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présentée par

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Hamiltonian Perturbation Methods for Magnetically Confined Fusion Plasmas

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Chapter 1

Introduction

Feedback effects are unavoidable in fusion plasmas: Maxwell's equations, describing the evolution of electromagnetic fields, involve the charge and current densities of the particles. In turn, particles trajectories are modified by the fields through the equations of motion. Then the cumulative effect of this feedback loop can lead to plasma deconfinement.

In this work we address the problem of improving plasma confinement by controlling turbulent transport and in particular we explore the opportunity of barrier formation. Self-consistent fluctuations of electromagnetic fields and particle densities lie at the origin of plasma instabilities [1] and turbulent transport phenomena. In order to understand their underlying mechanisms we study non-collisional plasma dynamics by applying Hamiltonian tools.

From a general point of view, plasma dynamics can be studied at different levels: in particular, kinetic and fluid. Both of these admit a Hamiltonian formulation. In the first case for example, a canonical Hamiltonian structure appears while constructing guiding-center model for particle motion in a six dimensional $(\mathbf{p}, \mathbf{q})$ phase space. Such a model permits us to study particle dynamics in an external electromagnetic field and does not take into account field-particle retroaction. The second approach, dual to the previous one[2], studies the evolution of the particle distribution function in 6 dimensional phase space. Here a non-canonical Hamiltonian formulation is possible for retroactive Maxwell–Vlasov model. Finally Hamiltonian structures are known for the group of fluid models of the evolution of a distribution function in 3 dimensional phase space. The use of the Hamiltonian approach implies that viscosity and other mechanisms of dissipations are not taken into account, for example this is a case of Charney–Hasegawa–Mima, two fluid model.

[1]Instabilities characterize by out of equilibrium state with exponential growth of fluctuations
[2]Here we imply Eulerian-Lagrangian particle-fluid description duality

1.1 Particle dynamics: guiding center approach

Fusion plasma represents a system with $N \sim 10^{23}$ particles, each of them governed by the fundamental equation of dynamics $md\mathbf{v}/dt = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$. Obviously tracking the trajectory of each particle is totally out of reach. This is why a dynamical description at a particular level is of interest for simplified models when neither interaction between particles nor between fields and particles is taken into account. Then the motion of a single particle (test-particle) in an external fields is considered. We will see that such a simplified model allows us to study some concrete physical effects. This is the case for example of the guiding-center model.

The strong magnetic field approach is relevant for fusion plasmas, this is why at the first approximation one can neglect fluctuations of magnetic field and consider only the electrostatic turbulence case. In this approach particle motion is multiscale: it consists of a fast gyration around magnetic field lines and the slow drift mainly across the magnetic field lines. The guiding-center approach arises from the separation of the fast dynamics component from its slow one. Such an approach provides the idea of dynamical reduction.

Below we illustrate how the $\mathbf{E} \times \mathbf{B}$ drift model, that is often used by physicists, arises from the Hamiltonian description for single particle motion inside the electromagnetic field, which is represented by the electromagnetic potentials $(\mathbf{A}, V)$.

$\mathbf{E} \times \mathbf{B}$ model

In canonical variables the autonomous Hamiltonian of the particle in external electromagnetic fields is given by:

$$H = \frac{\left(\mathbf{P} - e\mathbf{A}(\mathbf{q}, \tau)\right)^2}{2m} + eV(\mathbf{q}, \tau) + \mathcal{W} \tag{1.1}$$

Then the canonical Poisson bracket has a following expression:

$$\{f, g\} = \frac{\partial f}{\partial \mathbf{P}} \cdot \frac{\partial g}{\partial \mathbf{q}} - \frac{\partial f}{\partial \mathbf{q}} \cdot \frac{\partial g}{\partial \mathbf{P}} + \frac{\partial f}{\partial \mathcal{W}} \frac{\partial g}{\partial \tau} - \frac{\partial f}{\partial \tau} \frac{\partial g}{\partial \mathcal{W}} \tag{1.2}$$

It was remarked that such a variables are not very practical in use. In fact, canonical momentum $\mathbf{P}$ is not a physical variable of the particle, because it contains coupling with electromagnetic field. The following transformation permits us to pass from canonical variables to particle local variables $(\mathbf{v}, \mathbf{x})$.

$$\begin{aligned} \mathbf{x} &= \mathbf{q} && (1.3) \\ \mathbf{v} &= \frac{1}{m}\left(\mathbf{P} - e\mathbf{A}(\mathbf{q}, t)\right) && (1.4) \\ \mathcal{W} &= \mathcal{K}, \quad \tau = t && (1.5) \\ & && (1.6) \end{aligned}$$

Due to such a transformation, the field-particle coupling will be incorporated inside the Poisson bracket, which is no longer canonical:

$$
\begin{aligned}
\{f,g\} &= \frac{1}{m}\Big(\frac{\partial f}{\partial \mathbf{v}}\cdot\frac{\partial g}{\partial \mathbf{x}} - \frac{\partial f}{\partial \mathbf{x}}\cdot\frac{\partial g}{\partial \mathbf{v}}\Big) \\
&- \frac{e\mathbf{B}}{m^2}\cdot\Big(\frac{\partial f}{\partial \mathbf{v}}\times\frac{\partial g}{\partial \mathbf{v}}\Big) + \frac{\partial f}{\partial \mathcal{K}}\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}\frac{\partial g}{\partial \mathcal{K}}
\end{aligned} \tag{1.7}
$$

In order to pass from (1.2) to (1.7) we have used chain rule:

$$
\{f,g\}_{new} = \sum_{ij}\frac{\partial f}{\partial z_i}\Big\{z_i, z_j\Big\}_{old}\frac{\partial g}{\partial z_j} \tag{1.8}
$$

where $z_i = (\mathbf{x}, \mathbf{v}, \mathcal{K}, t)$ denotes new phase space variables. The expressions for $\{z_i, z_j\}_{old}$ are obtained by using the expression for the canonical Poisson bracket:

$$
\{\mathbf{x}_i, \mathbf{v}_j\} = -\frac{1}{m}\delta_{ij} \tag{1.9}
$$

$$
\{\mathbf{v}_i, \mathbf{v}_j\} = -\frac{e}{m^2}\Big(\frac{\partial \mathbf{A}_i}{\partial \mathbf{q}_j} - \frac{\partial \mathbf{A}_j}{\partial \mathbf{q}_i}\Big) = -\frac{e}{m^2}\epsilon_{ijk}\mathbf{B}_k \tag{1.10}
$$

$$
\{\mathcal{K}, t\} = 1 \tag{1.11}
$$

Here the magnetic field, that supposed to be constant and uniform is decomposed as follows: $\mathbf{B} \equiv B\hat{\mathbf{b}}$. The general case of non-uniform magnetic geometry will be discussed in Chapter 4. In order to decouple the fast dynamics from the slow one we produce the following decomposition for the particle position $\mathbf{x}$. This induces

