampere's law) .}
\end{aligned}
\tag{2.1}
$$

It turns out that these equations are Lorentz covariant, but this not immediately obvious in this form. To Maxwell's equations in a manifestly Lorentz covariant way, we start with Ampere's law and take its divergence

$$
\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \cdot \vec{j} + \vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t}
\tag{2.2}
$$

$$
0 = \vec{\nabla} \cdot \vec{j} + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{E})
\tag{2.3}
$$

$$
= \vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} .
\tag{2.4}
$$

This is the continuity equation, which we can express in terms of 4-vectors

$$
\partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla} \right) \ , \ j^\mu = (\rho, \vec{j}) \ ,
\tag{2.5}
$$

such that

$$
\partial_\mu j^\mu = 0 \ .
\tag{2.6}
$$

The continuity equation implies that the charge

$$
Q = \int_V \rho \, \mathrm{d}^3 x = \int_V j^0 \mathrm{d}^3 x
\tag{2.7}
$$

is locally conserved, *i.e.* the change in the charge in an arbitrarily small volume is only due to the current flow out of the volume.

At this point it is useful to introduce potentials, V and $\vec{A}$, such that

$$
\vec{B} = \vec{\nabla} \times \vec{A} \quad , \quad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V \ .
\tag{2.8}
$$

By writing the magnetic and electric field in this way, the Faraday-Lenz law and $\vec{\nabla} \cdot \vec{B} = 0$ are trivially satisfied. Re-writing Ampere's law to include $\vec{A}$ and V

$$
\begin{aligned}
\vec{\nabla} \times \vec{B} &= \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) \\
&= \vec{j} + \frac{\partial}{\partial t}\left(-\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}V\right) \\
&= \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} ,
\end{aligned}
\tag{2.9}
$$

where we have used the triple product expansion

$$
A \times (B \times C) = B(A \cdot C) - C(A \cdot B) .
\tag{2.10}
$$

Similarly, replacing $\vec{E}$ in Gauss' law gives

$$
-\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 V = \rho .
\tag{2.11}
$$

Maxwell's equations are also invariant under another type of transformation, Gauge transformations. If we transform

$$
\vec{A}' = \vec{A} - \nabla\chi \quad , \quad V' = V + \frac{\partial \chi}{\partial t} ,
\tag{2.12}
$$

where χ is a scalar field, this does not change the magnetic or electric fields. We can choose χ to simplify our expressions, by picking a solution that sets

$$
\vec{\nabla} \cdot \vec{A} + \frac{\partial V}{\partial t} = 0 .
\tag{2.13}
$$

This is the so-called *Lorenz gauge*. If we write, $A^\mu = (V, \vec{A})$, then the Lorentz gauge is manifestly Lorentz invariant,

$$
\partial_\mu A^\mu = 0
\tag{2.14}
$$

Back to

$$
-\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 V = \rho \Rightarrow \frac{\partial^2 V}{\partial t^2} - \vec{\nabla}^2 V = \rho
\tag{2.15}
$$

and

$$
\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \vec{j} - \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\frac{\partial V}{\partial t}
\tag{2.16}
$$

which gives

$$
\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}^2 \vec{A} = \vec{j} .
\tag{2.17}
$$

This can be written in the compact form

$$\partial^\mu \partial_\mu A^\nu = j^\nu \tag{2.18}$$

where

$$\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \ . \tag{2.19}$$

This expression is also manifestly Lorentz covariant. You will sometimes see this written with the replacement

$$\Box \equiv \partial_\mu \partial^\mu \quad \text{so} \quad \Box A^\mu = j^\mu. \tag{2.20}$$

In summary, we can write Maxwells equations in a compact, Lorentz covariant form as

$$\partial_\mu j^\mu = 0 \quad , \quad \partial^\mu \partial_\mu A^\nu = j^\nu \tag{2.21}$$

with

$$j^\mu = (\rho, \vec{j}) \quad , \quad A^\mu = (V, \vec{A}) \quad \text{and} \quad \partial_\mu A^\mu = 0 \ . \tag{2.22}$$

2.2   Quantum mechanics

So far we have focussed on special relativity. Now we want to see how this can be combined with quantum mechanics (QM). The fundamental building blocks we need from quantum mechanics are:

- Observables can be continuous or discrete.

- Variables are associated to operators, *i.e.* we can promote

$$E, \; x, \; p \to \hat{E}, \; \hat{x}, \; \hat{p} \; . \tag{2.23}$$

- Systems can be described by state vector $|\psi\rangle$ in Hilbert space.

- The operators act on the state vectors

$$\hat{A}|\psi\rangle = a|\psi\rangle \tag{2.24}$$

  where a is an eigenvalue and $|\psi\rangle$ and eigenvector.

- The eigenvectors can be made orthogonal such that

$$\langle \psi_m | \psi_n \rangle = \delta_{mn} \quad , \quad \langle \vec{x}_1 | \vec{x}_2 \rangle = \delta^3(\vec{x}_1 - \vec{x}_2) \tag{2.25}$$

- The eigenvectors form a complete set that spans the space such that

$$\sum_m |\psi_m\rangle\langle\psi_m| = 1 \tag{2.26}$$

$$\int \mathrm{d}^3\vec{x}|\vec{x}\rangle\langle\vec{x}| = 1 \tag{2.27}$$

- From the eigenvectors, we can define position-space wave-functions

$$\psi(\vec{x}) = \langle \vec{x} | \psi \rangle \tag{2.28}$$

  which project a state ψ onto a position $\vec{x}$.

- In the Born interpretation, $\psi(\vec{x})$ is interpreted in terms of position probability density $\rho(\vec{x}) = \psi^*\psi$

$$\int \psi^*(\vec{x})\psi(\vec{x})\mathrm{d}^3\vec{x} = 1 \tag{2.29}$$

- In the Schrödinger picture the state vectors are time-dependent,

$$|\psi\rangle \to |\psi(t)\rangle \text{ and } \psi(\vec{x},t) = \langle\vec{x}|\psi(t)\rangle \tag{2.30}$$

2.3   The Scrödinger equation

In classical mechanics

$$E = \frac{|\vec{p}|^2}{2m} + V(\vec{x}, t) \tag{2.31}$$

Promoting E and p to to operators gives

$$E \rightarrow \hat{E} = i\hbar \frac{\partial}{\partial t} \; , \tag{2.32}$$

$$\vec{p} \rightarrow \hat{p} = -i\hbar \vec{\nabla} \; ,$$

Dropping the $\hbar$, these can now be written in a neater 4-vector notation as

$$p^\mu = i\partial^\mu \; . \tag{2.33}$$

The operators $\hat{E}$ and $\hat{p}$ operate on a wave-function $\psi(\vec{x}, t)$ and this yields the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{x}, t) = -\frac{\hbar^2}{2m} \vec{\nabla}^2 \psi(\vec{x}, t) + V(\vec{x}, t)\psi(\vec{x}, t) \; , \tag{2.34}$$

where $V(\vec{x}, t)$ is an arbitrary potential.

We can also define a particle density by

$$\rho = \psi^* \psi \tag{2.35}$$

Differentiating ρ by parts yields

$$\frac{\partial \rho}{\partial t} = \psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t} \psi \; . \tag{2.36}$$

Using the complex conjugate of the Schrödinger equation

$$-i\hbar \frac{\partial}{\partial t} \psi^*(\vec{x}, t) = -\frac{\hbar^2}{2m} \vec{\nabla}^2 \psi^*(\vec{x}, t) + V^*(\vec{x}, t)\psi^*(\vec{x}, t) \; , \tag{2.37}$$

and assuming $V = V^*$ gives

$$\psi^* \frac{\partial \psi}{\partial t} - \frac{i}{2m} \psi^* \vec{\nabla}^2 \psi = -\psi \frac{\partial^* \psi}{\partial t} - \frac{i}{2m} \psi \vec{\nabla}^2 \psi^* \; , \tag{2.38}$$

which can be written (you will do something similar as an exercise) as

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = \partial_\mu j^\mu = 0 \; . \tag{2.39}$$

Here, ρ is the probability density and $\vec{j}$ is a probability current and $j^\mu = (\rho, \vec{j})$. This is the continuity equation for a conserved quantity (conserved probability in this case).

2.4 The Klein-Gordon equation

The Schrödinger equation was derived from the non-relativistic energy-momentum relation. It's non linear in E and $\vec{p}$, which does not allow us to use a Lorentz covariant notation. We know that there is a more appropriate relativistic equation. What happens if we take

$$E^2 = |\vec{p}|^2 + m^2 \tag{2.40}$$

and promote E and $\vec{p}$ to operators? This gives

$$\Rightarrow \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2 \right) \phi(\vec{x}, t) = 0 \tag{2.41}$$

which can be written in a compact notation as

$$(\partial^\mu \partial_\mu + m^2)\phi(x) = 0 \ . \tag{2.42}$$

This is the Klein-Gordon equation. The Klein-Gordon equation has plane wave solutions of the form

$$\phi(x) = N e^{-i(\omega t - \vec{k} \cdot \vec{x})} = N e^{-ip \cdot x} \tag{2.43}$$

Plugging $\phi(x^\mu)$ back into the equation yields

$$E = \pm \sqrt{|\vec{p}|^2 + m^2} \tag{2.44}$$

This is the equation for a free spin-0 particle. There are both positive and negative energy solutions (which should not come as a big surprise because we started with an expression for E^2).

Multiplying the Klein-Gordon equation by $-i\phi^*$ and its complex conjugate by $-i\phi$, gives

$$-i\phi^* \frac{\partial^2 \phi}{\partial t^2} + i\phi^* \vec{\nabla}^2 \phi - im^2 \phi^* \phi = -i\phi \frac{\partial^2 \phi^*}{\partial t^2} + i\phi \vec{\nabla}^2 \phi^* - im^2 \phi \phi^* = 0 \tag{2.45}$$

subtracting the two expressions

$$i\frac{\partial}{\partial t} \left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right) - i\vec{\nabla} \cdot \left(\phi^* \vec{\nabla} \phi - \phi \vec{\nabla} \phi^* \right) = 0 \tag{2.46}$$

this is the continuity equation for the Klein-Gordon equation,

$$\partial_\mu j^\mu = 0 \tag{2.47}$$

with

$$j^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) \ . \tag{2.48}$$

Plugging in the plane-wave solutions of the Klein-Gordon equation gives

$$j^\mu = 2|N|^2 p^\mu \ , \ p^\mu = (E, \vec{p}) \tag{2.49}$$

We now have a second problem, the negative energy solutions also have a negative probability density since $j^0 = 2|N|^2 E$. Unfortunately, the problem can't be simply ignored because in QM we need to work with a complete set of states.

2.5   The Feynman-Stückleberg interpretation

A prescription for handling negative energy states was proposed by Stückleberg (1941) and Feynman (1946):

> *Negative energy solutions describe a particle propagating backwards in time, or, a positive energy anti-particle propagating forwards in time.*

Suppose the Klein-Gordon equation describes a free spin-0 particle of charge $+e$, e.g. a π^+. Then the electromagnetic-current associated with the charged particle is

$$j^\mu(\pi^+) = 2e|N|^2(E, \vec{p}) \tag{2.50}$$

A π^- with the same energy an momentum has a current

$$j^\mu(\pi^-) = -2e|N|^2(E, \vec{p}) \tag{2.51}$$

which is the same current as the π^+ under $E \to -E$ and $\vec{p} \to -\vec{p}$. A negative-energy particle solution going backwards in time describes a positive energy anti-particle solution going forward in time. The reason this works is because $e^{-i(-E)(-t)} = e^{-iEt}$. Whilst the Klein-Gordon wave-functions are fundamentally single particle solutions, the Feynman-Stückleberg interpretation allows us to handle many particle states.

Lecture 3

Perturbation theory and particle scattering

3.1　Time dependent perturbation theory

In High Energy Physics we deal with situations that can be described by

1. free incoming particles of known $(E, \vec{p})$ (beam);

2. an interaction;

3. and free outgoing particles of known $(E, \vec{p})$ (measured in a detector).

The interaction acts over a short time T and can be described by a potential $V(t, \vec{x})$, where $V(t, \vec{x}) = 0$ outside of T. The interactions are also weak, such that the probability of an interaction occurring P(interaction) $\ll 1$. The smallness of the interaction allows us to make perturbative expansions.
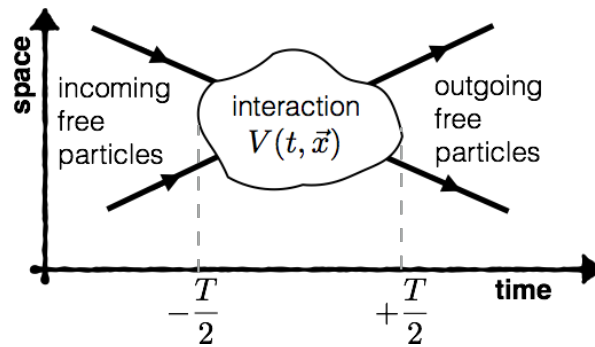

Figure 3.1: Illustration of a typical scattering problem in High Energy Physics.

We will use a non-relativistic framework to derive the tools we need to do scattering calculations. This might sound questionable, but a full treatment gives the same result. If you want to understand the reasons behind this, it is explained in *Relativistic Quantum Fields* by Bjorken and Drell.

We will start with the Schrödinger equation for a free particle. This is given by

$$\frac{i\partial\phi}{\partial t} = H_0(\vec{x})\phi \tag{3.1}$$

where the Hamiltonian, H_0, is constant in time, i.e.

$$H_0(\vec{x})\phi_m(\vec{x}) = E_m\phi_m(\vec{x}) \ , \tag{3.2}$$

but the states are time dependent,

$$\phi_m(t, \vec{x}) = \phi_m(\vec{x})e^{-iE_m t} \ . \tag{3.3}$$

The states are normalised such that

$$\int \phi_m^*(\vec{x})\phi_n(\vec{x})\mathrm{d}^3\vec{x} = \delta_{mn} \ . \tag{3.4}$$

The wave-functions ϕ form a complete basis so the solutions to this equation can be expressed as

$$\psi = \sum_m a_m(t)\phi(\vec{x})e^{-iE_m t} \tag{3.5}$$

The objective is then to solve

$$\left[H_0 + V(\vec{x},t)\right]\psi = i\frac{\partial \psi}{\partial t} \ , \tag{3.6}$$

where we have introduced a time-dependent potential $V(\vec{x},t)$. The normalisation of the wave-function ψ is given by

$$\int \psi^*\psi \,\mathrm{d}^4x = 1 \quad \text{implies} \quad \sum_m |a_m|^2 = 1 \ , \tag{3.7}$$

where $|a_m|^2$ is the probability to find the particle in state ϕ_m. This is time-independent for a free system.

