Relativisitc Quantum Mechanics

Problem Sheet 3

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Dirac equation

1. Using the particle spinors for the positive energy solutions of the Dirac equation, show that the spinors are orthogonal with a denisty of $2E$ particles per unit volume, i.e.
$$u^\dagger(p,r)u(p,s) = 2E\delta_{rs}$$
Further show that
$$\bar{u}(p,r)u(p,s) = 2m\delta_{rs}$$
.

[3]

2. Show that spinors u and v satisfy the following relations
$$\begin{aligned}\sum_s u(p,s)\overline{u}(p,s) &= \gamma^\mu p_\mu + m,\\ \sum_s v(p,s)\overline{v}(p,s) &= \gamma^\mu p_\mu - m.\end{aligned}$$

[3]

3. Consider the operator,
$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$
and show that its commutator with Hamiltonian $H_0 = \vec{\alpha}\cdot\vec{p} + \beta m$ is $-2i\vec{\alpha}\times\vec{p}$. (*i.e.* $[\vec{\Sigma}, H_0] = -2i\vec{\alpha}\times\vec{p}$ and $[H_0, \vec{\Sigma}] = +2i\vec{\alpha}\times\vec{p}$).

[4]

For the u-spinors take
$$u(p,s) = (E+m)^{1/2}\begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{(s)} \end{pmatrix}$$
where
$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;,\; \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;.$$