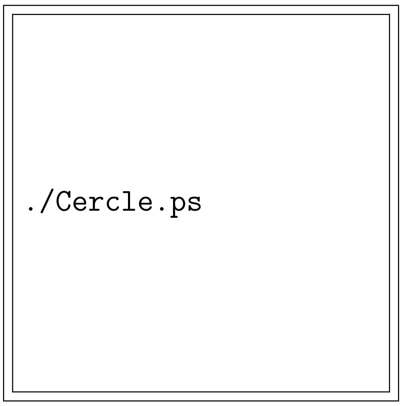

Figure 1.1: Guiding center

the second change of variables:

$$
\begin{aligned}
\mathbf{X} &= \mathbf{x} - \frac{m}{eB}\hat{\mathbf{b}}\times\mathbf{v} \\
\mathbf{u} &= \mathbf{v} \\
\mathcal{K} &= \mathcal{W},\ t = \tau
\end{aligned} \tag{1.12}
$$

Note that the ratio $m/eB \equiv \epsilon$ here plays the role of a small parameter because of considering strong magnetic field approach.

Then the Poisson bracket (1.7) transforms into:

$$
\begin{aligned}
\{f,g\} &= \frac{1}{m}\hat{\mathbf{b}}\hat{\mathbf{b}} : (\frac{\partial f}{\partial \mathbf{u}}\frac{\partial g}{\partial \mathbf{X}} - \frac{\partial f}{\partial \mathbf{X}}\frac{\partial g}{\partial \mathbf{u}}) + \frac{\partial f}{\partial \mathcal{W}}\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}\frac{\partial g}{\partial \mathcal{W}} \\
&- \frac{eB}{m^2}\,\hat{\mathbf{b}}\cdot(\frac{\partial f}{\partial \mathbf{u}} \times \frac{\partial g}{\partial \mathbf{u}}) + \frac{1}{eB}\hat{\mathbf{b}}\cdot(\frac{\partial f}{\partial \mathbf{X}} \times \frac{\partial g}{\partial \mathbf{X}}) \qquad (1.13)
\end{aligned}
$$

Where we have used tensor analysis notation $\mathbf{ab} : \mathbf{cd} \equiv \mathbf{b}\cdot\mathbf{c}\ \mathbf{d}\cdot\mathbf{a}$. The relations for the elementary brackets between new phase space variables (1.12) are obtained by using the noncanonical Poisson bracket (1.7):

$$
\begin{aligned}
\{\mathbf{X}_i, \mathbf{X}_j\} &= -\frac{1}{eB}\epsilon_{ijk}\hat{\mathbf{b}}_k \qquad (1.14) \\
\{\mathbf{X}_i, \mathbf{u}_j\} &= -\frac{1}{m}\hat{\mathbf{b}}_i\hat{\mathbf{b}}_j \qquad (1.15) \\
\{\mathbf{u}_i, \mathbf{u}_j\} &= -\frac{eB}{m^2}\epsilon_{ijk}\hat{\mathbf{b}}_k \qquad (1.16) \\
\{\mathcal{W}, t\} &= 1 \qquad (1.17)
\end{aligned}
$$

Here $\boldsymbol{\rho} \equiv \frac{m}{eB}\hat{\mathbf{b}}\times\mathbf{v}$ denotes the part of the particle position perpendicular to magnetic field that explicitly depends of fast gyroangle and $\mathbf{X}$ denotes the remaining part of the particle position, which is also called the guiding-center. Let us now consider one simple case when magnetic field is constant uniform and parallel to z direction $\mathbf{B} = B\mathbf{e}_z$. The expression for the Poisson bracket (1.13) becomes:

$$
\begin{aligned}
\{f,g\} = \frac{1}{eB}\Big(\frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X}\Big) - \frac{eB}{m^2}\Big(\frac{\partial f}{\partial u_x}\frac{\partial g}{\partial u_y} - \frac{\partial f}{\partial u_y}\frac{\partial g}{\partial u_x}\Big) \\
-\frac{1}{m}\Big(\frac{\partial f}{\partial Z}\frac{\partial g}{\partial u_z} - \frac{\partial f}{\partial u_z}\frac{\partial g}{\partial Z}\Big) + \frac{\partial f}{\partial \mathcal{W}}\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}\frac{\partial g}{\partial \mathcal{W}} \qquad (1.18)
\end{aligned}
$$

Note that in this simple case with $\hat{\mathbf{b}} = \mathbf{e}_z$ the final expression for the Poisson bracket is canonical. The canonically conjugate variables are $(X, Y), (u_x, u_y), (Z, u_z)$ and $(t, \mathcal{W})$. Then the equations of motion in the perpendicular to magnetic field plane become:

$$
\begin{aligned}
\dot{X} &= \{H, X\} = -\frac{1}{B}\frac{\partial V(X,Y,t)}{\partial Y} \qquad (1.19) \\
\dot{Y} &= \{H, Y\} = \frac{1}{B}\frac{\partial V(X,Y,t)}{\partial X} \qquad (1.20)
\end{aligned}
$$

Note that here only the first term of the Poisson bracket (1.18) is used in order to obtain dynamical equations.

Such a dynamics can be rewritten by introducing $\mathbf{E} = -\nabla V$ as follows:

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{1.21}$$

This is the $\mathbf{E} \times \mathbf{B}$ drift model that permits us in the electrostatic turbulence approximation to consider one of the possible mechanisms for plasma deconfinement. The fact that such a model possesses Hamiltonian structure gives us the possibility to implement Hamiltonian control tools in order to study barrier formation for reduction of such a drift motion and therefore to improve plasma confinement. It will be implemented in Chapter 2 while studying barrier formation.