After we have introduced the potential

$$\begin{aligned}
\left[H_0 + V(\vec{x},t)\right]\psi &= i\frac{\partial \psi}{\partial t} \\
&= i\frac{\partial}{\partial t}\sum_m a_m(t)\phi(\vec{x})e^{-iE_m t} \ ,
\end{aligned} \tag{3.8}$$

the $a_m(t)$ are time-dependent and we can get transitions between states. This can be expanded as

$$\begin{aligned}
\left[H_0 + V(\vec{x},t)\right]\sum_m a_m(t)\phi(\vec{x})e^{-iE_m t} &= \\
i\sum_m \left[\frac{\partial a_m(t)}{\partial t} - iE_m a_m(t)\right]&\phi(\vec{x})e^{-iE_m t}
\end{aligned} \tag{3.9}$$

giving

$$i\sum_m \frac{\partial a_m(t)}{\partial t}\phi(\vec{x})e^{-iE_m t} = V\left[\sum_m a_m(t)\phi(\vec{x})e^{-iE_m t}\right] \tag{3.10}$$

where we have used

$$H_0\phi_m(\vec{x}) = E_m\phi_m(\vec{x}) \ . \tag{3.11}$$

We can simplify this expression by multiplying through by $\phi_f^*(\vec{x})$ and integrating over the phasespace, *i.e.*

$$
i \int \sum_m \frac{\partial a_m(t)}{\partial t} \phi_f^*(\vec{x}) e^{+iE_f t} \phi_m(\vec{x}) e^{-iE_m t} \mathrm{d}^3 \vec{x}
$$
$$
= \int \sum_m a_m(t) \phi_f^*(\vec{x}) e^{+iE_f t} V(t, \vec{x}) \phi(\vec{x}) e^{-iE_m t} \mathrm{d}^3 \vec{x} \tag{3.12}
$$

and simplify further using

$$
\int \phi_f^*(\vec{x}) \phi_m(\vec{x}) \mathrm{d}^3 \vec{x} = \delta_{fm} \ . \tag{3.13}
$$

This gives us a series of coupled equations

$$
\frac{\partial a_f(t)}{\partial t} = -i \sum_m a_m(t) e^{i(E_f - E_m)t} \int \phi_f^*(\vec{x}) V(t, \vec{x}) \phi_m(\vec{x}) \mathrm{d}^3 \vec{x} \tag{3.14}
$$

from which we can define

$$
V_{fm} = \int \phi_f^*(\vec{x}) V(t, \vec{x}) \phi_m(\vec{x}) \mathrm{d}^3 \vec{x} \ . \tag{3.15}
$$

It is not possible to solve the system exactly, but we can exploit the weakness of the potential and choose to pull out a small scalar coupling constant κ, *i.e.* to replace $V(t, \vec{x})$ by $\kappa V(t, \vec{x})$ and V_{fm} by κV_{fm}. We can also express the $a_m(t)$ as a power series in terms of this small coupling constant as

$$
a_m(t) = \sum_j (\kappa)^j a_m^j(t) \ , \tag{3.16}
$$

where

$$
\frac{\mathrm{d} a_m^0}{\mathrm{d} t} = 0 \ . \tag{3.17}
$$

After comparing powers of κ, we have

$$
\frac{\mathrm{d} a_f^{j+1}(t)}{\mathrm{d} t} = -i \sum_m V_{fm} e^{i(E_f - E_m)t} a_m^j(t) \tag{3.18}
$$

At zeroth order

$$
\frac{\mathrm{d} a_f^0(t)}{\mathrm{d} t} = 0 \ \Rightarrow a_f = \delta_{fi} \tag{3.19}
$$

i.e. there is no interaction. At first order

$$\frac{\mathrm{d}a_f^1(t)}{\mathrm{d}t} = -i \sum_m V_{fm} e^{i(E_f - E_m)t} a_f^0 \tag{3.20}$$

$$= -i \sum_m V_{fm} e^{i(E_f - E_m)t} \delta_{mi} \ , \tag{3.21}$$

where we have substituted in the zeroth order result. This gives

$$a_f^1(t) = -i \int_{-\infty}^{t} V_{fi} e^{i(E_f - E_i)t'} \mathrm{d}t' \ . \tag{3.22}$$

At second order

$$\frac{\mathrm{d}a_f^2(t)}{\mathrm{d}t} = -i \sum_m \int_{-\infty}^{t} \mathrm{d}t' V_{fm} e^{i(E_f - E_m)t'} a_m^1(t') \tag{3.23}$$

$$a_f^2(t) = (-i)^2 \int_{-\infty}^{t} V_{fm} e^{i(E_f - E_m)t'} \mathrm{d}t' \int_{-\infty}^{t'} V_{mi} e^{i(E_m - E_i)t''} \mathrm{d}t'' \ . \tag{3.24}$$

where we have now substituted in the first order result. This process can then be repeated for higher-and-higher orders in κ. It is also important to note that this process is time-ordered.

If the potential is time-independent, i.e. $V = V(\vec{x})$, then

$$a_f^2(t) = -2\pi i \delta(E_f - E_i) \sum_{m \neq i} V_{fm} \frac{1}{E_i - E_m + i\varepsilon} V_{mi} \ , \tag{3.25}$$

which is integrated by introducing a small complex parameter $E_m - E_i \to E_m - E_i - i\varepsilon$ (where $\epsilon \to 0$). The zeroth, first and second order processes are shown schematically in Fig. 3.2. The first order contribution gives us an interaction with $f \neq i$.
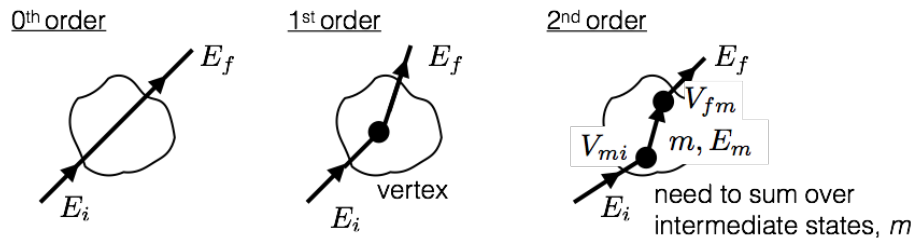

Figure 3.2: Zeroth, first and second-order contributions to the $i \to f$ transition.

As a simplification we can take the system to be in state i at $t = -T/2$

$$a_i(-T/2) = 1 \tag{3.26}$$
$$a_n(-T/2) = 0 \ \text{ for } n \neq i$$

which gives

$$\frac{\mathrm{d}a_f}{\mathrm{d}t} = -i \int \phi_f^*(\vec{x}) V \phi_i(\vec{x}) e^{i(E_f - E_i)t} \mathrm{d}^3\vec{x} \tag{3.27}$$

and at time $T/2$ after the interaction has ceased,

$$T_{fi} \equiv a_f(T/2) = -i \int_{-T/2}^{T/2} \phi_f^*(\vec{x}) V \phi_i(\vec{x}) e^{i(E_f - E_i)t} \mathrm{d}^3\vec{x}\,\mathrm{d}t \tag{3.28}$$

or more generally in a covariant form

$$T_{fi} = a_f(t) = -i \int \phi_f^*(x) V(x) \phi_i(x) \mathrm{d}^4 x \tag{3.29}$$

We would like to interpret $|T_{fi}|^2$ as the probability that a particle is scattered from state i into state f. For the time-independent potential

$$T_{fi} = -iV_{fi} \int_{-T/2}^{T/2} e^{i(E_f - E_i)t} \mathrm{d}t \tag{3.30}$$

such that

$$|T_{fi}|^2 = 4|V_{fi}|^2 \frac{\sin^2(\frac{T}{2}(E_f - E_i))}{(E_f - E_i)^2} \tag{3.31}$$

This function is tightly peaked around $E_f = E_i$. If we take $T \to \infty$

$$T_{fi} = 2\pi i V_{fi} \delta(E_f - E_i) \tag{3.32}$$

the δ-function expresses energy conservation in the transition. Typically we want to know the transition rate from a known initial state to a group of final states (e.g. a particles decay width),

$$|T_{\mathrm{tot},fi}|^2 = \sum_f |T_{fi}|^2 = \int |T_{fi}|^2 \mathrm{d}N(E_f) = \int |T_{fi}|^2 \rho(E_f) \mathrm{d}E_f \tag{3.33}$$

Assuming $\rho(E_f)$ and V_{fi} are constant over a narrow integration window,

$$|T_{\mathrm{tot},fi}|^2 = 4\rho(E_f)|V_{fi}|^2 \int \frac{\sin^2((E_f - E_i)\frac{T}{2})}{(E_f - E_i)^2} \mathrm{d}E_f \tag{3.34}$$
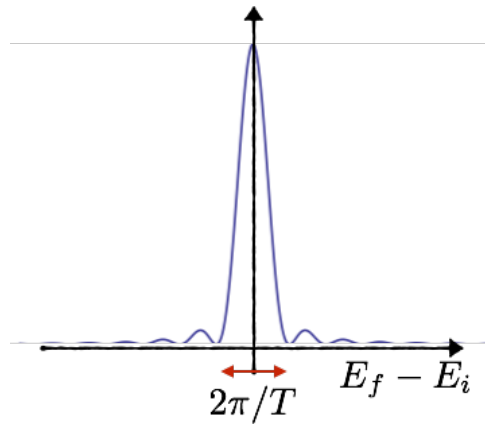
Figure 3.3: $|T_{fi}|^2$ for fixed T as a function of $E_f - E_i$. It is sharply peaked at $E_f \approx E_i$.

$$\lim_{T \to \infty} |T_{\text{tot},fi}|^2 = 2\pi T |V_{fi}|^2 \rho(E_f) \tag{3.35}$$

T appears on the right-hand side so this is not a transition rate. We can instead define a transition rate per unit time as

$$W = \lim_{T \to \infty} \frac{|T_{\text{tot},fi}|^2}{T} = 2\pi |V_{fi}|^2 \rho(E_f) \tag{3.36}$$

This is Femi's golden rule. We made a lot of approximations to get here, but as long as the interaction is weak this rule holds. In the QFT course, you will re-visit time-dependent perturbation theory using operators and the Dirac interaction picture.

Lecture 4

Coulomb scattering of charged spin-0 particles

4.1   Motion of a charged particle in an EM field

We have already seen that free spin-0 particles can be described by the Klein-Gordon equation

$$(\partial^\mu \partial_\mu + m^2)\phi = 0 \,. \tag{4.1}$$

In classical EM, the motion of a particle with charge $(-e)$ in a field A^μ can be obtained by substituting

$$p^\mu \to p^\mu + eA^\mu \tag{4.2}$$

The QM analogue is then

$$i\partial^\mu \to i\partial^\mu + eA^\mu \tag{4.3}$$

This is Quantum-Electro-Dynamics (QED). Plugging this into the KG equation gives

$$\left[(\partial^\mu - ieA^\mu)(\partial_\mu - ieA_\mu) + m^2) \right] \phi = 0$$
$$\left[\partial^\mu \partial_\mu - e^2 A^\mu A_\mu - ie(\partial^\mu A_\mu + A^\mu \partial_\mu) + m^2 \right] \phi = 0 \tag{4.4}$$

where $ie\partial^\mu A_\mu$ acts on ϕ, *i.e.* $ie\partial^\mu(A_\mu \phi)$. This can be written in the form of an equation of motion that depends on a potential

$$(\partial^\mu \partial_\mu + m^2)\phi = -V\phi \,, \tag{4.5}$$

where

$$V = -ie(\partial^\mu A_\mu + A^\mu \partial_\mu) - e^2 A^\mu A_\mu \,. \tag{4.6}$$

This is the interaction potential for the charged particle in the field. Since $e \ll 1$,

$$\alpha_{\text{EM}} = \frac{e^2}{4\pi} \sim \frac{1}{137} \,, \tag{4.7}$$

we can simplify the expression by taking only the first order terms

$$V \approx -ie(\partial^\mu A_\mu + A^\mu \partial_\mu) \tag{4.8}$$

We now want to compute the transition current

$$T_{fi} = -i \int \phi_f^*(x) V(x) \phi_i(x) \mathrm{d}^4 x \tag{4.9}$$

$$T_{fi} = i^2 \int \phi_f^*(x) e(A^\mu \partial_\mu + \partial_\mu A^\mu) \phi_i(x) \mathrm{d}^4 x$$
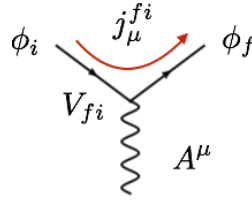
Figure 4.1: A spineless particle interacting with a field A^μ.

This can be integrated by parts

$$\int\limits_{-\infty}^{+\infty} \phi_f^*(x)\partial^\mu(A_\mu\phi_i(x))\mathrm{d}^4x = \phi_f^* A_\mu\phi_i\Big|_{-\infty}^{+\infty} - \int\limits_{-\infty}^{+\infty}(\partial^\mu\phi_f^*(x)A_\mu\phi_i(x)\mathrm{d}^4x \tag{4.10}$$

Assuming $A_\mu \to 0$ as $x_\mu \to \infty$

$$\phi_f^* A_\mu\phi_i\Big|_{-\infty}^{+\infty} \to 0 \tag{4.11}$$

Giving

$$T_{fi} = -i\int(-ie)(\phi_f^*\partial_\mu\phi_i - \phi_i\partial_\mu\phi_f^*)A^\mu\mathrm{d}^4x \tag{4.12}$$

$$\tag{4.13}$$

This should look familiar, it is the Klein-Gordon current, only having different states for ϕ_i and ϕ_f. From this we can define a transition current

$$j_\mu^{fi} = -ie(\phi_f^*\partial_\mu\phi_i - \phi_i\partial_\mu\phi_f^*) \tag{4.14}$$

and write

$$T_{fi} = -i\int j_\mu^{fi}A^\mu\mathrm{d}^4x \tag{4.15}$$

$$\tag{4.16}$$

Now we can look at the initial and final states. These are both free spin-0 particles, which obey the Klein-Gordon equation with solutions

$$\phi = Ne^{-ip\cdot x} \tag{4.17}$$

Using

$$\partial_\mu\phi = -ip_\mu\phi \tag{4.18}$$

the current takes the form

$$j_\mu^{fi} = -e(p_i + p_f)_\mu\phi_f^*\phi_i \tag{4.19}$$

The final ingredient we need is the field A^μ. For coulomb scattering the source of this field is the other particle. We already know that there will be a transition current associated with this particle, $j_\mu^{(2)}$, and can use Maxwell's equations to find an expression for the field,

$$\partial^\mu \partial_\mu A^\mu = j_{(2)}^\mu = -ie(\phi_D^* \partial^\mu \phi_B - \phi_B \partial^\mu \phi_D^*) \tag{4.20}$$
$$= -e(p_B + p_D)^\mu N_B N_D e^{i(p_D - p_B)\cdot x}$$
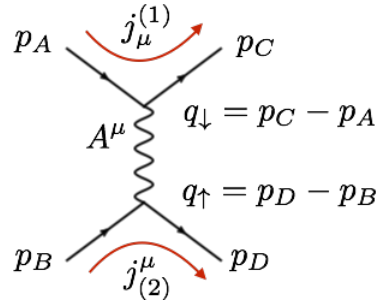
We often write $p_D - p_B = q$.

Figure 4.2: Two particles scattering off each other. The particle four-momentum are indicated by p_A, p_B, p_C and p_D. For coulomb scattering the internal line corresponds to the exchange of a photon.

By realising that

$$\partial^\mu \partial_\mu e^{-iq\cdot x} = -q^2 e^{-iq\cdot x} \tag{4.21}$$

we can express A^μ in terms of $j_{(2)}^\mu$ as

$$A^\mu = -\frac{j_{(2)}^\mu}{q^2} \ . \tag{4.22}$$

Finally, putting the ingredients together

$$T_{fi} = -i \int j_\mu^{(1)} \left(-\frac{1}{q^2}\right) j_{(2)}^\mu \mathrm{d}^4 x \tag{4.23}$$

and plugging in the the plane wave solutions to the Klein-Gordon equation

$$j_\mu^{(1)} = -eN_A N_C (p_A + p_C)_\mu e^{i(p_C - p_A)\cdot x} \tag{4.24}$$
$$j_{(2)}^\mu = -eN_B N_D (p_B + p_D)_\mu e^{i(p_D - p_B)\cdot x}$$

$$\tag{4.25}$$

gives

$$T_{fi} = -ie^2 N_A N_B N_C N_D (p_A + p_C)_\mu \frac{1}{q^2} (p_B + p_D)^\mu$$
$$\times \int e^{i(p_C - p_A + p_D - p_B) \cdot x} \mathrm{d}^4 x \tag{4.26}$$

The integral results in the δ-function

$$\int e^{i(p_C - p_A + p_D - p_B) \cdot x} \mathrm{d}^4 x = (2\pi)^4 \delta^4 (p_C - p_A + p_D - p_B), \tag{4.27}$$

which ensures 4-momentum conservation. We can also pull out an Lorentz invariant amplitude

$$-i\mathcal{M} = ie(p_A + p_C)_\mu \left(-\frac{ig^{\mu\nu}}{q^2} \right) ie(p_B + p_D)_\nu \tag{4.28}$$

It is interesting to note that in this process $q^2 \neq 0$, which implies that $m_\gamma \neq 0$. The photon is a "virtual" or "off-mass-shell" particle.