Idea of dynamical reduction

On the other hand considering test particle motion in the electromagnetic field is of interest because of possibility to explicitly illustrate dynamical reduction related to elimination of the fast scale motion. At the first approximation, we neglect fast dynamics dependence inside the electric potential:

$$H = \frac{1}{2} m \mathbf{u}^2 + eV\left(\mathbf{X} + \epsilon\, \hat{\mathbf{b}} \times \mathbf{u}; t\right) + \mathcal{W} \to \widetilde{H} = \frac{1}{2} m \mathbf{u}^2 + eV\left(\mathbf{X}; t\right) + \mathcal{W} \tag{1.22}$$

By separating directions parallel and perpendicular to magnetic field, and by introducing the coordinates:

$$\mu = m \frac{\mathbf{u}_x^2 + \mathbf{u}_y^2}{2B} \tag{1.23}$$

$$\zeta = \arctan \frac{\mathbf{u}_x}{\mathbf{u}_y} \tag{1.24}$$

we can rewrite the perpendicular velocities part of the Poisson bracket as

$$-\frac{eB}{m^2}\Big(\frac{\partial f}{\partial u_x}\frac{\partial g}{\partial u_y} - \frac{\partial f}{\partial u_y}\frac{\partial g}{\partial u_x}\Big) \to \frac{e}{m}\big(\frac{\partial f}{\partial \mu}\frac{\partial g}{\partial \zeta} - \frac{\partial f}{\partial \zeta}\frac{\partial g}{\partial \mu}\big) \tag{1.25}$$

The variables μ and ζ are canonically conjugate: $\{\mu, \zeta\} = \frac{e}{m}$ (up to a constant factor).

The reduced Hamiltonian is given by:

$$\widetilde{H} = \frac{1}{2}\mu B + \frac{1}{2m}\mathbf{u}_z^2 + eV(x, y, z; t) + \mathcal{W} \tag{1.26}$$

Finally we find that μ has a trivial dynamics $\dot{\mu} = 0$, i.e. μ is a constant of motion, and $\dot{\zeta} = \{H, \zeta\} = \dfrac{eB}{m} \equiv \dfrac{1}{\epsilon}$ is the fast gyroangle.

A systematic derivation of the expression for constant of motion at each order of small parameter ϵ, as well as geometrical aspects related to the dynamical reduction, will be discussed in Chapter 4.

1.2 Kinetic approach

Plasma kinetics studies plasma evolution on six dimensional phase space. It is well known that such an approach is very demanding numerically and needs reduction of number of dynamical variables. One of the possible way to realize it, is to remove fast gyrophase dependence from dynamics. Such an approach is named "Gyrokinetics". Particle numerical simulations based on the use of nonlinear gyrokinetic equations have experienced an important expansion over the last several decades. It represents now a powerful tool for studying various aspects of turbulence, instabilities and its associated anomalous transport.

1.3 Perturbation methods leading to the Gyrokinetic Maxwell-Vlasov equations

There exists two principal groups of methods that permits us to get reduced dynamical equations implemented inside those codes. The first one, referred to also as the standard method, consists in dealing with explicit gyroaveraging of the Vlasov equation expressed in lowest order reduced (guiding-center) coordinates. This is followed by separation of equilibrium and perturbed parts of the guiding-center distribution function. One of the serious disadvantages of such a method is its failure to provide a clear iterative algorithm.

Another group of methods do not deal with Vlasov equation directly, but start with consideration of a single particle Lagrangian. They use Lie-transform techniques which provide near-identity coordinate transformations that decouple the gyration from the slower dynamics of interest. Such a method was formally introduced in [1] and applied for stationary electrostatic turbulence case. Later its application was expanded on the problem of a single particle motion in an external non-uniform magnetic [2] and electromagnetic [3] fields as well as to study of mechanics of magnetic field line flow [4]. Their first advantage with respect to the first group of reduction methods is that such a transformation is reversible, so the information about the fast dynamics is not lost and can be recovered when it is needed. The second strong point of such approaches is existence of a well defined iterative procedure that permits us at each order to derive gyroangle-independent dynamics. The more general among those methods, is the action-variational Lie perturbation method. This method deals with the phase-space Lagrangian (Poincaré-Cartan fundamental one-form), which couples the symplectic structure and the Hamiltonian[3]:

$$\Gamma \equiv L\,dt = \mathbf{p}\cdot d\mathbf{q} - Hdt \tag{1.27}$$

where $\mathbf{p}$ and $\mathbf{q}$ represents canonical phase space variables. Then the Hamiltonian equations are obtained according to the variational principle when that the phase space variables are varied independently of each other:

$$\delta\Gamma \equiv \delta\mathbf{p}\cdot\left(d\mathbf{q} - \frac{\partial H}{\partial \mathbf{p}}dt\right) - \delta\mathbf{q}\cdot\left(d\mathbf{p} + \frac{\partial H}{\partial \mathbf{q}}dt\right) = 0 \tag{1.28}$$

so that

$$\forall\delta\mathbf{q},\ \delta\mathbf{p} \Leftrightarrow \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} \tag{1.29}$$

Note that the independent variation of the phase space variables here represents the main difference between the traditional variational principle, when the Lagrangian is defined on configuration space $(\mathbf{q}, \dot{\mathbf{q}})$, and the variational principle using the phase-space Lagrangian.

The first step here in obtaining gyrophase-independent dynamics is to pass from the canonical variables $(\mathbf{p}, \mathbf{q})$ into the local particle variables $z^\alpha = (\mathbf{X}, \mu, \zeta, u_z)$, introduced into the previous section. The next step consists in performing a set of transformations given by:

$$\tau_\epsilon z^a \equiv \bar{z^a} = z^a + \epsilon G_1^a + \epsilon^2\left(G_2^a + \frac{1}{2}G_1^b\frac{\partial G_1^a}{\partial z^b}\right) + \ldots \tag{1.30}$$

where z^a denotes initial set of non-reduced coordinates and $\bar{z^a}$ reduced correspondingly the n-th order transformation is driven by phase-space vector field $G_n^a\partial/\partial z^a$. Such a phase space change of variables induces phase-space Lagrangian transformation as follows

$$\bar{\Gamma} = \mathsf{T}_\epsilon^{-1}\Gamma + dS \equiv \epsilon^{-1}\bar{\Gamma}_0 + \bar{\Gamma}_1 + \epsilon\,\bar{\Gamma}_2 + \epsilon^2\,\bar{\Gamma}_3 \tag{1.31}$$

where $\bar{\Gamma}_n = \bar{\Gamma}_{na}d\bar{z^a} - \bar{H}_n dt$ and the push forward operator $\mathsf{T}_\epsilon^{-1} = \ldots \exp(-\epsilon^2 \pounds_2)\ \exp(-\epsilon\pounds_1)$ is expressed in terms of Lie-derivatives. According to Cartan's formula, Lie derivative of one-form yields one-form

$$\pounds_G\Gamma \equiv i_G\cdot d\Gamma + d(i_G\cdot\Gamma) = G^a\,\omega_{ab}dz^b + d(G^a\Gamma_a) \tag{1.32}$$

[3]Phase space Lagrangian expression arises from the transition between configuration space and phase space given by Legendre transformation

$$L(\mathbf{q}, \dot{\mathbf{q}}; t) = \mathbf{p}\frac{d\mathbf{q}}{dt} - H(\mathbf{p}, \mathbf{q}; t)$$

where $\mathbf{p} \equiv \partial_{\dot{\mathbf{q}}}L$

here $d\Gamma \equiv \omega_{ab} dz^a \wedge dz^b$. Then by applying two first order decomposition for pull-back operator to the phase-space Lagrangian we obtain the iterative procedure up to the ϵ^3:

$$\begin{aligned}\bar{\Gamma} &\equiv \exp(-\epsilon^2 \mathcal{L}_2)\ \exp(-\epsilon \mathcal{L}_1)\big(\epsilon^{-1}\Gamma_0 + \Gamma_1 + \epsilon\ \Gamma_2 + \epsilon^2\ \Gamma_3\big) \\ &= \big(1 - \epsilon^2 \mathcal{L}_2\big)\ \big(1 - \epsilon \mathcal{L}_1 + \frac{\epsilon^2}{2}\mathcal{L}_1^2\big)\big(\epsilon^{-1}\Gamma_0 + \Gamma_1 + \epsilon\ \Gamma_2 + \epsilon^2\ \Gamma_3\big) \end{aligned} \tag{1.33}$$

then at each order we obtain

$$\epsilon^{-1}\ :\ \bar{\Gamma}_0 = \Gamma_0 \tag{1.34}$$

$$\epsilon^0 \quad \bar{\Gamma}_1 = \Gamma_1 - \mathcal{L}_1\Gamma_0 + dS_1 \tag{1.35}$$

$$\epsilon^1 \quad \bar{\Gamma}_2 = \Gamma_2 - \mathcal{L}_2\Gamma_0 - \mathcal{L}_1\Gamma_1 + \frac{1}{2}\mathcal{L}_1^2\Gamma_0 + dS_2 \tag{1.36}$$

$$\epsilon^2 \quad \bar{\Gamma}_3 = \Gamma_3 - \mathcal{L}_3\Gamma_0 - \mathcal{L}_2\Gamma_1 - \mathcal{L}_1\Gamma_2 + \frac{1}{2}\mathcal{L}_1^2\Gamma_1 + \mathcal{L}_2\mathcal{L}_1\Gamma_0 - \frac{1}{6}\mathcal{L}_1^3\Gamma_0 \tag{1.37}$$

This iterative procedure is started with Γ_0 and Γ_1 expressed by:

$$\Gamma = \left(\frac{e}{\epsilon}\mathbf{A} + \left(p_{\,||}\hat{\mathbf{b}} + \mathbf{p}_\perp\right)\right) \cdot d\mathbf{x} - \gamma\ m dt \equiv \epsilon^{-1}\Gamma_0 + \Gamma_1 \tag{1.38}$$

where we assume that $c = 1$ and $m\gamma = \sqrt{p^2 + m^2}$, here p is kinetic particle momentum.

Here the goal is to define the vector fields G_{ia} components that provides the expression for reduced set of phase space coordinates according to the expression (1.30).

Further procedure of gyroangle dependence removing is explicitly detailed in [5].

Such a methods are referred as modern gyrokinetic methods. In the Chapter 4, methods developed by Littlejohn [2, 3, 6] and generalized by Cary and Brizard [5] was implemented during variational derivation of Gyrokinetic Maxwell-Vlasov equations.

The general structure of the action-variational Lie perturbation method can be summarized in two principal stages. At the first stage dynamics of a single charged particle moving in a non-uniform time-independent magnetic field is considered. Then the fast dynamics (gyroangle dependence)is removed when applying near-identity phase space transformation (guiding-center) resulting from application of Lie derivatives. At the end of this procedure guiding-center model for reduced dynamics is obtained.

At the second stage the reduced system is perturbed by electromagnetic fluctuations. These perturbations reintroduces gyrophase dependence inside it one more time. The goal of a new phase space transformation (gyrocenter) is to eliminate second time fast dynamical dependence.

./Lie_transform_picture.eps

Figure 1.2: Lie transform

Ones dynamical reduction is accomplished for a single particle motion, the reduced Vlasov equation can be derived by implementing the pull back transformation. The general idea of such a transformation is presented on the figure below.

Then the Maxwell equations are obtained as a result of calculation of zeroth (Poisson equation) and first (Ampère equation) velocity moments of reduced Vlasov distribution function. It is important to note that that this reduction procedure preserves energy.

In Chapter 3 we use implementation of the Lie transform perturbation method for the gyrocenter Hamiltonian. Then the reduced Vlasov-Maxwell equations are derived using a variational principle with constrained variations that will be explicitly introduced.

1.4 Continuous systems Hamiltonian formalism

Here we propose to consider the problem of Maxwell-Vlasov dynamical reduction from another point of view, by making use of its non-canonical Hamiltonian structure.

Systems that possess Hamiltonian structure are of special interest in physics. Originally, systems endowed with a *canonical* Hamiltonian bracket were recognized. Later, after finding Hamiltonian structure for such systems as the Korteweg–de Vries equation, the usefulness of non-canonical variables was realized. More precisely in [7] the idea to introduce Hamiltonian structure on space of functionals defined over the dynamical variables, appears.

Functional derivative

Here we will employ the notion of the functional derivative. There are some subtitle differences between its mathematical and physical definition. Traditionally functional derivative appears as a generalization of the directional derivative. At the place to take derivative in the direction of a vector, it produces differentiation in the direction of a function. It describes how the entire functional, $F[f(x)]$, changes as a result of a small change in the test function $\varphi(x)$. The mathematical definition gives a relationship independently of the choice of the test function φ and its variation it is defined as:

$$\left\langle \frac{\delta F\,[f]}{\delta f}, \varphi \right\rangle = \int \frac{\delta F[f(x)]}{\delta f(x')}\ \varphi(x')\ dx' \equiv \frac{d}{d\varepsilon} F\,[f + \varepsilon\varphi]\Big|_{\varepsilon=0} \tag{1.39}$$

The physical definition, that we will use in what follows, make choice of the specific test function as Dirac δ- function. It means that we are varying the test function $\varphi(x) = \delta(x - y)$ only about some neighborhood of y. Consequently, there is no variation of $\varphi(x)$ outside of this neighborhood.

$$\frac{\delta F[f(x)]}{\delta f(y)} = \lim_{\varepsilon \to 0} \frac{F[f(x) + \epsilon\ \delta(x - y)] - F[f(x)]}{\varepsilon} \tag{1.40}$$

During the calculations it is convenient to use the following expression:

$$F[f(x) + \delta(x - y)] - F[f(x)] = \int \frac{\delta F}{\delta f} \delta(x - y) dy \tag{1.41}$$

Then we use (1.40) during the derivation of the Maxwell–Vlasov equations as the equations of motion for the Hamiltonian system defined by (1.56) and (1.57).