4.2   Feynman rules for spin-0 particles

The building blocks we need to perform Coulomb scattering calculations are:

Element	Description	Factor
———•	external spin 0 particle	1
•———•	internal spin 0 particle	$\dfrac{1}{q^2 - m^2}$
∿∿∿•	external photon	$\varepsilon_\mu^*(p, \lambda)$
•∿∿∿•	internal photon	$\dfrac{-ig^{\mu\nu}}{q^2}$
	vertex	$ie(p_1 + p_2)^\mu$

Figure 4.3: Feynman rules for charged spin-0 particles

External photons are accompanied by a polarisation vector $\varepsilon_\mu^*(p, \lambda)$, where p is the photon momentum and λ is the spin projection of the photon along $\vec{p}$. We won't derive the

origin of this factor in these notes. There are also two different internal lines depending on whether a photon or a spin-0 particle are exchanged. We refer to these internal lines as propagators.

4.3   Propagators and time-ordering

If we look again at the diagram in Fig 4.2 we actually have a diagram with two vertices, *i.e.*

$$-i\mathcal{M} \propto e^2 \ . \tag{4.29}$$

This process should correspond to a second order term in the perturbative expansion. Recalling, Eq. 3.25, this should be related to

$$-2\pi i\delta(E_f - E_i) \sum_{m \neq i} V_{fm} \frac{1}{E_i - E_m + i\varepsilon} V_{mi} \ . \tag{4.30}$$

The term

$$\frac{1}{E_i - E_m} \tag{4.31}$$

in the second order expression looks a lot like a propagator. To make the connection to the propagator its important to realise that when we draw a Feynman diagram we really mean a sum over different possible time-ordered diagrams. When we do the calculation we integrate over space-time and enforce energy conservation for the external particles.
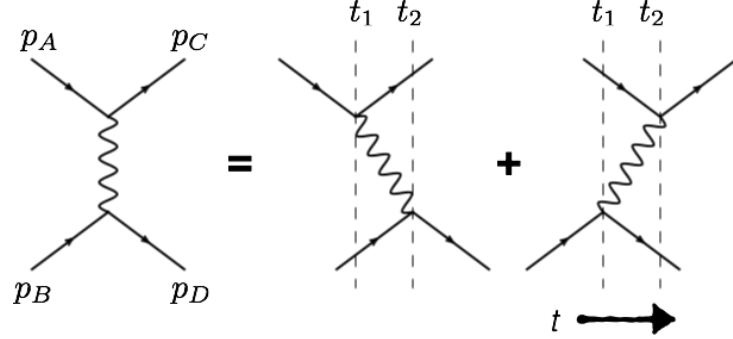
Figure 4.4: Feynman diagrams are a sum over the different possible time-ordered diagrams.

For the first time ordering in Fig. 4.4, the energy of the intermediate state is

$$E_m = E_C + E_B + \sqrt{|\vec{p}_A - \vec{p}_C|^2 + m^2} \ , \tag{4.32}$$

where the exchanged particle has energy $E_P = \sqrt{|\vec{p}_A - \vec{p}_C|^2 + m^2}$ and mass m. The energy of the initial state is

$$E_i = E_A + E_B \ . \tag{4.33}$$

This gives

$$E_i - E_m = (E_A + E_B) - E_C - E_B - \sqrt{|\vec{p}_A - \vec{p}_C|^2 + m^2} \tag{4.34}$$
$$= (E_A - E_C) - \sqrt{|\vec{p}_A - \vec{p}_C|^2 + m^2} \ .$$

The same exercise for the second time ordering in Fig. 4.4 gives

$$E_m = E_A + E_D + \sqrt{|\vec{p}_B - \vec{p}_D|^2 + m^2} \tag{4.35}$$

and

$$E_i - E_m = -(E_A - E_C) - \sqrt{|\vec{p}_A - \vec{p}_C|^2 + m^2} \ . \tag{4.36}$$

Putting the two parts together

$$\frac{1}{(E_i - E_m)_{(1)}} + \frac{1}{(E_i - E_m)_{(2)}} = \frac{2\sqrt{|\vec{p}_A - \vec{p}_C|^2 + m^2}}{(p_A - p_C)^2 - m^2} = \frac{2\,E_P}{q^2 - m^2} \tag{4.37}$$

The transition amplitude, T_{fi}, is then proportional to the propagator we expected. You can also see how this relates to the propagator for an internal spin-0 particle.

Lecture 5

Observable quantities

5.1 Box normalisation

From the Klein-Gordon equation, the plane wave solutions for a free particle are

$$\phi = N e^{-ip \cdot x} \, , \tag{5.1}$$

which correspond to a current

$$j^\mu = 2|N|^2 p^\mu \tag{5.2}$$

such that the density is

$$\rho = j^0 = 2E|N|^2 \, . \tag{5.3}$$

The proportionality of ρ to E is needed to compensate for the Lorentz contraction of the volume element $\mathrm{d}^3\vec{x}$, leaving the number of particles $\rho \mathrm{d}^3\vec{x}$ unchanged. We usually choose a normalisation of $2E$ particles in the arbitrary normalisation volume, V,

$$\rho = \frac{2E}{V} \quad \Rightarrow \quad N = \frac{1}{\sqrt{V}} \, . \tag{5.4}$$

We will see later that V drops out in our calculations and you will often see this written as $\rho = 2E$. The density of states is given by

$$\mathrm{d}n = \rho(E)\mathrm{d}E \, . \tag{5.5}$$

$$p_x = \frac{2\pi}{L} n_x$$

$$n_x \epsilon \ \mathbb{Z}$$

$$L$$
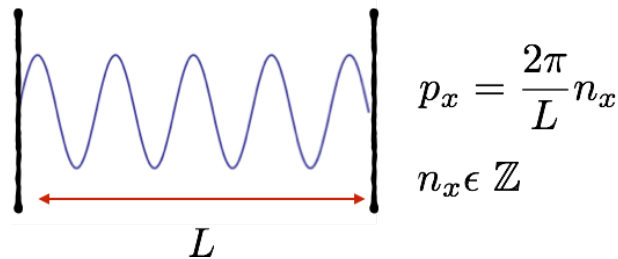
Figure 5.1: Box normalistion of states. States exist in box with sidelength L.

If the states exist in a box of side L, we know from the infinite square well problem in Quantum Mechanics that in one dimension (x),

$$p_x = \frac{2\pi}{L} n_x \ ,$$

(5.6)

where n_x is an integer. This gives

$$\mathrm{d}p_x = \frac{2\pi}{L} \mathrm{d}n_x \ .$$

(5.7)

The total number of states in a cube of side L with momentum between $\vec{p}$ and $\vec{p} + \mathrm{d}^3\vec{p}$ is then

$$\mathrm{d}n = \mathrm{d}n_x \mathrm{d}n_y \mathrm{d}n_z$$

(5.8)

$$= \left(\frac{L}{2\pi}\right)^3 \mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z$$

$$= \left(\frac{L}{2\pi}\right)^3 \mathrm{d}^3\vec{p} = \frac{V}{(2\pi)^3} \mathrm{d}^3\vec{p}$$

We have $2E$ particles in the volume V, so the number of states per particle is

$$\mathrm{d}n = \frac{V}{(2\pi)^3 2E} \mathrm{d}^3\vec{p} \ .$$

(5.9)

5.2   Computing observable quantities from invariant amplitudes

The task is now to turn the Lorentz covariant amplitude $\mathcal{M}$ into a measurable quantity. In particle physics we typically want to compute:

1. Cross-sections, σ, for a process (e.g. $A + B \rightarrow C + D$)

2. Decay rates, Γ, for a particle (e.g. $A \rightarrow B + C$)

To compute σ or Γ, we start with T_{fi} written in terms of $\mathcal{M}$

$$T_{fi} = -i\mathcal{M}\delta^4(p_f - p_i) \prod_f N_f \prod_i N_i \tag{5.10}$$

Here, p_f and p_i are the final and initial state 4-momentum summing over all of the particles. The N_f and N_i are the normalisations associated to each of the initial and final state particles. The δ-function ensures energy and momentum conservation in the process.
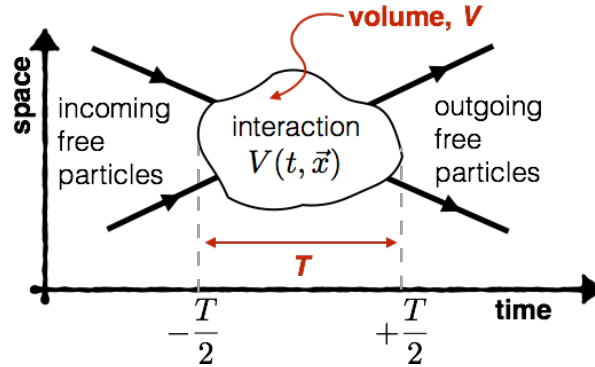

Figure 5.2: Illustration of a typical scattering problem in High Energy Physics, occuring in a local volume V over time T.

We can turn this amplitude into a probability for a transition from $i \rightarrow f$ by taking its square,

$$p_{fi} = |T_{fi}|^2 = \left| -i\mathcal{M}\delta^4(p_f - p_i) \prod_f N_f \prod_i N_i \right|^2 . \tag{5.11}$$

This includes the square of a δ-function. To deal with this can replace the δ-function with the identity

$$(2\pi)^4 \delta^4(p_f - p_i) = \int e^{i(p_f - p_i)\cdot x} \mathrm{d}^4 x \,, \tag{5.12}$$

such that

$$|(2\pi)^4 \delta^4(p_f - p_i)|^2 \approx (2\pi)^4 \delta^4(p_f - p_i) \int_{T,V} e^{i(p_f - p_i)\cdot x} \mathrm{d}^4 x \tag{5.13}$$

where the integral is over the volume of the interaction region, V, and the extent of the interaction in time, T. The first δ-function forces $p_f = p_i$, so this is simply

$$|(2\pi)^4 \delta^4(p_f - p_i)|^2 \approx VT (2\pi)^4 \delta^4(p_f - p_i) \tag{5.14}$$

Putting the ingredients back into $|T_{fi}|^2$,

$$\begin{aligned} |T_{fi}|^2 &= |\mathcal{M}|^2 VT (2\pi)^4 \delta^4(p_f - p_i) \prod_f |N_f|^2 \prod_i |N_i|^2 \\ &= |\mathcal{M}|^2 VT (2\pi)^4 \delta^4(p_f - p_i) \frac{1}{V^{n_f}} \frac{1}{V^{n_i}} \end{aligned} \tag{5.15}$$

where n_f and n_i are the number of initial and final state particles. For a process involving $A + B \to C + D$, we would have

$$\begin{aligned} |T_{fi}|^2 &= |\mathcal{M}|^2 VT (2\pi)^4 \delta^4(p_C + p_D - p_A - p_B)|N_A|^2 |N_B|^2 |N_C|^2 |N_D|^2 \\ &= |\mathcal{M}|^2 VT (2\pi)^4 \delta^4(p_C + p_D - p_A - p_B) \frac{1}{V^4} \end{aligned} \tag{5.16}$$

The transition rate per unit volume can be computed from $|T_{fi}|^2$ as

$$W_{fi} = \frac{|T_{fi}|^2}{VT} = |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_f - p_i) \frac{1}{V^{n_f}} \frac{1}{V^{n_i}} \,. \tag{5.17}$$

The transition rate per unit volume into a small fixed final state phase-space is

$$\mathrm{d}W = W_{fi} \, \mathrm{d}n \tag{5.18}$$

$$= |\mathcal{M}|^2 \frac{1}{V^{n_i}} \frac{1}{V^{f_i}} (2\pi)^4 \delta^4(p_f - p_i) \prod_f V \frac{\mathrm{d}^3 \vec{p}}{2 E_f (2\pi)^3}$$

$$= |\mathcal{M}|^2 \frac{1}{V^{n_i}} (2\pi)^4 \delta^4(p_f - p_i) \prod_f \frac{\mathrm{d}^3 \vec{p}}{2 E_f (2\pi)^3}$$

$$\tag{5.19}$$

We call

$$\mathrm{d}Q = (2\pi)^4 \delta^4(p_f - p_i) \prod_f \frac{\mathrm{d}^3 \vec{p}}{2 E_f (2\pi)^3} \tag{5.20}$$

the Lorentz invariant phase-space.

5.3   Decay rates

Lets now consider a decay $A \to B + C$ and compute

$$\text{decay rate} = \frac{\text{decays per unit time}}{\text{decaying particles}} \tag{5.21}$$

from the number of transitions per unit volume per unit time, $\mathrm{d}W$. The number of particles of type A per unit volume is $2E_A/V$, yielding the differential decay rate

$$\mathrm{d}\Gamma = \mathrm{d}W \, \frac{V}{2E_A} \tag{5.22}$$

$$= \frac{V}{2E_A} |\mathcal{M}|^2 \frac{1}{V} (2\pi)^4 \delta^4(p_A - p_B - p_C) \frac{\mathrm{d}^3\vec{p}_B}{2E_B(2\pi)^3} \frac{\mathrm{d}^3\vec{p}_C}{2E_C(2\pi)^3} \ .$$

The size of the arbitrary normalisation volume, V, drops out. The differential decay rate, $\mathrm{d}\Gamma$ is covariant so we can pick a frame to make this calculation easier. In the rest frame of A,

$$p_A = (m_A, 0) \quad , \quad p_B = (E_B, \vec{p}) \quad , \quad p_C = (E_C, -\vec{p}) \ . \tag{5.23}$$

The decay rate is then

$$\Gamma = \int \frac{1}{8(2\pi)^2 m_A} |\mathcal{M}|^2 \delta^4(p_A - p_B - p_C) \frac{\mathrm{d}^3\vec{p}_B}{E_B} \frac{\mathrm{d}^3\vec{p}_C}{E_C} \ . \tag{5.24}$$

We can integrate over one of the final state momenta using the δ-function to fix $\vec{p}_B = -\vec{p}_C$. This gives

$$\Gamma = \frac{1}{8(2\pi)^2 m_A} \int |\mathcal{M}|^2 \delta(m_A - E_B - E_C) \frac{\mathrm{d}^3\vec{p}}{E_B E_C} \ . \tag{5.25}$$

To put this into a more usable form we can use polar coordinates

$$\mathrm{d}^3\vec{p} = |\vec{p}|^2 \mathrm{d}|\vec{p}| \mathrm{d}\Omega \ . \tag{5.26}$$

We can also use the relativistic energy momentum relationship

$$E^2 = |\vec{p}|^2 + m^2 \tag{5.27}$$

to write

$$\mathrm{d}E = \frac{|\vec{p}|\mathrm{d}|\vec{p}|}{E} \quad \text{and} \quad \mathrm{d}(E_B + E_C) = \frac{|\vec{p}_B|\mathrm{d}|\vec{p}_B|}{E_B} + \frac{|\vec{p}_C|\mathrm{d}|\vec{p}_C|}{E_C} \tag{5.28}$$

but since $|\vec{p}_B| = |\vec{p}_C|$,

$$\mathrm{d}(E_B + E_C) = \frac{|\vec{p}|\mathrm{d}|\vec{p}|}{E_B} + \frac{|\vec{p}|\mathrm{d}|\vec{p}|}{E_C} \tag{5.29}$$

$$\tag{5.30}$$

rearranging gives

$$|\vec{p}|\mathrm{d}|\vec{p}| = \frac{E_B E_C}{E_B + E_C}\mathrm{d}(E_B + E_C) \ . \tag{5.31}$$

We can use this to write

$$\frac{\mathrm{d}^3\vec{p}}{E_B E_C} = \frac{|\vec{p}|^2\mathrm{d}|\vec{p}|\mathrm{d}\Omega}{E_B E_C} = \frac{|\vec{p}|\mathrm{d}(E_B + E_C)\mathrm{d}\Omega}{E_B + E_C} \ , \tag{5.32}$$

and get

$$\Gamma = \frac{1}{8(2\pi)^2 m_A}\int|\mathcal{M}|^2\delta(m_A - E_B - E_C)\frac{|\vec{p}|\mathrm{d}(E_B + E_C)\mathrm{d}\Omega}{E_B + E_C} \tag{5.33}$$

which can again be integrated by exploiting the δ-function to fix $m_A = E_B + E_C$. Finally we arrive at

$$\Gamma = \frac{1}{32\pi^2 m_A^2}\int|\mathcal{M}|^2|\vec{p}|\mathrm{d}\Omega \tag{5.34}$$