1.4.1 Korteweg–de Vries

Korteweg–de Vries equation is a mathematical model of waves on shallow water surfaces.

$$u_t = uu_x + u_{xxx} \tag{1.42}$$

This equation was at the center of interest for many reasons. First of all it represents an exactly solvable model, it means that the solutions of such a partial differential equation can be exactly specified; it possesses solitons solutions; it can be solved by means of inverse scattering transform. Here we will address our attention to this model because of its Lagrangian (variational) and Hamiltonian structures. The variational formulation of the eq. (1.42) is given by introducing the Lagrangian :

$$\mathcal{L} = \int dx\ \left[\frac{1}{2} u\ \phi_t - \frac{1}{6} u^3 + \frac{1}{2} u_x^2\right] \tag{1.43}$$

then by writing the corresponding Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial}{\partial t}\Big(\frac{\partial \mathcal{L}}{\partial \phi_t}\Big) + \frac{\partial}{\partial \mathbf{x}}\Big(\frac{\partial \mathcal{L}}{\partial u}\Big) \tag{1.44}$$

and introducing the functional:

$$F\left[u\right] = \int_0^{2\pi} f(u, u_x)dx = \int_0^{2\pi} \left(\frac{1}{6}u^3 - \frac{1}{2}u_x^2\right) dx \tag{1.45}$$

we obtain the eq.(1.42) in the following form:

$$u_t = \frac{\partial}{\partial \mathbf{x}}\Big(\frac{\delta}{\delta u} F\left[u\right]\Big) \tag{1.46}$$

The Hamiltonian formulation for the Korteweg–de Vries equation follows from introduction of the Poisson bracket on the functionals of u:

$$\{G_1, G_2\} = \int_0^{2\pi} dx\ \frac{\delta G_1\left[u\right]}{\delta u}\ \frac{\partial}{\partial x}\left[\frac{\delta G_2\left[u\right]}{\delta u}\right] \tag{1.47}$$

with Hamiltonian $H = F\left[u\right]$. Finally, we can rewrite (1.42)in its Hamiltonian form $u_t = -\{H, u\}$.

Really by applying (1.47) and (1.45) with further integration by parts we obtain $u_t = \displaystyle\int_0^{2\pi} dx\ \frac{\delta F\left[u\right]}{\delta u} \frac{\partial}{\partial x} \frac{\delta u(x)}{\delta u(x')} = -\int_0^{2\pi} \frac{\partial}{\partial x} \frac{\delta F\left[u\right]}{\delta u} \delta(x - x')dx$, where we have used that $\dfrac{\delta u(x)}{\delta u(x')} = \delta(x - x')$.

We will see that the example of Korteweg–de Vries system was pioneering in discovery of Hamiltonian Maxwell-Vlasov structure.

1.4.2 Maxwell-Vlasov

In the case of the Maxwell-Vlasov system, one of the principal difficulties was related to the necessity to describe field-particle interaction, which involves the coupling between fields variables and the canonical phase space variables $\mathbf{P} = m\dot{\mathbf{q}} + e\mathbf{A}(\mathbf{q})$.

The principal ideas that lie behind the discovery of Hamiltonian structure for Maxwell-Vlasov system can be formulated as follows:

- Use of the infinite dimensional phase space realized as space of the functionals $F(f, \mathbf{E}, \mathbf{B})$ on the gauge-invariant (non-canonical) variables: Electromagnetic fields $\mathbf{E} = \mathbf{E}(\mathbf{q})$, $\mathbf{B} = \mathbf{B}(\mathbf{q})$ and Vlasov distribution function $f = f(\mathbf{p}, \mathbf{q})$ with $\mathbf{p}$-kinetic particle momentum

- Translation of the field-particle coupling from the phase space inside the Hamiltonian bracket.

The corresponding non-canonical Hamiltonian structure obtained involving physical intuition and symplectic geometry methods was presented in [8, 9].

Later the relativistic Hamiltonian formulation of Maxwell-Vlasov equations was proposed by Bialynicki–Birula in [10]. It uses the Klimontovich (discrete) representation of particle distribution function: Such a representation expresses each distribution function as a sum of contributions from isolated particles. Here $\xi_A(t)$ and $\pi_A(t)$ denotes the position and kinetic momentum of the A-th particle and S_α represents the set of particles of type α.

$$f_\alpha(\mathbf{p}, \mathbf{q}; t) = \sum_{A \in S_\alpha} \delta\left(\mathbf{q} - \xi_A(t)\right) \delta\left(\mathbf{p} - \pi_A(t)\right) \tag{1.48}$$

The general idea of this work is to obtain the Maxwell-Vlasov Hamiltonian structure using elementary Poisson bracket relations for the set of non-canonical phase space variables, composed of electromagnetic fields $(\mathbf{E}, \mathbf{B})$ and (π, ξ), kinetic particle momentum and position correspondingly. Then we apply the general rule:

$$\{F, G\} = \sum_{i,j} \frac{\partial F}{\partial \chi^i} \{\chi^i, \chi^j\} \frac{\partial G}{\partial \chi^j} \tag{1.49}$$

The Poisson bracket for electromagnetic field was proposed by Born and Infeld (1935)

$$\{\mathbf{B}_i(\mathbf{q}), \mathbf{E}_j(\mathbf{q}')\} = \epsilon_{ijk}\ \partial_k \delta(\mathbf{q}' - \mathbf{q}) \tag{1.50}$$

where ∂_k designs k-th component of spatial gradient. The Poisson brackets that introduces coupling between fields and particles uses the expression (1.48) for particle distribution function:

This coupling elementary Poisson brackets are

$$\{\xi^i_A, \pi^j_B\} = \delta_{AB}\delta_{ij} \tag{1.51}$$

$$\{\pi^i_A, \pi^j_B\} = \delta_{AB} e_A \epsilon_{ijk} \mathbf{B}^k(\xi_A) \tag{1.52}$$

$$\{\pi^i_A, E^j(\mathbf{q})\} = e_A \delta_{ij} \delta(\mathbf{q} - \xi_A) \tag{1.53}$$

Further generalization to the continuous case of Vlasov distribution function is realized by replacing the partial derivatives by the functional ones and the sum by an integral in (1.49).

Another important remark that we should make there is about the physical constraints that are imposed on the phase space in each of the methods leading to the Maxwell-Vlasov Hamiltonian formulation. As we have mentioned above, in such an approach the phase space is infinity dimensional, composed by particle distribution function that obey Vlasov equation $f(\mathbf{p}, \mathbf{q})$, and electromagnetic fields $\mathbf{E}(\mathbf{q})$ and $\mathbf{B}(\mathbf{q})$. Two physical constraints, expressed by two of Maxwell's equations, are imposed on this phase space:

$$\nabla \cdot \mathbf{B} = 0 \tag{1.54}$$

$$\nabla \cdot \mathbf{E} = e \int d^{\,3}p\, f(\mathbf{p}, \mathbf{q}) \tag{1.55}$$

Note that such a constraints are preserved by the time evolution of the system. Two others Maxwell's equations play the role of dynamical ones. The observables forms the vector space of "smooth" functionals over the functions $f(\mathbf{p},\mathbf{q}), \mathbf{E}(\mathbf{q}), \mathbf{B}(\mathbf{q})$. Maxwell-Vlasov Poisson bracket preserves this vector space, so that the observables form a Poisson algebra. In this approach the interaction between the plasma and the electromagnetic field is introduced entirely through the following Poisson bracket:

$$\begin{aligned}
\{F,G\} = \iint d^{\,3}q\, d^{\,3}p\, f\, &\left[\frac{\partial}{\partial \mathbf{p}}\frac{\delta F}{\delta f}\cdot\frac{\partial}{\partial \mathbf{q}}\frac{\delta G}{\delta f} - \frac{\partial}{\partial \mathbf{q}}\frac{\delta F}{\delta f}\cdot\frac{\partial}{\partial \mathbf{p}}\frac{\delta G}{\delta f}\right] \\
+\int d^{\,3}q\, &\left[\nabla\times\frac{\delta F}{\delta \mathbf{B}}\cdot\frac{\delta G}{\delta \mathbf{E}} - \frac{\delta F}{\delta \mathbf{E}}\cdot\nabla\times\frac{\delta G}{\delta \mathbf{B}}\right] \\
+\iint d^{\,3}q\, d^{\,3}p\, \frac{\partial f}{\partial \mathbf{p}}\cdot &\left[\frac{\delta F}{\delta f}\frac{\delta G}{\delta \mathbf{E}} - \frac{\delta F}{\delta \mathbf{E}}\frac{\delta G}{\delta f}\right] \\
-e\iint d^{\,3}q\, d^{\,3}p\, f\, \mathbf{B}\cdot &\left[\frac{\partial}{\partial \mathbf{p}}\frac{\delta F}{\delta f}\times\frac{\partial}{\partial \mathbf{p}}\frac{\delta G}{\delta f}\right]
\end{aligned} \tag{1.56}$$

Here fluid approach is used: $(\mathbf{p},\mathbf{q})$ do not undergo time evolution and play the role of labels permitting to mark degrees of freedom. The first term in this expression represents particle bracket, the second one-field bracket and the last two terms introduces the retroaction between fields and particles.

The Hamiltonian is given by the kinetic energy of particles plus the energy of the electromagnetic fields[4]:

$$H\left[f,\mathbf{E},\mathbf{B}\right] = \int\int d^{\,3}q\, d^{\,3}p\, f\, m\gamma + \int d^{\,3}q\, \frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{2} \tag{1.57}$$

where $m\gamma = \sqrt{p^2+m^2}$ and $|\mathbf{B}|^2 \equiv \mathbf{B}\cdot\mathbf{B}$ is the field norm.

Equations of motion

We start by obtaining the expression for the Liouville operator which is derived from the Hamiltonian (1.57) and the Poisson bracket above (1.56). By taking into account the expressions for functional derivatives:

$$\frac{\delta H}{\delta f} = m\gamma,\ \frac{\delta H}{\delta \mathbf{E}} = \mathbf{E},\ \frac{\delta H}{\delta \mathbf{B}} = \mathbf{B} \tag{1.58}$$

[4]Here we suppose that $c = 1$

and by integrating by parts, we have

$$\{H\} = -\int\int d\,^3q\,d\,^3p\Big(\mathbf{v}\cdot\nabla f + e\ (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{p}}\Big)\frac{\delta}{\delta f} \tag{1.59}$$

$$+ \int\int d\,^3q\,\Big(\nabla\times\mathbf{B}\cdot\frac{\delta}{\delta\mathbf{E}} - \nabla\times\mathbf{E}\cdot\frac{\delta}{\delta\mathbf{B}}\Big) \tag{1.60}$$

$$+ \int\int d\,^3q\,d\,^3p\;m\gamma\,\frac{\partial f}{\partial\mathbf{p}}\cdot\frac{\delta}{\delta\mathbf{E}} \tag{1.61}$$

Then the Maxwell-Vlasov equations are:

$$\dot{\mathbf{E}} = \{H,\mathbf{E}\} = \nabla\times\mathbf{B} - \int d\,^3p\;\mathbf{v}\;f \tag{1.62}$$

$$\dot{\mathbf{B}} = \{H,\mathbf{B}\} = -\nabla\times\mathbf{E} \tag{1.63}$$

$$\dot{f} = \{H,f\} = -\mathbf{v}\;\partial_{\mathbf{q}}f(\mathbf{p},\mathbf{q}) - e\ (\mathbf{E}+\mathbf{v}\times\mathbf{B})\ \partial_{\mathbf{p}}f(\mathbf{p},\mathbf{q}) \tag{1.64}$$

1.5 Hamiltonian perturbation theory

The general idea of our approach is to treat coupling between fields and particles as a perturbation of some uncoupled motion. Let us consider the system with simplified Hamiltonian:

$$H_0\,[f,\mathbf{E},\mathbf{B}] = \int\int d\,^3q\,d\,^3p\;f\;m\gamma + \int d\,^3q\;\frac{|\mathbf{B}|^2}{2} \tag{1.65}$$

The dynamics of this system possesses one remarkable property: the magnetic field does not evaluate under the flow generated by the Hamiltonian H_0.

By substituting the expression for the Hamiltonian H_0 in the Maxwell-Vlasov Poisson bracket we obtain:

$$\dot{\mathbf{B}} = \{H_0,\mathbf{B}\} = 0 \tag{1.66}$$

$$\dot{\mathbf{E}} = \{H_0,\mathbf{E}\} = \nabla\times\mathbf{B} - e\int d\,^3p\;\mathbf{v}\;f(\mathbf{p},\mathbf{q}) \tag{1.67}$$

$$\dot{f} = \{H_0,f\} = -\mathbf{v}\cdot\frac{\partial f(\mathbf{p},\mathbf{q})}{\partial\mathbf{q}} - e\ (\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f(\mathbf{p},\mathbf{q})}{\partial\mathbf{p}} \tag{1.68}$$

where $\mathbf{v}\equiv\mathbf{p}/m\gamma$ denotes the relativistic particle velocity.