5.4   Cross sections

We can also compute the cross-section for a scattering process $A + B \to C + D$. Lets start by considering a beam of particles A incident on a target of type B. The number of particles of type A passing through unit area per unit time is

$$|\vec{v}_A|\frac{2E_A}{V} \ , \tag{5.35}$$

where $\vec{v}_A$ is the velocity of the particles in the beam of A,

$$\vec{v}_A = \frac{\vec{p}_A}{E_A}. \tag{5.36}$$

The number of particles per unit volume in the target is

$$\frac{2E_B}{V} \ . \tag{5.37}$$

The cross-section for the process is then given by

$$\sigma(A + B \to C + D) = W_{fi} \times \frac{\text{number of final states}}{\text{initial flux}} \ , \tag{5.38}$$

where we have

$$W_{fi} = |T_{fi}|^2 = (2\pi)^4\delta^4(p_C + p_D - p_A - p_D)\frac{|\mathcal{M}|^2}{V^4} \tag{5.39}$$

The initial flux of particles in the lab-frame is

$$|\vec{v}_A|\frac{2E_A}{V}\frac{2E_B}{V},\tag{5.40}$$

and the number of final states between $\vec{p}_C$ and $\vec{p}_C + \mathrm{d}^3\vec{p}_C$ and between $\vec{p}_D$ and $\vec{p}_D + \mathrm{d}^3\vec{p}_D$ is

$$\frac{V\mathrm{d}^3\vec{p}_C}{(2\pi)^3 2E_C}\frac{V\mathrm{d}^3\vec{p}_D}{(2\pi)^3 2E_D}\ .\tag{5.41}$$

This gives

$$\mathrm{d}\sigma = \frac{1}{4E_A E_B |\vec{v}_A|}|\mathcal{M}|^2\frac{1}{(2\pi)^2}\delta^4(p_C + p_D - p_A - p_D)\frac{\mathrm{d}^3\vec{p}_C}{2E_C}\frac{\mathrm{d}^3\vec{p}_D}{2E_D}\ ,\tag{5.42}$$

where once again the arbitrary normalisation volume cancels. It's useful to write this in terms of the Lorentz invariant phase-space

$$\mathrm{d}\sigma = \frac{|\mathcal{M}|^2}{F}\mathrm{d}Q\ ,\tag{5.43}$$

where F is the incident flux

$$F = |\vec{v}_A|2E_A 2E_B\ .\tag{5.44}$$

More generally, for a collinear collision between A and B,

$$F = |\vec{v}_A - \vec{v}_B|2E_A 2E_B\ .\tag{5.45}$$

From which can can derive,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2(p_A + p_B)^2}\frac{|\vec{p}_C|}{|\vec{p}_A|}|\mathcal{M}|^2\ .\tag{5.46}$$

5.5   Mandelstam variables

At this point it is useful to introduce the kinematic variables

$$\begin{aligned}
s &= (p_A + p_B)^2 = (p_C + p_D)^2\\
t &= (p_A - p_C)^2 = (p_B - p_D)^2\\
u &= (p_A - p_D)^2 = (p_B - p_C)^2
\end{aligned}\tag{5.47}$$

These are called Mandelstam variables. They will appear in scattering calculations and are Lorentz covariant, they can be used to describe a process in a frame independent way.

Lecture 6

Calculating cross-sections
for spin-0 particle scattering

6.1   Spinless scattering: $e^- \mu^- \to e^- \mu^-$

The procedure to compute a scattering cross-section is the same for all processes:

1. At a given order, draw all possible Feynman diagrams for the process.

2. Using the Feynman rules, label the elements in each diagram with the appropriate factor.

3. Form an invariant amplitude $-i\mathcal{M}$ by multiplying all of the factors in the proper order. This is the order which follows the lines in the diagram against the direction of the particle flow.

4. Compute the transition probability from $|\mathcal{M}|^2$.

The transition probability can then be used to calculate cross-sections or decay rates using the formulæ we derived earlier.

The first concrete example we'll work through is spin-0 $e^- \mu^- \to e^- \mu^-$ scattering. This has a single Feynamn diagram at the lowest order, see Fig. 6.1. In reality, we know that electrons and muons are both spin-$\frac{1}{2}$ particles and we'll do the full calculation later in this course.
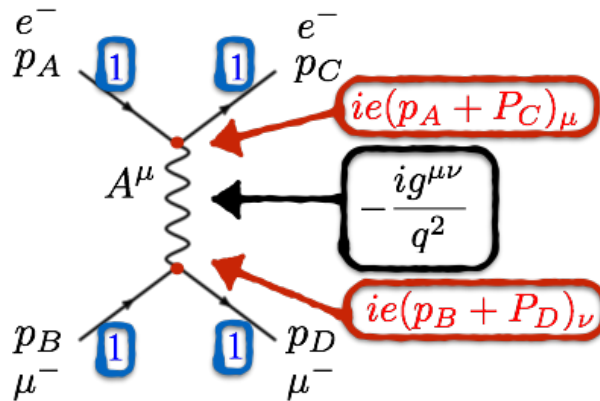

Figure 6.1: Feynman diagram for electron-muon scattering.

From the diagram we can write the invariant transition amplitude as

$$-i\mathcal{M}_{e^- \mu^-} = [1][ie(p_A + p_C)_\mu][1]\left[\frac{-ig^{\mu\nu}}{q^2}\right][1][ie(p_B + p_D)_\nu][1] \tag{6.1}$$

$$= ie^2(p_A + p_C)_\mu(p_B + p_D)^\mu \frac{1}{(p_A - p_C)^2} \ ,$$

where we have used $q = p_A - p_C$. If there were more diagrams, we would sum over them in the amplitude. The transition probability

$$|\mathcal{M}_{e^-\mu^-}|^2 = e^4 \left(\frac{(p_A + p_C)_\mu (p_B + p_D)^\mu}{(p_A - p_C)^2} \right)^2 \tag{6.2}$$

$$= e^4 \left(\frac{p_A \cdot p_B + p_A \cdot p_D + p_C \cdot p_B + p_C \cdot p_D}{t} \right)^2$$

We can now write this expression in terms of the Mandelstam variables

$$\begin{aligned} s &= (p_A + p_B)^2 = (p_C + p_D)^2 \\ &= p_A^2 + p_B^2 + 2p_A \cdot p_B = m_A^2 + m_B^2 + 2p_A \cdot p_B \\ &= p_C^2 + p_B^2 + 2p_C \cdot p_D = m_C^2 + m_D^2 + 2p_C \cdot p_D \end{aligned} \tag{6.3}$$

$$\begin{aligned} u &= (p_A - p_D)^2 = (p_B - p_C)^2 \\ &= p_A^2 + p_D^2 - 2p_A \cdot p_D \\ &= p_B^2 + p_C^2 - 2p_B \cdot p_C \ , \end{aligned}$$

as

$$\begin{aligned} p_A \cdot p_B + p_A \cdot p_D + p_C \cdot p_B + p_C \cdot p_D = \\ \tfrac{1}{2}(s - m_A^2 - m_B^2) + \tfrac{1}{2}(s - m_C^2 - m_D^2) - \tfrac{1}{2}(u - m_A^2 - m_D^2) - \tfrac{1}{2}(u - m_B^2 - m_C^2) \end{aligned} \tag{6.4}$$

such that

$$|\mathcal{M}_{e^-\mu^-}|^2 = e^4 \left(\frac{s - u}{t} \right)^2 \ .$$

This finally brings us to the expression for the differential cross-section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{e^4}{64\pi^2 s} \frac{|\vec{p}_C|}{|\vec{p}_A|} \left(\frac{s - u}{t} \right)^2 \ . \tag{6.5}$$

In the centre of mass system $|\vec{p}_A| = |\vec{p}_C|$. Further, in the very high energy limit $m_e \ll E_e$ and $m_\mu \ll E_\mu$ and the differential cross-section becomes

$$\left. \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right|_{\mathrm{CM}} = \frac{e^4}{64\pi^2 s} \left(\frac{3 + \cos\theta}{1 - \cos\theta} \right)^2 \ , \tag{6.6}$$

where, θ is the angle between $\vec{p}_A$ and $\vec{p}_C$. The cross-section diverges as $\theta \to 0$.
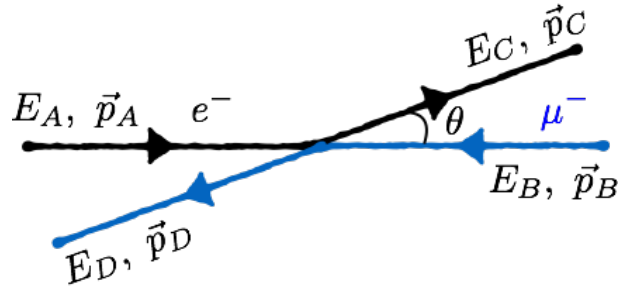
Figure 6.2: Centre of mass system for $e^-\mu^- \to e^-\mu^-$ scattering.

6.2   Spinless scattering: $e^-e^- \to e^-e^-$

Now lets consider a slightly more complex situation where we have electron-electron scattering, $e^-e^- \to e^-e^-$. The complication here is that we have identical particles in the initial and final state. We therefore cannot distinguish C from D (and A from B). The resulting amplitude should be symmetric under the exchanges $A \leftrightarrow B$ and $C \leftrightarrow D$. The invariant amplitude is the sum of the two diagrams in Fig. 6.3. The first diagram is the same one we had for $e^-\mu^- \to e^-\mu^-$ scattering. The second diagram interchanges C and D.
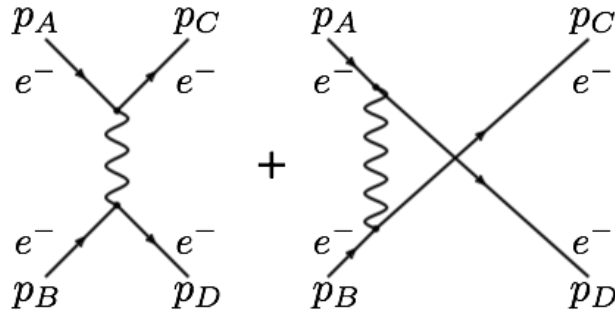
Figure 6.3: Diagrams for $e^-e^- \to e^-e^-$ scattering. Note, in the second diagram the two lines do not cross.

The invariant amplitude corresponding to these diagrams is

$$-i\mathcal{M}_{e^-e^-} = ie^2 \frac{(p_A + p_C)_\mu (p_B + p_D)^\mu}{(p_A - p_C)^2} + ie^2 \frac{(p_A + p_D)_\mu (p_B + p_C)^\mu}{(p_A - p_D)^2} \ . \tag{6.7}$$

It's easy to see that this is invariant under $C \to D$. It's also invariant under $B \to D$. This

can be written using the Mandelstam variables as

$$-i\mathcal{M}_{e^-e^-} = ie^2 \left(\frac{s-u}{t} + \frac{s-t}{u} \right) \qquad (6.8)$$

6.3 Infrared divergences

The invariant amplitude for $e^-e^- \to e^-e^-$ scattering diverges as $t \to 0$ and $u \to 0$. This corresponds to a divergence at scattering angles close to zero and π. The Mandelstam variables t and u in this case correspond to the 4-momentum transferred by the virtual photon and $t(u) \to 0$ implies $q^2 \to 0$ such that the photon approaches the *on mass-shell* condition. To solve the problem of divergences in the theory, it is important to both include higher order diagrams and to renormalise the theory. You will come across this in later courses.
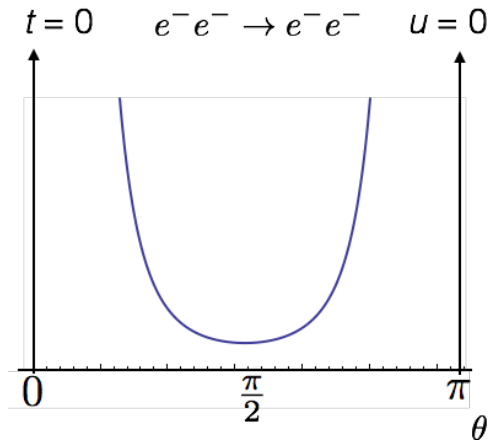

Figure 6.4: Differential cross-section for $e^-e^- \to e^-e^-$ scattering as a function of the scattering angle, θ.

6.4   Spinless scattering: $e^+e^- \rightarrow e^+e^-$

Up-to now we have just looked at particle-particle scattering. Now let's try to compute $e^+e^- \rightarrow e^+e^-$ scattering. In this case the particles are distinguishable in the initial and final state, but we again have two diagrams. The second diagram corresponds to particle-antiparticle annihilation. The two diagrams are shown in Fig. 6.5.
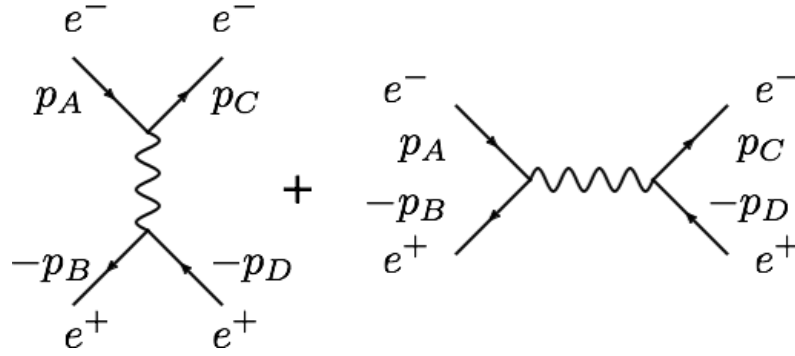
Figure 6.5: Feynman diagrams for spinless electron-positron scattering.

In order to write down the invariant amplitude we need Feynman rules involving antiparticles. If we recall the Feynman-Stückelberg picture then a positive energy antiparticle travelling forwards in time is equivalent to a negative energy particle travelling backwards in time.

We can use the Feynamn-Stückelberg picture for example to write the vertex term for the lower vertex in the first diagram as $ie(-p_B - p_D)_\nu$. For the first diagram, the 4-momentum of the exchanged photon as $q = p_A - p_C$. For the second diagram, the 4-momentum of the photon is $q = p_A + p_B$. From the diagrams, the invariant amplitude is then

$$-i\mathcal{M}_{e^+e^-} = ie^2 \frac{(p_A + p_C)_\mu(-p_B - p_D)^\mu}{(p_A - p_C)^2} + ie^2 \frac{(p_A - p_B)_\mu(p_C - p_D)^\mu}{(p_A + p_B)^2} \tag{6.9}$$

Comparing this to the expression we had earlier we find that

$$\mathcal{M}_{e^-e^-}(p_A, p_B, p_C, p_D) = \mathcal{M}_{e^+e^-}(p_A, -p_D, p_C, -p_B) \tag{6.10}$$

i.e. it is the same expression just exchanging $p_B \leftrightarrow -p_D$. We can exploit this to short cut to the result

$$s = (p_A + p_B)^2 \rightarrow u$$
$$t = (p_A - p_C)^2 \rightarrow t \tag{6.11}$$
$$u = (p_A - p_D)^2 \rightarrow s$$

such that

$$-i\mathcal{M}_{e^+e^-} = ie^2 \left(\frac{u-s}{t} + \frac{u-t}{s} \right) \tag{6.12}$$

We refer to the diagram that contributes $(u-s)/t$ to the amplitude as a *t-channel* process and the diagram that contributes $(u-t)/s$ as an *s-channel* process.