Another important property of such a system is that the electric field dynamics is now uncoupled from the particle dynamics. Then now field and particles can be considered separately.

Using Euler-Lagrangian duality we can project particle dynamics on the 6 dimensional phase space $(\mathbf{p},\mathbf{q})$. The key property that we will use during realization of such a projection is the fact that magnetic field $\mathbf{B}$ is constant under the simplified Hamiltonian flow.

Euler-Lagrange duality

In this thesis we adopt both Eulerian and Lagrangian viewpoints, summarized in Table 1.1.

Euler	Lagrange
Observables	
$F[f]$	$f(p,q)$
Phase space	
$\mathbf{E}(\mathbf{q}), \mathbf{B}(\mathbf{q}), f(\mathbf{p},\mathbf{q})$	$(\mathbf{p},\mathbf{q})$
Poisson bracket	
Maxwell-Vlasov	Gyroscopic
Kinetic energy	
$\int d\,^3\mathbf{q}\, f(\mathbf{p},\mathbf{q}) m\gamma$	$m\gamma$
Equation of motion	
$\dot{f} = -\left(\mathbf{v}\cdot\partial_{\mathbf{q}} + e\ \left(\mathbf{v}\times\mathbf{B}\right)\cdot\partial_{\mathbf{p}}\right)$	$\dot{\mathbf{p}} = e\,\mathbf{v}\times\mathbf{B},\ \dot{\mathbf{q}} = \mathbf{v}$

Table 1.1: Summary of the Eulerian and Lagrangian descriptions

Discussion

The next step in our perturbative construction is to consider dynamical reduction for particle motion in a non-uniform external magnetic field $\mathbf{B}$. This problem is considered in Chapter 4 of this dissertation. The next step of such a reduction procedure will consist of perturbative field-particle coupling reintroduction into the system.

Overview of the dissertation

The text of this dissertation is organized as follows.

In Chapter 2 Hamiltonian control method is implemented in order to study barrier formation in $\mathbf{E} \times \mathbf{B}$ drift model. Chapter 3 deals with investigation of momentum transport through derivation of the momentum conservation law for Maxwell-Vlasov equations.

Chapter 4 explores the fundamental geometrical problems related to the dynamical reduction of charged particle motion in an non-uniform magnetic field. This work represents an important step in the construction of the alternative method for dynamical reduction of the Maxwell-Vlasov system.

Chapter 2

Barriers for the reduction of transport due to the $E \times B$ drift in magnetized plasmas

Abstract.

We consider a $1\frac{1}{2}$ degrees of freedom Hamiltonian dynamical system, which models the chaotic dynamics of charged test-particles in a turbulent electric field, across the confining magnetic field in controlled thermonuclear fusion devices. The external electric field $\mathbf{E} = -\nabla V$ is modeled by a phenomenological potential V and the magnetic field $\mathbf{B}$ is considered uniform. It is shown that, by introducing a small additive control term to the external electric field, it is possible to create a transport barrier for this dynamical system. The robustness of this control method is also investigated. This theoretical study indicates that alternative transport barriers can be triggered without requiring a control action on the device scale as in present Internal Transport Barriers (ITB).

2.1 Introduction

It has long been recognized that the confinement properties of high performance plasmas with magnetic confinement are governed by electromagnetic turbulence that develops in microscales [11]. In that framework various scenarios are explored to lower the turbulent transport and therefore improve the overall performance of a given device. The aim of such a research activity is two-fold.

First, an improvement with respect to the basic turbulent scenario, the so-called L-mode (L for low) allows one to reduce the reactor size to achieve a given fusion power and to improve the economical attractiveness of fusion energy production. This line of thought has been privileged for ITER that considers the H-mode (H for high) to achieve an energy amplification factor of 10 in its reference scenario [12]. The H-mode scenario is based on a local reduction of the turbulent transport in a narrow regime in the vicinity of the outermoster confinement surface [13].

Second, in the so-called advanced tokamak scenarios, Internal Transport Barriers are considered [12]. These barriers are characterised by a local reduction of turbulent transport with two important consequences, first an improvement of the core fusion performance, second the generation of bootstrap current that provides a means to generate the required plasma current in regime with strong gradients [14]. The research on ITB then appears to be important in the quest of steady state operation of fusion reactors, an issue that also has important consequences for the operation of fusion reactors.

The H-mode appears as a spontaneous bifurcation of turbulent transport properties in the edge plasma [13], the ITB scenarios are more difficult to generate in a controlled fashion [15]. Indeed, they appear to be based on macroscopic modifications of the confinement properties that are both difficult to drive and difficult to control in order to optimise the performance.

In this paper, we propose an alternative approach to transport barriers based on a macroscopic control of the $E \times B$ turbulence. Our theoretical study is based on a localized hamiltonian control method that is well suited for $E \times B$ transport. In a previous approach [16], a more global scheme was proposed with a reduction of turbulent transport at each point of the phase space. In the present work, we derive an exact expression to govern a local control at a chosen position in phase space. In principle, such an approach allows one to generate the required transport barriers in the regions of interest without enforcing large modification of the confinement properties to achieve an ITB formation [15]. Although the application of such a precise control scheme remains to be assessed, our approach shows that local control transport barriers can be generated without requiring macroscopic changes of the plasma properties to trigger such barriers. The scope of the present work is the theoretical demonstration of the control scheme and consequently the possibility of generating transport barriers based on more specific control schemes than envisaged in present advanced scenarios.

In Section 2.2, we give the general description of our model and the physical motivations for our investigation. In Section 2.3, we explain the general method of *localized control* for Hamiltonian systems and we estimate the size of the control term. Section 2.4 is devoted to the numerical investigations of the control term, and we discuss its robustness and its energy cost. The last section 2.5 is devoted to conclusions and discussion.