Lecture 7

The Dirac equation

7.1   The Dirac equation

The Klein-Gordon equation for a free, spin-0, particle with wave-function ϕ is given by

$$\frac{\partial^2 \phi}{\partial t^2} - \vec{\nabla}^2 \phi + m^2 \phi = 0 \ . \tag{7.1}$$

When deriving this expression, we came across two problems:

1. The equation has solutions with $E < 0$,

2. The $E < 0$ solutions have a negative probability density, *i.e.* $\rho < 0$.

Technically these problems arise because the Klein-Gordon equation is second order in the time derivative. Dirac argued that to solve this problem, we need to find an equation that is linear in $\partial/\partial t$. By Lorentz covariance this also has to be linear in $\vec{\nabla}$. The most general form of the equation, for a particle of mass m and wave-function $\psi(\vec{x}, t)$, is

$$i \frac{\partial}{\partial t} \psi(\vec{x}, t) = \left(-i \vec{\alpha} \cdot \vec{\nabla} + \beta m \right) \psi(\vec{x}, t) \ . \tag{7.2}$$

To be a consistent relativistic equation, ψ also has to obey $E^2 = |\vec{p}|^2 + m^2$ and be invariant under Lorentz transformations. So, what are $\vec{\alpha}$ and β?

Let's start by requiring that ψ also satisfies the Klein-Gordon equation,

$$-\frac{\partial^2 \psi(\vec{x}, t)}{\partial t^2} = \left(-\vec{\nabla}^2 \psi(\vec{x}, t) + m^2 \psi(\vec{x}, t) \right) \ . \tag{7.3}$$

Taking the square of the Dirac equation

$$\left(i \frac{\partial}{\partial t} \right) \left(i \frac{\partial}{\partial t} \right) \psi(\vec{x}, t) = \left(-i \vec{\alpha} \cdot \vec{\nabla} + \beta m \right) \left(-i \vec{\alpha} \cdot \vec{\nabla} + \beta m \right) \psi(\vec{x}, t) \ , \tag{7.4}$$

$$= \left(-i \alpha_i \nabla_i + \beta m \right) \left(-i \alpha_j \nabla_j + \beta m \right) \psi(\vec{x}, t) \ , \tag{7.5}$$

$$\tag{7.6}$$

where in the second line we have explicitly included the indices i and j that are being summed over. This gives

$$-\frac{\partial^2 \psi(\vec{x}, t)}{\partial t^2} = \left[-\alpha_i^2 \nabla_i^2 + \beta^2 m^2 - (\alpha_i \alpha_j + \alpha_j \alpha_i) \nabla_i \nabla_j \right. \tag{7.7}$$

$$\left. -i(\alpha_i \nabla_i \beta + \beta \alpha_j \nabla_j) m \right] \psi(\vec{x}, t) \ , \tag{7.8}$$

where the case that $i = j$ has been separated from the case where $i \neq j$. We can then write down the following relations for $\vec{\alpha}$ and β

$$\begin{aligned} \beta^2 &= 1 \\ \alpha_i^2 &= 1 \\ \alpha_i \alpha_j + \alpha_j \alpha_i &= 2\delta_{ij} \\ \alpha_i \beta + \alpha_j \beta &= 0 \end{aligned} \tag{7.9}$$

An important observation is that α_i and β anti-commute *i.e.* $\{\alpha_i, \alpha_j\} = 0$ if $i \neq j$ and $\{\alpha_i, \beta\} = 0$. They cannot be numbers and instead must be matrices operating on a multi-component wave-function.

It can be shown that the matrices

- Are Hermitian ($\alpha_i = a_i^\dagger$);

- Are Traceless, i.e. $\mathrm{Tr}(\alpha_i) = \mathrm{Tr}(\beta) = 0$;

- Have eigenvalues of ± 1;

- Have even dimensionality and that the lowest dimension is four.

One popular choice for α_i and β is

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} , \quad \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} , \tag{7.10}$$

where the σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{7.11}$$

The motivation for this choice will become clear later. The Pauli matrices obey the commutation relation

$$[\sigma^i, \sigma^j] = 2i\varepsilon^{ijk}\sigma^k \tag{7.12}$$

where

$$\varepsilon^{ijk} = \begin{cases} 0 & \text{if } i, j, k \text{ repeat} \\ +1 & \text{for cyclic permutations} \\ -1 & \text{for non-cyclic permutations} \end{cases} \tag{7.13}$$

If α and β are 4×4 matrices, ψ must be a 4-component vector,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} , \quad \psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) . \tag{7.14}$$

We can now write the Dirac equation in a more compact form as

$$(i\gamma^\mu \partial_\mu - m)\psi = 0, \tag{7.15}$$

with $\gamma^\mu = (\beta, \beta\alpha_i)$. Explicitly writing out the sum over the space-time indices

$$i\gamma^0 \frac{\partial \psi}{\partial t} + i\gamma^1 \frac{\partial \psi}{\partial x^1} + i\gamma^2 \frac{\partial \psi}{\partial x^2} + i\gamma^3 \frac{\partial \psi}{\partial x^3} - m\mathbb{1}_4\psi = 0 . \tag{7.16}$$

Here, γ^μ is not a 4-vector, its components are matrices. Explicitly,

$$\sum_{k=1}^{4}\left[\sum_{\mu}(i\gamma^\mu_{jk}\partial_\mu - m\delta_{jk})\right]\psi_k = 0 \; . \tag{7.17}$$

The γ matrices have the following useful properties that we'll need in our later calculations

$$\begin{aligned}
\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu &= 2g^{\mu\nu}\mathbb{1} \\
\gamma^{0\dagger} &= \gamma^0 \\
\gamma^0\gamma^0 &= +\mathbb{1}_4 \\
\gamma^{i\dagger} &= -\gamma^i \\
\gamma^i\gamma^i &= -\mathbb{1}_4 \\
\gamma^{\mu\dagger} &= \gamma^0\gamma^\mu\gamma^0
\end{aligned} \tag{7.18}$$

We can also define an adjoint spinor $\bar\psi = \psi^\dagger\gamma^0$ as the *appropriate* conjugate of ψ. The equivalent to the Dirac equation is then

$$i\partial_\mu\bar\psi\gamma^\mu + m\bar\psi = 0 \; . \tag{7.19}$$

To illustrate where this expression comes from, we can take the Hermitian conjugate of the Dirac equation

$$-i\partial_\mu\psi^\dagger\gamma^{\mu\dagger} - m\psi^\dagger\mathbb{1} = 0 \; . \tag{7.20}$$

Multiplying from the right by γ^0 gives

$$-i\partial_\mu\psi^\dagger\gamma^{\mu\dagger}\gamma^0 - m\psi^\dagger\gamma^0 = 0 \tag{7.21}$$
$$-i\partial_\mu\psi^\dagger\gamma^0\gamma^\mu - m\psi^\dagger\gamma^0 = 0$$
$$\tag{7.22}$$

i.e.

$$i\partial_\mu\bar\psi\gamma^\mu + m\bar\psi = 0 \tag{7.23}$$

Using ψ and $\bar\psi$ we can form a continuity equation for a current,

$$\partial_\mu(\bar\psi\gamma^\mu\psi) = \partial_\mu j^\mu = 0 \; . \tag{7.24}$$

This solves the problem with the negative probability solutions

$$\begin{aligned}
\rho = j^0 = \bar\psi\gamma^0\psi &= \psi^\dagger\gamma^0\gamma^0\psi \\
&= \psi^\dagger\psi > 0 \; .
\end{aligned} \tag{7.25}$$

7.2   Free particle solutions

When we derived the Dirac equation we required the wave-function ψ also satisfied the Klein-Gordon equation. The components ψ_k of the Dirac spinor, much east satisfying the Klein-Gordon equation separately, *i.e.*

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2 \right) \psi_k = 0 \ . \tag{7.26}$$

The solutions of this equation will have the general form

$$\psi = u(p) e^{-ip \cdot x} \ , \tag{7.27}$$

where we call $u(p)$ a *spinor* and $e^{-ip \cdot x}$ is associated with the usual plane wave solution. Substituting this solution into the Dirac equation yields

$$(i\gamma^\mu \partial_\mu - m) u(p) e^{-ip \cdot x} = 0 \tag{7.28}$$

which collapses to

$$(\gamma^\mu p_\mu - m) u(p) = 0 \tag{7.29}$$

This is sometimes written with

$$\gamma^\mu p_\mu = \not{p} \quad \text{and} \quad (\not{p} - m) u(\vec{p}) = 0 \ . \tag{7.30}$$

To learn something about the spinor $u(p)$ it is useful to express the γ matrices once again in terms of $\vec{\alpha}$ and β,

$$H\, u(p) = (\vec{\alpha} \cdot \vec{p} + \beta m) u(p) = E\, u(p) \ . \tag{7.31}$$

There are four independent solutions to this equation, two with $E > 0$ and two with $E < 0$. For a particle at rest

$$\beta\, m\, u(p) = \begin{pmatrix} m\mathbb{1}_2 & 0 \\ 0 & -m\mathbb{1}_2 \end{pmatrix} u(p) = E\, u(p) \tag{7.32}$$

which has eigenvalues $m, m, -m, -m$ and eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \ . \tag{7.33}$$

For $\vec{p} \neq 0$,

$$\begin{pmatrix} m\mathbb{1}_2 & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -m\mathbb{1}_2 \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = E \begin{pmatrix} u_A \\ u_B \end{pmatrix} \tag{7.34}$$

where $u(p)$ has been divided into two two-component spinor u_A and u_B. This can be written more compactly as

$$(\vec{\sigma} \cdot \vec{p})\, u_B = (E - m)\, u_A \tag{7.35}$$
$$(\vec{\sigma} \cdot \vec{p})\, u_A = (E + m)\, u_B \tag{7.36}$$
$$\tag{7.37}$$

where

$$\vec{\sigma} \cdot \vec{p} = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}. \tag{7.38}$$

Solutions only exist if

$$\det \begin{pmatrix} -(E - m)\mathbb{1} & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -(E + m)\mathbb{1} \end{pmatrix} = 0 \tag{7.39}$$

and we again arrive at $E^2 = |\vec{p}|^2 + m^2$, which has positive and negative energy solutions. The two component spinors are, at this point, arbitrary *i.e.* the two component spinors can be written as

$$u_A = a_1\, \chi^1 + a_2\, \chi^2 \tag{7.40}$$
$$u_B = b_1\, \chi^2 + b_2\, \chi^2$$

with

$$\chi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{7.41}$$

For the $E > 0$ solutions, we can take $u_A(s) = \chi^{(s)}$ (where $s = 1, 2$) and

$$u_B(s) = \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \chi^{(s)} \tag{7.42}$$

The positive-energy four spinors solutions to the Dirac equation are then

$$u(p, s) = N \begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \chi^{(s)} \end{pmatrix} \tag{7.43}$$

where N is a normalisation constant. For $E < 0$, we can instead write $u_B(s) = \chi^{(s)}$ and solve for u_A such that

$$u(p, s + 2) = N \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E - m} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix} = N \begin{pmatrix} -\frac{\vec{\sigma} \cdot \vec{p}}{|E| + m} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix} \tag{7.44}$$

The four solutions are orthogonal, such that $u^\dagger(r)u(s) = 0$ if $r \neq s$.

For the normalisation, we once again choose N such that we have $2E$ particles per unit volume,

$$u^\dagger(p, s)u(p, s) = |N|^2 \left(1 + \left(\frac{\vec{\sigma} \cdot \vec{p}}{E + m} \right)^2 \right) = 2E \ . \tag{7.45}$$

This sets the normalisation of the spinors to as $N = (E + m)^{1/2}$.

Lecture 8

Spin and antiparticles

8.1   Spin

In addition to the positive and negative energy solutions we have an additional two-fold degeneracy (which gives four eigenvectors/eigenvalues). To understand where this degeneracy comes from, we need to find out if there is another observable that commutes with the Hamiltonian.

The Hamiltonian for a free particle is

$$H_0 = \vec{\alpha} \cdot \vec{p} + \beta m \tag{8.1}$$

First let's check to see if angular momentum $\vec{L} = \vec{x} \times \vec{p}$ commutes with H_0,

$$[H_0, \vec{L}] = [\vec{\alpha} \cdot \vec{p} + \beta m, \vec{x} \times \vec{p}] \tag{8.2}$$

$$= [\vec{\alpha} \cdot \vec{p}, \vec{x} \times \vec{p}] \tag{8.3}$$

Using

$$[AB, C] = A[B, C] + [A, C]B \quad \text{and} \quad [p_i, p_j] = 0, \ [x_i, p_j] = i\delta_{ij} \tag{8.4}$$

gives

$$[H_0, \vec{L}] = -i\vec{\alpha} \times \vec{p} \tag{8.5}$$

i.e. angular momentum is not a good quantum number. The Dirac equation does not conserve orbital angular momentum and therefore cannot describe a spin-0 particle. Now lets consider another operator

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{8.6}$$

which should be familiar from your quantum mechanics classes.

$$[H_0, \vec{\Sigma}] = 2i\vec{\alpha} \times \vec{p} \tag{8.7}$$

The result is equivalent up-to a factor of two and

$$[H_0, \vec{J}] = [H_0, \vec{L} + \frac{1}{2}\vec{\Sigma}] = 0 \tag{8.8}$$

The operator $\vec{S} = \frac{1}{2}\vec{\Sigma}$ is intrinsic to the particle. For a particle at rest $\vec{S}$ commutes with H_0 and is a good quantum number, e.g. operating through with $S_3 = \frac{1}{2}\Sigma_3$

$$
\begin{aligned}
E > 0: \quad & S_3 \, u(E = +m, 1) = +\tfrac{1}{2} \, u(E = +m, 1) \\
& S_3 \, u(E = +m, 2) = -\tfrac{1}{2} \, u(E = +m, 2)
\end{aligned}
$$

$$
\begin{aligned}
E < 0: \quad & S_3 \, u(E = -m, 3) = +\tfrac{1}{2} \, u(E = -m, 3) \\
& S_3 \, u(E = -m, 4) = -\tfrac{1}{2} \, u(E = -m, 4)
\end{aligned}
\tag{8.9}
$$

which has eigenvalues of $\pm\frac{1}{2}$. We already know that for $\vec{p} \neq 0$ that $\vec{S}$ does not commute with H_0. Can we find an operator that does? Let's try

$$\frac{1}{2}\vec{\Sigma} \cdot \frac{\vec{p}}{|\vec{p}|} = \frac{1}{2}\vec{\Sigma} \cdot \hat{p} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} \cdot \hat{p} & 0 \\ 0 & \vec{\sigma} \cdot \hat{p} \end{pmatrix} \tag{8.10}$$

which is the projection of the spin along the particles momentum vector. We refer to this as the particles Helicity. If we choose $\vec{p} = (0, 0, p)$

$$
\begin{aligned}
E > 0 : \quad & \tfrac{1}{2}\Sigma_3 \, u(E > 0, 1) = +\tfrac{1}{2}\, u(E > 0, 1) \\
& \tfrac{1}{2}\Sigma_3 \, u(E > 0, 2) = -\tfrac{1}{2}\, u(E > 0, 2) \\
\\
E < 0 : \quad & \tfrac{1}{2}\Sigma_3 \, u(E < 0, 3) = +\tfrac{1}{2}u(E < 0, 3) \\
& \tfrac{1}{2}\Sigma_3 \, u(E < 0, 4) = -\tfrac{1}{2}\, u(E < 0, 4)
\end{aligned}
\tag{8.11}
$$

Eigenvalues of $+\frac{1}{2}$ have spin aligned with the particles momentum vector. We have found that the Dirac equation describes a spin-$\frac{1}{2}$ particle and conserves, $\vec{J}$, the total angular momentum.
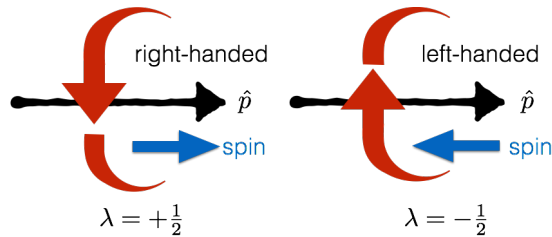
Figure 8.1: Helicity and handiness of a particle with a momentum vector pointing along the direction $\hat{p}$.

Asside:

Helicity is also not a very well defined quantity. We can always boost to the rest-frame of the particle where the helicity is undefined. A more clearly defined observable (relativistically covariant observable) is the particles chirality, related to the operator

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ . \tag{8.12}$$

In the Pauli-Dirac basis, which we are working with for the γ-matrices, this is

$$\gamma^5 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \ . \tag{8.13}$$

We can then define left- and right-handed projection operators, which for relativistic particles project out the left- and right-handed helicities as

$$\begin{aligned} P_L &= \tfrac{1}{2}(1 - \gamma^5) \\ P_R &= \tfrac{1}{2}(1 + \gamma^5) \ . \end{aligned} \tag{8.14}$$

8.2   Antiparticles

It's interesting to look at how Dirac interpreted the negative energy solutions. He postulated the existence of a *sea* of negative energy states and that the vacuum (or ground state of the system) has all of these negative states full. An additional electron must fill one of the positive energy states because the Pauli exclusion principal forbids it from occupying a filled negative energy states. If we promote a negative energy state to a positive one, an electron-hole pair is created (a positive energy electron and a hole in the negative sea). The hole appears as a positive energy anti-particle (the positron). This mechanism is essentially particle and antiparticle pair production. The positron was only observed experimentally four years later by Anderson in a cloud chamber experiment. Dirac's mechanism only works for fermions because bosons do not obey the Pauli exclusion principal.

We have solutions for the Dirac equation that can be written for the $E > 0$ solutions as

$$\psi(p_+) = u(p_+, s)e^{-ip\cdot x} \quad , \quad u(p_+, s) = (E + m)^{1/2} \begin{pmatrix} \chi^s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^s \end{pmatrix} \tag{8.15}$$

and for the $E < 0$ solutions as

$$\psi(p_-) = u(p_-, s+2)e^{-ip\cdot x} \quad , \quad u(p_-, s+2) = (E + m)^{1/2} \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{|E|+m}\chi^s \\ \chi^s \end{pmatrix} \, , \tag{8.16}$$

where

$$p_\pm = (\pm E, \vec{p}) \, . \tag{8.17}$$

If we think of the Feynman-Stückelberg prescription, then we also need to swap the sign of $\vec{p}$ to interpret the negative energy solutions in terms of anti-particles *i.e.*

$$\psi(p)_{\text{anti}} = (E + m)^{1/2} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^s \\ \chi^s \end{pmatrix} e^{+ip\cdot x} \, . \tag{8.18}$$

We can introduce a second type of spinor for the antiparticles, $v(p, s)$. The v-spinors are spinors for anti-particles with 4-momentum $(E, \vec{p})$ and we keep the u-spinors for particles with $E > 0$. The v-spinors are also orthogonal,

$$v^\dagger(p, r)v(p, s) = 2E\delta_{rs} \, . \tag{8.19}$$

The two particle solutions of the Dirac equation are

$$\psi(p, 1) = u(p, 1)e^{-ip\cdot x} = (E + m)^{1/2} \begin{pmatrix} \chi^1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^1 \end{pmatrix}$$
$$\psi(p, 2) = u(p, 2)e^{-ip\cdot x} = (E + m)^{1/2} \begin{pmatrix} \chi^2 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^2 \end{pmatrix} \, . \tag{8.20}$$

The two antiparticle solutions of the Dirac equation are

$$
\begin{aligned}
\psi(p, 1) &= v(p, 1)e^{+ip \cdot x} = (E + m)^{1/2} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi^2 \\ \chi^2 \end{pmatrix} \\
\psi(p, 2) &= v(p, 2)e^{+ip \cdot x} = (E + m)^{1/2} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi^1 \\ \chi^1 \end{pmatrix} .
\end{aligned}
\tag{8.21}
$$

Note, the association of $v(p, 1)$ with $u(p, 4)$ and $v(p, 2)$ with $u(p, 3)$. This is due to the change in the momentum direction and ensures that the indices 1 and 2 in $\psi(p, s)$ refer to the positive and negative helicity states, respectively.

The current associated to the Dirac equation is

$$
j^\mu = \bar{\psi}\gamma^\mu\psi
\tag{8.22}
$$

where $\bar{\psi}$ is an adjoint spinor defined as $\bar{\psi} = \psi^\dagger\gamma^0$. We also therefore need adjoint u- and v-spinors. The u-spinors satisfy

$$
(\gamma^\mu p_\mu - m)u(p, s) = (\not{p} - m)u(p, s) = 0 .
\tag{8.23}
$$

The u-spinors for $E < 0$ with $p \to -p$ satisfy

$$
(-\gamma^\mu p_\mu - m)u(-p, s) = 0
\tag{8.24}
$$

or

$$
(\gamma^\mu p_\mu + m)v(p, s) = (\not{p} + m)v(p, s) = 0 .
\tag{8.25}
$$

We also need to define adjoint spinors for our Dirac current $j^\mu = \bar{\psi}\gamma^\mu\psi$. To obtain an adjoint spinor we can take the Hermittian conjugate,

$$
\left[(\not{p} - m)u(p, s)\right]^\dagger = \bar{u}(p, s)(\not{p} - m) = 0
\tag{8.26}
$$

and similarly

$$
\bar{v}(p, s)(\not{p} + m) = 0
\tag{8.27}
$$

where

$$
\bar{u}(p, s) = u(p, s)^\dagger\gamma^0 \quad \text{and} \quad \bar{v}(p, s) = v(p, s)^\dagger\gamma^0 .
\tag{8.28}
$$

The combination

$$
u(p, r)^\dagger u(p, s) = 2E\delta_{rs} ,
\tag{8.29}
$$

is not Lorentz invariant. However,

$$
\bar{u}(p, r)u(p, s) = 2m\delta_{rs} \quad \text{and} \quad \bar{v}(p, r)v(p, s) = -2m\delta_{rs} .
\tag{8.30}
$$

The orthogonality of the spinors also means that

$$\bar{u}(p,r)v(p,s) = \bar{v}(p,r)u(p,s) = 0 \ . \tag{8.31}$$

With the adjoint spinors we have a complete set of states and can write the completeness relation, summing over the spins

$$\sum_{s=1,2} u(p,s)\bar{u}(p,s) = \slashed{p} + m$$
$$\sum_{s=1,2} v(p,s)\bar{v}(p,s) = \slashed{p} - m \tag{8.32}$$

This will be useful in our calculations later.

8.3 Coupling spin-$\frac{1}{2}$ particles to an EM field

We have already seen how to do this for the Klein-Gordon equation by making the minimal substitution

$$i\partial^\mu \to i\partial^\mu + eA^\mu \ , \tag{8.33}$$

for a particle of charge $-e$. If we apply this to the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{8.34}$$

we get

$$(\gamma^\mu(i\partial^\mu + eA_\mu) - m)\psi = 0 \tag{8.35}$$

There should also be an equivalent expression for a spin-$\frac{1}{2}$ particle with a charge of $+e$

$$(\gamma^\mu(i\partial^\mu - eA_\mu) - m)\psi_C = 0 \ . \tag{8.36}$$

The form of the equation has to be the same because nature cannot care about how we define the charges. An obvious question is, what is the relationship between ψ and ψ_C (and can we define an operator that transforms between the two)? For the Klein-Gordon equation, we simply took the complex conjugate. Taking the complex conjugate of the Dirac equation gives

$$\left(\gamma^{\mu*}(-i\partial_\mu + eA_\mu) - m\right)\psi^* = 0 \tag{8.37}$$

We can then define,

$$\psi_C = \hat{C}\psi^* \ . \tag{8.38}$$

The operator $\hat{C}$ must be a 4×4 matrix and satisfy

$$-\hat{C}\gamma^{\mu*} = \gamma^\mu\hat{C} \ , \tag{8.39}$$

such that

$$-\hat{C}\left[(\gamma^{\mu*}(-i\partial_\mu + eA_\mu) - m)\psi^*\right] = (\gamma^\mu(i\partial_\mu - eA_\mu) - m)\hat{C}\psi^* \tag{8.40}$$

It must also be Hermitian, *i.e.* $\hat{C}^\dagger\hat{C} = \mathbb{1}$. We discussed earlier that the γ-matrices are not unique and the exact form of $\hat{C}$ will depend on the representation of the γ-matrices. In the Pauli-Dirac representation it is

$$\hat{C} = i\gamma^2 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} & & & +1 \\ & & -1 & \\ & -1 & & \\ +1 & & & \end{pmatrix} \tag{8.41}$$

It is instructive to look at what ψ_C is for our particle wave function, e.g.

$$\psi_C(p,1) = \hat{C}\psi^*(p,1) = i\gamma^2 u(p,1)^* e^{+ip\cdot x}$$

$$= Ni\gamma^2 \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{(\vec{\sigma}\cdot\vec{p})^*}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} e^{+ip\cdot x}$$

$$= N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} e^{+ip\cdot x}$$

$$= u(-p,4)e^{-i(-p)\cdot x}$$

This is just

$$v(p,1)e^{+ip\cdot x} \ . \tag{8.42}$$

The operator $\hat{C}$ is the charge conjugation operator. It changes a particle into an antiparticle and vice versa, flipping the sign of all of the particles charges.

Asside:

The choice of γ matrices is not unique. In this course we are working with the so-called Pauli-Dirac basis,

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \ , \ \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \ . \tag{8.43}$$

Another popular representation is the Weyl basis

$$\alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \ , \ \beta = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \ . \tag{8.44}$$

The Pauli-Dirac basis disgonalises in the particle energy in the non-relativistic limit. This leads to a particularly simple looking set of eigenvalues and eignevectors for a particle at rest. The Weyl basis is not diagonalised in the particle energy (the unit matrices in β appear off-diagonal) and instead diagonalises the particle helicity in the relativistic limit (such that the unit matrices in γ^5 appear on the diagonal).

8.4 Gyromagnetic ratio

If we come back to the expression

$$\begin{pmatrix} m & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -m \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = E \begin{pmatrix} u_A \\ u_B \end{pmatrix} \tag{8.45}$$

which we can write as a series of coupled equations

$$\vec{\sigma} \cdot \vec{p} \, u_B = (E - m) \, u_A \tag{8.46}$$
$$\vec{\sigma} \cdot \vec{p} \, u_A = (E + m) \, u_B \ ,$$

we can rearrange this to write

$$u_B = \frac{\vec{\sigma} \cdot \vec{p}}{E + m} u_A \ . \tag{8.47}$$

Re-inserting this into the first of the coupled equations gives

$$\frac{1}{E + m} (\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{p}) \, u_A = (E - m) \, u_A \ , \tag{8.48}$$

or

$$\left(E - m - \frac{1}{E + m} (\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{p}) \right) u_A = 0 \tag{8.49}$$

In the non-relativisitc limit $E \approx m$, such that

$$\frac{1}{2m} (\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{p}) u_A = 0 \tag{8.50}$$

Further,

$$u_B = \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \ll u_A \tag{8.51}$$

We can couple the electron, with charge $-e$, to an EM field by making the transformation

$$p^\mu \to p^\mu + eA^\mu \tag{8.52}$$

leading to

$$\left(\frac{1}{2m} [\vec{\sigma} \cdot (\vec{p} + e\vec{A})][\vec{\sigma} \cdot (\vec{p} + \vec{A})] \right) u_A = 0 \tag{8.53}$$

Using $\vec{p} = -i\vec{\nabla}$ and $\vec{p} \times \vec{\nabla} + \vec{A} \times \vec{p} = i\vec{\nabla} \times \vec{A}$, we get

$$\frac{1}{2m} (\vec{p} + e\vec{A})^2 u_A + \frac{e}{2m} (\vec{\nabla} \times \vec{A}) \cdot \vec{\sigma} u_A = 0 \tag{8.54}$$

The second term involves a coupling to the magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ and the spin of the electron. This is usually written in terms of the magnetic moment as $-\vec{\mu} \cdot \vec{B}$, such that

$$\vec{\mu} = -\frac{e}{2m} \vec{\Sigma} = -g \frac{e}{2m} \frac{\vec{\Sigma}}{2} \tag{8.55}$$

where g is the gyromagnetic ratio of the electron. The prediction that $g = 2$ is a triumph of the Dirac equation.

Lecture 9

Coulomb scattering of charged spin-$\frac{1}{2}$ particles

9.1 Spin-$\frac{1}{2}$ scattering

If you recall for spin-0 particles, the transition amplitude came from first order in perturbation theory and was given by

$$T_{fi} = -i \int \phi_f^*(x) V(x) \phi_i(x) \mathrm{d}^4 x \tag{9.1}$$

For spin-$\frac{1}{2}$ particles we need to make the substitution

$$\begin{aligned} T_{fi} &= -i \int \psi_f^\dagger(x) V(x) \psi_i(x) \mathrm{d}^4 x \\ &= -i \int j_\mu^{fi} A^\mu \mathrm{d}^4 x \end{aligned} \tag{9.2}$$

Once again we need to find the form of the potential, $V(x)$. We can start by writing the Dirac equation in a form that separates out the Hamiltonian. The Dirac equation is

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{9.3}$$

and we want to write it in a form

$$H_0 \psi = E \psi = i \frac{\partial \psi}{\partial t} \ . \tag{9.4}$$

We start by expanding the sum over the space-time indices

$$i\gamma^0 \frac{\partial \psi}{\partial t} + \gamma^i \nabla_i \psi - m\psi = 0 \ , \tag{9.5}$$

which can be arranged to give

$$\begin{aligned} i\gamma^0 \frac{\partial \psi}{\partial t} &= (-i\gamma^j \nabla_j \psi + m)\psi \\ i\gamma^0 \gamma^0 \frac{\partial \psi}{\partial t} &= (-i\gamma^0 \gamma^j \nabla_j \psi + m\gamma^0)\psi \\ i\frac{\partial \psi}{\partial t} &= (-i\gamma^0 \gamma^j \nabla_j \psi + m\gamma^0)\psi \end{aligned} \tag{9.6}$$

If we introduce an EM potential by writing $p^\mu \to p^\mu + eA^\mu$ for a particle of charge $-e$

$$\left(i\gamma^0 \partial_0 + i\gamma^j \nabla_j + e\gamma^\mu A_\mu - m \right) \psi = 0 \tag{9.7}$$

This can be written as

$$i\frac{\partial \psi}{\partial t} = (-i\gamma^0 \gamma^j \nabla_j \psi + m\gamma^0 - e\gamma^0 \gamma^\mu A_\mu)\psi \tag{9.8}$$
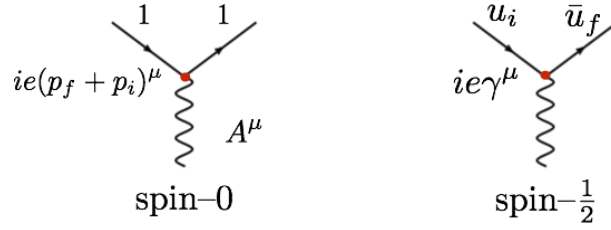
Figure 9.1: Comparison of Feynman rules for spin-0 and spin-$\frac{1}{2}$ particles.

Comparing to the free particle case (with $H = H_0 + V$), we can make the association

$$V = -e\gamma^0\gamma^\mu A_\mu \ . \tag{9.9}$$

Plugging the potential into the transition amplitude expression gives

$$T_{fi} = -i \int \psi_f^\dagger(x) V(x) \psi_i(x) \mathrm{d}^4 x \tag{9.10}$$

$$= ie \int \psi_f^\dagger \gamma^0 \gamma^\mu A_\mu \psi_i \mathrm{d}^4 x$$

$$= ie \int \bar{\psi} \gamma^\mu A_\mu \psi_i \mathrm{d}^4 x$$

Again comparing to the spin-0 case, we can identify a transition current

$$j_\mu^{fi} = -e\bar{\psi}_f \gamma_\mu \psi_i \tag{9.11}$$

$$= -e\bar{u}_f \gamma_\mu u_i e^{+i(p_f - p_i)\cdot x}$$

For Spin-0 scattering we had

$$j_\mu^{fi} = -e(p_f + p_i)_\mu e^{+i(p_f - p_i)\cdot x} \ . \tag{9.12}$$

Comparing the expressions we see that the vertex factor is $ie\gamma^\mu$ and their are now spinors on the incoming and outgoing legs. The vertex factor is a 4×4 matrix.

This brings us to the Feynman rules for spin-$\frac{1}{2}$ particle scattering,

description	spin-0	spin-$\frac{1}{2}$
incoming particle	1	$u(p_i, s_i)$
outgoing particle	1	$\bar{u}(p_f, s_f)$
incoming antiparticle	1	$\bar{v}(p_i, s_i)$
outgoing antiparticle	1	$v(p_f, s_f)$
vertex factor	$ie(p_i + p_f)^\mu$	$ie\gamma^\mu$

We also have internal lines (propagators),

$$
\begin{array}{ll}
\text{spin-0} & \dfrac{i}{p^2 - m^2} \\[2mm]
\text{spin-}\tfrac{1}{2} & \dfrac{i(\not{p} + m)}{p^2 - m^2} \\[2mm]
\text{photon (spin-1)} & -\dfrac{ig^{\mu\nu}}{q^2}
\end{array}
$$

We have exactly the same expression for the spin-0 propagator and the photon propagator. Does it make sense that these are the same? For a spin-0 particle we derived the photon propagator from

$$\Box A^\mu = j_{fi}^\mu \tag{9.13}$$

where j_{fi}^μ was the transition current associated to the other particle. The dependence on $1/q^2$ came from the realisation that

$$\partial^\mu \partial_\mu e^{-iq\cdot x} = -\frac{1}{q^2} e^{-iq\cdot x} \tag{9.14}$$

where $q = p_i - p_f$.

Finally, we get the expression for the transition amplitude for $A + B \to C + D$,

$$T_{fi} = -i \int j_\mu^{(1)}(x) \left(-\frac{1}{q^2}\right) j_{(2)}^\mu(x) \mathrm{d}^4 x \tag{9.15}$$

$$= -i(-e\bar{u}_C \gamma_\mu u_A) \left(-\frac{1}{q^2}\right) (-e\bar{u}_D \gamma^\mu u_B)(2\pi)^4 \delta^4(p_A + p_B - p_C - p_D)$$

where $q = p_A - p_C$. In terms of the invariant amplitude,

$$T_{fi} = -i(2\pi)^4 \delta^4(p_A + p_B - p_C - p_D)\mathcal{M} \tag{9.16}$$

where

$$-i\mathcal{M} = (ie\bar{u}_C \gamma^\mu u_A) \left(\frac{-ig_{\mu\nu}}{q^2}\right) (ie\bar{u}_D \gamma^\nu u_B) \tag{9.17}$$

All of the dynamics of the system is in the invariant amplitude, $\mathcal{M}$.

At this point it is useful to look in more detail at the structure of the Dirac current, $\bar{u}_f \gamma^\mu u_i$. To do this we make a so-called Gordon-Decomposition,

$$\bar{u}_f \gamma^\mu u_i = \frac{1}{2m} \bar{u}_f \left((p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu\right) u_i \tag{9.18}$$

where

$$\sigma^{\mu\nu} = \tfrac{i}{2}[\gamma^\mu, \gamma^\nu] = \tfrac{i}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \tag{9.19}$$

The first term in the decomposition looks identical to what we had in the spin-0 case for the vertex factor. The second term is a magnetic moment and is related to the spin of the particle. This is new for the spin-$\frac{1}{2}$ case. If you would like to prove the Gordon-Decomposition works you can expand the right-hand side and then simplify using

$$(\not{p} - m)u(p) = 0 \tag{9.20}$$
$$\bar{u}(p)(\not{p} - m) = 0 \ .$$

9.2 Interfering diagrams and relative signs

If multiple diagrams need to be included, when calculating an invariant amplitude, it is also important to account for the relative signs between the different diagrams. Additional factors of (-1) need to be included in the following scenarios:

1. an anti-fermion line runs continuously from the initial to the final-state;

2. and between diagrams with identical fermions in the final state.

These factors come from the anticommuation properties of the fermionic operators (and appear when ordering the fermionic fields in a full Quantum Field Theory treatment).

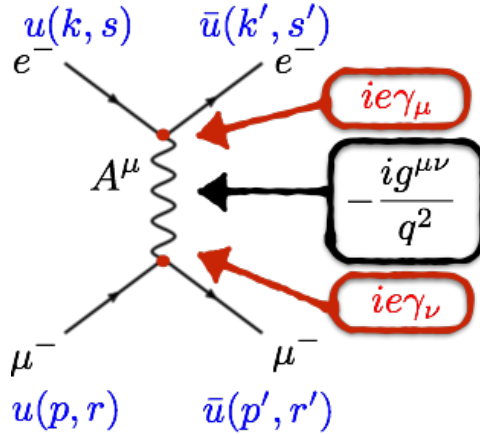# 9.3 Spin-$\frac{1}{2}$ scattering: $e^-\mu^- \to e^-\mu^-$

Figure 9.2: Feynman diagram for spin-$\frac{1}{2}$ $e^-\mu^- \to e^-\mu^-$ scattering.

The invariant amplitude for $e^-\mu^- \to e^-\mu^-$ scattering is given by

$$-i\mathcal{M} = e^2 \bar{u}(k',s')\gamma^\mu u(k,s) \frac{1}{q^2} \bar{u}(p',r')\gamma_\mu u(p,r) \tag{9.21}$$

where $q = k - k'$. We want the transition probability for our cross-section calculations and squaring the invariant amplitude gives

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}[\bar{u}(k',s')\gamma^\mu u(k,s)\,\bar{u}(p',r')\gamma_\mu u(p,r)] \tag{9.22}$$
$$\times [\bar{u}(k',s')\gamma^\nu u(k,s)\,\bar{u}(p',r')\gamma_\nu u(p,r)]^*$$

This looks complicated to expand. Don't worry, it's not as complicated as it looks. The structure of $\bar{u}\gamma^\mu u$ is

$$(\text{row})(\text{matrix})(\text{column}) = (\text{number}) \tag{9.23}$$

and we can write

$$[\bar{a}\gamma^\nu b]^* = [\bar{a}\gamma^\nu b]^\dagger \,, \tag{9.24}$$

More generally we can exploit the structure to split the expression as

$$[(\bar{a}\gamma^\mu b)(\bar{c}\gamma^\nu d)]^\dagger = (\bar{a}\gamma^\mu b)^\dagger(\bar{c}\gamma^\nu d)^\dagger \tag{9.25}$$

Looking a bit closer at the structure we can also write

$$
\begin{aligned}
(\bar{a}\gamma^\mu b)^\dagger &= b^\dagger \gamma^{\mu\dagger}\bar{a}^\dagger \ , \\
&= b^\dagger \gamma^{\mu\dagger}\gamma^{0\dagger}a \ , \\
&= b^\dagger (\gamma^0 \gamma^\mu \gamma^0)\gamma^0 a \ , \\
&= b^\dagger \gamma^0 \gamma^\mu a = \bar{b}\gamma^\mu a \ .
\end{aligned}
\tag{9.26}
$$

We can now use these blocks to re-write our original expression,

$$
|\mathcal{M}|^2 = \frac{e^4}{q^4}\bar{u}(k',s')\gamma^\mu u(k,s)\,\bar{u}(p',r')\gamma_\mu u(p,r)
\tag{9.27}
$$

$$
\times\,\bar{u}(p,r)\gamma_\nu u(p',r')\bar{u}(k,s)\gamma^\nu u(k',s')
$$

We can once again exploit the fact that the blocks $\bar{a}\gamma^\mu b$ are numbers to re-arrange this expression as

$$
|\mathcal{M}|^2 = \frac{e^4}{q^4}\bar{u}(k',s')\gamma^\mu u(k,s)\,\bar{u}(k,s)\gamma^\nu u(k',s')
\tag{9.28}
$$

$$
\times\,\bar{u}(p',r')\gamma_\mu u(p,r)\bar{u}(p,r)\gamma_\nu u(p',r')
$$

Why is this a useful thing to do? It identifies a useful feature of the calculation, we can separate the electron and muon system and write the invariant amplitude

$$
|\mathcal{M}|^2 = \frac{e^4}{q^4}L^{\mu\nu}_{(e)}L^{(\mu)}_{\mu\nu}
\tag{9.29}
$$

Tensor associated with the electron vertex is

$$
L^{\mu\nu}_{(e)} = \bar{u}(k',s')\gamma^\mu u(k,s)\,\bar{u}(k,s)\gamma^\nu u(k',s')
\tag{9.30}
$$

and the muon vertex is

$$
L^{(\mu)}_{\mu\nu} = \bar{u}(p',r')\gamma^\mu u(p,r)\,\bar{u}(p,r)\gamma^\nu u(p',r')
\tag{9.31}
$$

In most experiments we are interested in measuring an unpolarised cross-section, i.e. a cross-section in which we have no information on the spin of the incoming and outgoing particles and our incoming beams are a equal mixture of the different spin states. To allow for all possible spin configurations we average over the spins of the incoming particles and sum over all the possible configurations of the outgoing particles, i.e.

$$
|\mathcal{M}|^2 \to \overline{|\mathcal{M}|^2} \equiv \frac{1}{\text{incoming spin-states}}\sum_{\text{spin-states}}|\mathcal{M}|^2
\tag{9.32}
$$

For spin-$\frac{1}{2}$ particles, there are two possible spin-states. We have already seen that summing of the spins

$$
\sum_s u(p,s)\bar{u}(p,s) = (\not{p}+m)
\tag{9.33}
$$

This completeness relation will come in handy for our calculations.

If we look at the structure of the electron and muon tensors, then they have the form

$$L^{\mu\nu} = \bar{a}\gamma^\mu b\,\bar{b}\gamma^\nu a \tag{9.34}$$

which in terms of the basic building blocks is

$$(\text{row})(\text{matrix})(\text{column})(\text{row})(\text{matrix})(\text{column}) = (\text{row})(\text{matrix})(\text{column}) \tag{9.35}$$
$$= (\text{number}) \tag{9.36}$$

Labelling the indices, we can write this as

$$L = \bar{a}_i M_{ij} a_j \; . \tag{9.37}$$

Since these are just numbers, they can be re-arranged as

$$L = a_j \bar{a}_i M_{ij} \; . \tag{9.38}$$

The product $a_j \bar{a}_i$ is another matrix A_{ji}.

$$A_{ji} M_{ij} = B_{jj} \tag{9.39}$$

Summing the indices, this is the trace of a matrix,

$$L = \bar{a} M a = \text{Tr}(a\bar{a}M) \tag{9.40}$$

Lecture 10

Trace techniques and spin sums

10.1 Trace techniques

As introduced in Sec. 7.1, the Dirac gamma matrices obey the following relations

$$
\begin{aligned}
\gamma^0 \gamma^0 &= +\mathbb{1} \\
\gamma^i \gamma^i &= -\mathbb{1} \\
\{\gamma^\mu, \gamma^\nu\} &= \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}
\end{aligned}
\tag{10.1}
$$

It is also useful to define

$$
\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3
\tag{10.2}
$$

such that

$$
\begin{aligned}
\gamma^5 \gamma^5 &= \mathbb{1}_4 \\
\gamma^{5\dagger} &= \gamma^5 \\
\{\gamma_\mu, \gamma_5\} &= \gamma^\mu \gamma^5 + \gamma^5 \gamma^\mu = 0 \; .
\end{aligned}
\tag{10.3}
$$

You will come across the γ^5 matrix again when you deal with interactions involving the weak interaction which have both vector $\bar{u}\gamma^\mu u$ and axialvector $\bar{u}\gamma^\mu \gamma^5 u$ couplings.

The trace of an $n \times n$ matrix is

$$
\mathrm{Tr}(A) = \sum_{i=1}^{N} A_{ii} \; ,
\tag{10.4}
$$

where the A_{ii} are the diagonal elements of the matrix. The 4×4 unit matrix has a trivial trace

$$
\mathrm{Tr}(\mathbb{1}) = 4 \; .
\tag{10.5}
$$

Traces are also the same for cyclic permutations, such that

$$
\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB) \; .
\tag{10.6}
$$

Using the notation $\slashed{a} = \gamma^\mu a_\mu$ we can also write down the following useful rules

- The trace of an odd number of γ matrices vanishes

- $\mathrm{Tr}(\slashed{a}\slashed{b}) = 4a \cdot b = 4a^\mu b_\mu$

- $\mathrm{Tr}(\slashed{a}\slashed{b}\slashed{c}\slashed{d}) = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]$

- $\mathrm{Tr}(\gamma^5) = 0$

- $\mathrm{Tr}(\gamma^5 \slashed{a}\slashed{b}) = 0$

- $\mathrm{Tr}(\gamma^5 \slashed{a}\slashed{b}\slashed{c}\slashed{d}) = 4i\varepsilon_{\mu\nu\lambda\sigma} a^\mu b^\nu c^\lambda d^\sigma$

Example: To show that the Trace of an odd number of γ matrices vanishes

$$
\begin{aligned}
\mathrm{Tr}(\gamma^\alpha \gamma^\sigma \gamma^\rho \ldots \gamma^\mu) &= \mathrm{Tr}(\gamma^\alpha \gamma^\sigma \gamma^\rho \ldots \gamma^\mu \gamma^\nu \gamma^5 \gamma^5) \\
&= \mathrm{Tr}(\gamma^5 \gamma^\alpha \gamma^\sigma \gamma^\rho \ldots \gamma^\mu \gamma^\nu \gamma^5) \\
&= \mathrm{Tr}(\gamma^5 \gamma^\alpha \gamma^\sigma \gamma^\rho \ldots \gamma^\mu \gamma^5 \gamma^\nu) \times (-1) \\
&= \mathrm{Tr}(\gamma^5 \gamma^5 \gamma^\alpha \gamma^\sigma \gamma^\rho \ldots \gamma^\mu \gamma^5 \gamma^\nu) \times (-1)^n
\end{aligned}
\tag{10.7}
$$

where n is the number of γ matrices and we have used the anti-commutation relation $\{\gamma^\mu, \gamma^5\} = 0$ and $\gamma^5 \gamma^5 = \mathbb{1}$. If n is odd the trace has to be zero for the left- and right-hand sides to be equal.

Example: To show that $\mathrm{Tr}(\not{a}\not{b}) = 4a \cdot b$

$$
\begin{aligned}
\mathrm{Tr}(\not{a}\not{b}) &= \mathrm{Tr}(\not{b}\not{a}) \\
&= \tfrac{1}{2}\mathrm{Tr}(\not{a}\not{b} + \not{b}\not{a}) \\
&= \tfrac{1}{2}a_\mu b_\nu \mathrm{Tr}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \\
&= \tfrac{1}{2}a_\mu b_\nu \mathrm{Tr}(2g^{\mu\nu}\mathbb{1}) \\
&= g^{\mu\nu} a_\mu b_\nu \mathrm{Tr}(\mathbb{1}) \\
&= 4g^{\mu\nu} a_\mu b_\nu = 4a^\nu b_\nu
\end{aligned}
\tag{10.8}
$$

10.2   Spin-$\frac{1}{2}$ scattering: $e^- \mu^- \to e^- \mu^-$

To progress any further we need to do the spin sums. For the incoming particles we need to average over the two possible spin configurations. For the outgoing particles we need to sum over the spin configurations. Taking the expression,

$$L^{\mu\nu}_{(e)} = \mathrm{Tr}(u(k',s')\bar{u}(k',s')\gamma^\mu u(k,s)\bar{u}(k,s)\gamma^\nu) \tag{10.9}$$

Applying the spin sums this becomes

$$\overline{L^{\mu\nu}_{(e)}} = \frac{1}{2}\mathrm{Tr}((\slashed{k}' + m_{(e)})\gamma^\mu(\slashed{k} + m_{(e)})\gamma^\nu) \ . \tag{10.10}$$

The factor of $\frac{1}{2}$ comes from averaging over the two initial spin states the electron has. The invariant amplitude can then be written as

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\frac{e^4}{q^4}\mathrm{Tr}\left[(\slashed{k}' + m_{(e)})\gamma^\mu(\slashed{k} + m_{(e)})\gamma^\nu\right] \times \mathrm{Tr}\left[(\slashed{p}' + m_{(\mu)})\gamma_\mu(\slashed{p} + m_{(\mu)})\gamma_\nu\right] \ . \tag{10.11}$$

Both traces have the same structure, so we only need to solve it once,

$$\overline{L^{\mu\nu}_{(e)}} = \frac{1}{2}\mathrm{Tr}\left[(\slashed{k}' + m_{(e)})\gamma^\mu(\slashed{k} + m_{(e)})\gamma^\nu\right] \tag{10.12}$$

$$= \frac{1}{2}\mathrm{Tr}\left[\slashed{k}'\gamma^\mu\slashed{k}\gamma^\nu + \slashed{k}'\gamma^\mu m_{(e)}\gamma^\nu + m_{(e)}\gamma^\mu\slashed{k}\gamma^\nu + m_e^2\gamma^\mu\gamma^\nu\right] \ .$$

The easiest term to handle is

$$\mathrm{Tr}\left(\slashed{p}\gamma^\mu m\gamma^\nu\right) \ . \tag{10.13}$$

This contains an odd number of γ matrices and so the trace is zero. It is also easy to handle the term

$$\mathrm{Tr}\left(m^2\gamma^\mu\gamma^\nu\right) = \frac{m^2}{2}\mathrm{Tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) \tag{10.14}$$

$$= \frac{m^2}{2}\mathrm{Tr}(2g^{\mu\nu}\mathbb{1}) = 4m^2 g^{\mu\nu} \ . \tag{10.15}$$

The remaining term is

$$\mathrm{Tr}(\slashed{p}'\gamma^\mu\slashed{p}\gamma^\nu) = \mathrm{Tr}(\gamma^\sigma p'_\sigma\gamma^\mu\gamma^\rho p_\rho\gamma^\nu) \tag{10.16}$$

$$= p'_\sigma p_\rho\mathrm{Tr}(\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu)$$

$$= 4(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p' \cdot p)) \ .$$

Combing the terms

$$\overline{L^{\mu\nu}_{(e)}} = 2(k'^\mu k^\nu + k'^\nu k^\mu - g^{\mu\nu}(k' \cdot k - m^2_{(e)})) \ . \tag{10.17}$$

Finally,

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{q^4} \overline{L_{(e)}^{\mu\nu}} \; \overline{L_{\mu\nu}^{(\mu)}} \tag{10.18}$$

$$= \frac{4e^4}{q^4} (k'^\mu k^\nu + k'^\nu k^\mu - g^{\mu\nu}(k' \cdot k - m^2))(p'_\mu p_\nu + p'_\nu p_\mu - g_{\mu\nu}(p' \cdot p - m_{(\mu)}^2)) \ . \tag{10.19}$$

We now need to multiply all the pieces and keep track of the indices we are summing, *e.g.*

$$k^\mu k'^\nu p_\mu p_\nu = (k \cdot p)(k' \cdot p') \ . \tag{10.20}$$

We can also use $g^{\mu\nu} g_{\mu\nu} = 4$ to simplify the expression. This gives

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4} \left[(k \cdot p)(k' \cdot p') + (k \cdot p')(p' \cdot k) - m_{(e)}^2 (p \cdot p') - m_{(\mu)}^2 (k \cdot k') + 2m_{(e)}^2 m_{(\mu)}^2 \right] \tag{10.21}$$

If we work in the high energy limit ($E \gg m$) we can neglect terms proportional to m or m^2 and write this in the compact form

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4} \left[(k \cdot p)(k' \cdot p') + (k \cdot p')(p' \cdot k) \right] \ , \tag{10.22}$$

and as before this can be expressed in a frame invariant form in terms of the Mandelstam variables,

$$s = +2k \cdot p = +2k' \cdot p' \tag{10.23}$$
$$t = -2k \cdot k' = -2p \cdot p'$$
$$u = -2k \cdot p' = -2p \cdot k'$$

as

$$\overline{|\mathcal{M}|^2} = 2e^4 \left(\frac{s^2 + u^2}{t^2} \right) \tag{10.24}$$

10.3   Spin-$\frac{1}{2}$ scattering: $e^- e^+ \to \mu^- \mu^+$

We can obtain the result for $e^- e^+ \to \mu^- \mu^+$ by applying *crossing* rules (and the Feynman Stuckelberg interpretation) to the result for $e^- \mu^- \to e^- \mu^-$. This is demonstrated in Fig. 10.1. The right-hand figure looks a lot like the bottom left-hand figure just viewed from a different perspective. The *crossing* tells us we need to exchange

$$p_B \leftrightarrow -p_C \tag{10.25}$$

which is equivalent to swapping $s \leftrightarrow t$. The unpolarised result is then

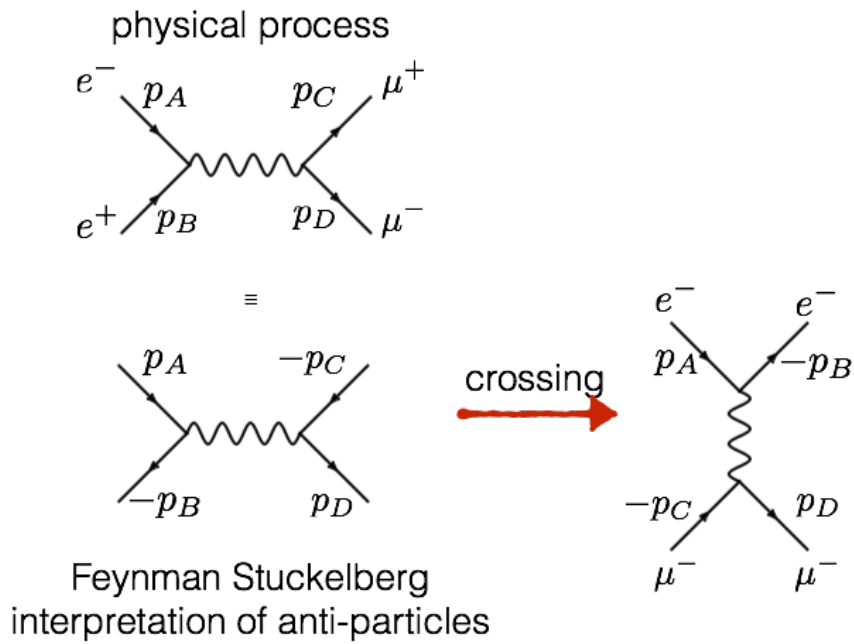$$\overline{|\mathcal{M}|^2} = 2e^4 \left(\frac{t^2 + u^2}{s^2} \right) \tag{10.26}$$

Figure 10.1: The relationship between $e^- e^+ \to \mu^- \mu^+$ and $e^- \mu^- \to e^- \mu^-$ scattering.

The transition probability can be translated into a differential cross-section for $e^- e^+ \to \mu^- \mu^+$ scattering using the expression we had from earlier. In the centre of mass system

$$\left. \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right|_{\mathrm{CM}} = \frac{1}{64\pi^2 s} \overline{|\mathcal{M}|^2} \tag{10.27}$$

which in terms of a scattering angle, θ, can be written as

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \frac{1}{64\pi^2 s} 2e^4 \left[\tfrac{1}{2}(1 + \cos^2\theta)\right] \tag{10.28}$$

$$= \frac{\alpha^2}{4s}(1 + \cos^2\theta)$$

The total cross section integrating over solid angle (θ and ϕ) is

$$\sigma(e^- e^+ \to \mu^- \mu^+) = \frac{4\pi\alpha^2}{3s} \tag{10.29}$$

This is the first result that we can compare to data (see Fig. 10.2). It agrees quite nicely, but only in this energy range. At higher energies we can also draw diagrams involving the Z boson that will have a significant impact on the cross-section (there will be a pole-like enhancement when $\sqrt{s} = m_Z$).
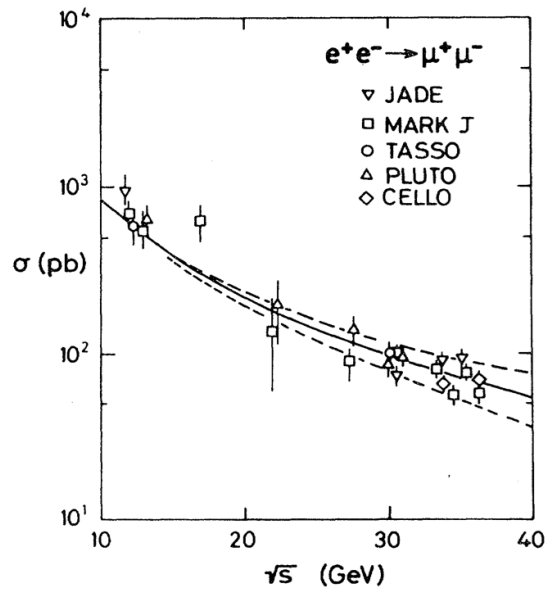

Figure 10.2: Total cross section for $e^- e^+ \to \mu^- \mu^+$ measured at the PETRA accelerator at DESY. The solid-curve corresponds to our QED expectation.

10.4 Photons and polarisation vectors

For spin-0 particle scattering we introduced the rule for an incoming/outgoing photon. We have also seen that Maxwells equations can be written in the compact form

$$\Box A^\mu = j^\mu \quad with \quad \partial_\mu A^\mu = 0 \tag{10.30}$$

The requirement $\partial_\mu A^\mu = 0$ is known as the Lorentz condition. In quantum mechanics, the wavefunction of the free photon will satisfy

$$\Box A^\mu = 0 \tag{10.31}$$

i.e. setting the current density to zero. This equation has solutions of the form

$$A^\mu = \varepsilon^\mu(\vec{q})e^{-iq\cdot x} \tag{10.32}$$

Substituting this back into $\Box A^\mu = 0$ gives $q^2 = 0$, i.e. $m_\gamma = 0$ (as expected). The four-vector $\varepsilon^\mu(\vec{q})$ deals with the polarisation of the photon. The photon is a spin-1 particle, which has only two transverse polarisation states and so we expect to be able to cancel two of the four possible polarisation vectors. In general, a spin-1 particle with non-zero mass will have three polarisation states. How do we remove two of the polarisation states? The Lorenz condition $\partial_\mu A^\mu$ gives

$$q_\mu \varepsilon^\mu = 0 \ , \tag{10.33}$$

reducing the number of independent components to three. We also have freedom in out choice of gauge, transforming

$$A_\mu \to A_\mu + \partial_\mu \Lambda \ . \tag{10.34}$$

Choosing

$$\Lambda = iae^{-iq\cdot x} \tag{10.35}$$

shows that that the underlying physics is unchanged by

$$\varepsilon \to \varepsilon + aq_\mu \tag{10.36}$$

i.e. two polarisation vectors that differ by multiples of q_μ describe the same photon. We can ue this freedom to ensure the time component of the polaristion vanishes, i.e. $\varepsilon^0 = 0$. The Lorentz condition reduces to

$$\vec{\varepsilon} \cdot \vec{q} = 0 \tag{10.37}$$

For a photon travelling along the z-axis, the two polarisation vectors are

$$\varepsilon_1 = (1, 0, 0) \quad , \quad \varepsilon_1 = (0, 1, 0) \tag{10.38}$$

It can be shown that the linear combinations

$$\varepsilon_R = -\sqrt{\frac{1}{2}}(\varepsilon_1 + i\varepsilon_2) \quad , \quad \varepsilon_L = +\sqrt{\frac{1}{2}}(\varepsilon_1 - i\varepsilon_2) \; , \tag{10.39}$$

correspond to a photon of helicity of $\lambda = \pm 1$. The polarisation vectors $\varepsilon_{L,R}$ are known as circular polarisation vectors. The circular polarisation vectors obey the completeness relation

$$\sum_\lambda (\varepsilon^\lambda)_i^* (\varepsilon^\lambda)_j = \delta_{ij}\hat{q}_i\hat{q}_j \tag{10.40}$$

In general a spin-1 particle of helicity λ, mass m and momentum $\vec{p} = (0, 0, p_z)$ can be described by three polarisation vectors

$$\varepsilon^{\lambda=\pm 1} = \mp\sqrt{\frac{1}{2}}(0, 1, \pm i, 0) \quad , \quad \varepsilon^{\lambda=0} = \frac{1}{m}(|\vec{p}|, 0, 0, E) \tag{10.41}$$

with the completeness relation

$$\sum_\lambda \varepsilon_\mu^{\lambda*} \varepsilon_\nu^\lambda = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \tag{10.42}$$

Problem set (1)

Special relativity, Lorentz covariance and the Klein-Gordon equation

1. Show that length
$$A^2 = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2$$
   is invariant under Lorentz transformation.

   [2]

2. Show that $g_{\mu\nu}g^{\mu\nu} = 4$.

   [2]

3. Using Schrödinger's equation and the definition of particle density, $\rho = \psi^*\psi$, show that the system satisfies the continuity equation with a current defined as

$$\vec{j} = \frac{1}{2mi}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right).$$

   [4]

4. Show that plane waves
$$Ne^{-i(wt - \vec{k}\cdot\vec{x})}$$
   are solutions of the Klein-Gordon equation. Obtain expressions for the energies of the solutions.

   [2]

Problem set (2)

Transition amplitudes and particle scattering

1. Show that solution of the Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = \left[H_0(\vec{x}) + \kappa V(t, \vec{x}) \right] \psi$$

   written in form of linear superposition of stationary states $\phi_m(\vec{x})$

$$\psi = \sum_m a_m(t)\phi_m(\vec{x})e^{-iE_m t}$$

   yields a system of differential equations

$$\frac{\mathrm{d}a_f(t)}{\mathrm{d}t} = -i\kappa \sum_m a_m(t)e^{i(E_f - E_m)t} \int \phi_f^* V(t, \vec{x})\phi_m(\vec{x})\mathrm{d}^3\vec{x}$$

   .

   [3]

2. Show that the Lorentz invariant phase-space for $A + B \to C + D$ scaterring

$$\mathrm{d}Q = (2\pi)^4 \delta^4(p_C + p_D - p_A - p_B)\frac{\mathrm{d}^3\vec{p}_C}{(2\pi)^3 2E_C}\frac{\mathrm{d}^3\vec{p}_D}{(2\pi)^3 2E_D}$$

   can be written in polar coordinates as

$$\mathrm{d}Q = \frac{1}{4\pi^2}\frac{|\vec{p}_C|}{4\sqrt{s}}\mathrm{d}\Omega \ ,$$

   where $\mathrm{d}\Omega$ is the element of solid angle and $s = (p_A + p_B)^2$. Hence, show that the differential cross-section for the process is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{p}_C|}{|\vec{p}_A|}|\mathcal{M}|^2 \ .$$

   [4]

3. In the very high-energy limit $(E \gg m)$, show that the differential cross section for spinless electron-muon scattering in the centre of mass system becomes

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\mathrm{CM}} = \frac{\alpha^2}{4s}\left(\frac{3 + \cos\theta}{1 - \cos\theta}\right)^2 \ ,$$

   where $\alpha = e^2/4\pi$ and θ is the scattering angle.

   [3]

Problem set (3)

Dirac equation

1. Using the particle spinors for the positive energy solutions of the Dirac equation, show that the spinors are orthogonal with a denisty of $2E$ particles per unit volume, i.e.
$$u^\dagger(p, r)u(p, s) = 2E\delta_{rs}$$

   Further show that
$$\bar{u}(p, r)u(p, s) = 2m\delta_{rs}$$

   .

2. Show that spinors u and v satisfy the following relations
$$\sum_s u(p, s)\bar{u}(p, s) = \gamma^\mu p_\mu + m,$$
$$\sum_s v(p, s)\bar{v}(p, s) = \gamma^\mu p_\mu - m.$$

   [3]

3. Consider the operator,
$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

   and show that its commutator with Hamiltonian $H_0 = \vec{\alpha} \cdot \vec{p} + \beta m$ is $-2i\vec{\alpha} \times \vec{p}$ (*i.e.* $[\vec{\Sigma}, H_0] = -2i\vec{\alpha} \times \vec{p}$ and $[H_0, \vec{\Sigma}] = +2i\vec{\alpha} \times \vec{p}$).

   [4]

For the u-spinors you can take

$$u(p, s) = (E + m)^{1/2} \begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma} \cdot \vec{p}}{E+m}\chi^{(s)} \end{pmatrix}$$

where

$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Problem set (4)

Spin-$\frac{1}{2}$ particle scattering

1. Calculate the spin averaged amplitude squared $(\overline{|\mathcal{M}|^2})$ for the process $e^+e^- \to e^+e^-$ and in the high energy limit. Express the result in terms of the Mandelstam variables.

   Hint: include both diagrams

   [10]