2.2 Physical motivations and the $E \times B$ model

2.2.1 Physical motivations

Fusion plasma are sophisticated systems that combine the intrinsic complexity of neutral fluid turbulence and the self-consistent response of charged species, both electrons and ions, to magnetic fields. Regarding magnetic confinement in a tokamak, a large external magnetic field and a first order induced magnetic field are organised to generate the so-called magnetic equilibrium of nested toroidal magnetic surfaces [17]. On the latter, the plasma can be sustained close to a local thermodynamical equilibrium. In order to analyse turbulent transport we consider plasma perturbations of this class of solutions with no evolution of the magnetic equilibrium, thus excluding MHD instabilities. Such perturbations self-consistently generate electromagnetic perturbations that feedback on the plasma evolution. Following present experimental evidence, we shall assume here that magnetic fluctuations have a negligible impact on turbulent transport [18]. We will thus concentrate on electrostatic perturbations that correspond to the vanishing β limit, where $\beta = p/(B^2/2\mu_0)$ is the ratio of the plasma pressure p to the magnetic pressure. The appropriate framework for this turbulence is the Vlasov equation in the gyrokinetic approximation associated with the Maxwell-Gauss equation that relates the electric field to the charge density. When considering the Ion Temperature Gradient instability [19] that appears to dominate the ion heat transport, one can further assume the electron response to be adiabatic so that the plasma response is governed by the gyrokinetic Vlasov equation for the ion species.

Let us now consider the linear response of such a distribution function $\widehat{f}$, to a given electrostatic perturbation, typically of the form $T_e\ \widehat{\phi}\ e^{-i\omega t+i\vec{k}\vec{r}}$, (where $\widehat{f}$ and $\widehat{\phi}$ are Fourier amplitudes of distribution function and electric potential). To leading orders one then finds that the plasma response exhibits a resonance:

$$\widehat{f} = \left(\frac{\omega + \omega^*}{\omega - k_{||}\ v_{||}} - 1 \right) \widehat{\phi} f_{eq} \tag{2.1}$$

Here f_{eq} is the reference distribution function, locally Maxwellian with respect to $v_{||}$ and ω^* is the diamagnetic frequency that contains the density and temperature gradient that drive the ITG instability [19]. T_e is the electronic temperature. This simplified plasma response to the electrostatic perturbation allows one to illustrate the turbulent control that is considered to trigger off transport barriers in present tokamak experiments.

Let us examine the resonance $\omega - k_{||}\ v_{||} = 0$ where $k_{||} = (n - m/q)/R$ with R being the major radius, q the safety factor that characterises the specific magnetic equilibrium and m and n the wave numbers of the perturbation that yield the wave vectors of the perturbation in the two periodic directions of the tokamak equilibrium. When the turbulent frequency ω is small with respect to $v_{th}/(qR)$, (where $v_{th} =$

$\sqrt{k_B T/m}$ is the thermal velocity), the resonance occurs for vanishing values of $k_{||}$, and as a consequence at given radial location due to the radial dependence of the safety factor. The resonant effect is sketched on figure 2.1. In a quasilinear approach,

./Fig0.eps

Figure 2.1: Resonances for $q = \frac{m}{n}$ and $q = \frac{m+1}{n}$ for two different widths, narrow resonances empedding large scale turbulent transport and broad resonances favouring strong turbulent transport.

the response to the perturbations will lead to large scale turbulent transport when the width of the resonance δ_m is comparable to the distance between the resonances $\Delta_{m,m+1}$ leading to an overlap criterion that is comparable to the well known Chirikov criterion for chaotic transport $\sigma_m = (\delta_m + \delta_{m+1})/\Delta_{m,m+1}$ with $\sigma > 1$ leading to turbulent transport across the magnetic surfaces and $\sigma < 1$ localising the turbulent transport to narrow radial regions in the vicinity of the resonant magnetic surfaces.

The present control schemes are two-fold. First, one can consider a large scale radial electric field that governs a Doppler shift of the mode frequency ω. As such the Doppler shift $\omega - \omega_E$ has no effect. However a shear of the Doppler frequency ω_E, $\omega_E = \bar{\omega}_E + \delta r \omega'_E$ will induce a shearing effect of the turbulent eddies and thus control the radial extent of the mode δ_m, so that one can locally achieve $\sigma < 1$ in order to drive a transport barrier.

Second, one can modify the magnetic equilibrium so that the distance between the resonant surfaces is strongly increased in particular in a magnetic configuration with weak magnetic shear ($dq/dr \approx 0$) so that $\Delta_{m,m+1}$ is strongly increased, $\Delta_{m,m+1} \gg \delta_m$, also leading to $\sigma < 1$.

Both control schemes for the generation of ITBs can be interpreted using the situation sketched on figure 2.1. The initial situation with large scale radial transport across the magnetic surfaces (so called L-mode) is indicated by the dashed lines and is governed by significant overlap between the resonances. The ITB control scheme aims at either reducing the width of the islands or increasing the distance between the resonances yielding a situation sketeched by the plain line in figure 2.1 where the overlap is too small and a region with vanishing turbulent transport, the ITB, develops between the resonances.

Experimental strategies in advanced scenarios comprising Internal Transport Barriers are based on means to enforce these two control schemes. In both cases they aim at modifying macroscopically the discharge conditions to fulfill locally the

$\sigma < 1$ criterion. It thus appears interesting to devise a control scheme based on a less intrusive action that would allow one to modify the chaotic transport locally by the choice of an appropriate electrostatic perturbation hence leading to a local transport barrier.

2.2.2 The $E \times B$ model

For fusion plasmas, the magnetic field B is slowly variable with respect to the inverse of the Larmor radius ρ_L i.e: $\rho_L|\nabla \ln B| \ll 1$. This fact allows the separation of the motion of a charged test particle into a slow motion (parallel to the lines of the magnetic field) and a fast motion (Larmor rotation). This fast motion is named gyromotion, around some gyrocenter. In first approximation the averaging of the gyromotion over the gyroangle gives the approximate trajectory of the charged particle. This averaging is the guiding-center approximation.

In this approximation, the equations of motion of a charged test particle in the presence of a strong uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, (where $\hat{\mathbf{z}}$ is the unit vector in the z direction) and of an external time-dependent electric field $\mathbf{E} = -\nabla V_1$ are:

$$\begin{aligned} \frac{d}{dT}\begin{pmatrix} X \\ \\ Y \end{pmatrix} &= \frac{c\mathbf{E}\times\mathbf{B}}{B^2} = \frac{c}{B}\,\mathbf{E}(X,Y,T)\times\hat{\mathbf{z}} \\ &= \frac{c}{B}\begin{pmatrix} -\partial_Y V_1(X,Y,T) \\ \\ \partial_X V_1(X,Y,T) \end{pmatrix} \end{aligned} \tag{2.2}$$

where V_1 is the electric potential. The spatial coordinates X and Y play the role of canonically-conjugate variables and the electric potential $V_1(X,Y,T)$ is the Hamiltonian for the problem. Now the problem is placed into a parallelepipedic box with dimensions $L \times \ell \times (2\pi/\omega)$, where L and ℓ are some characteristic lengths and ω is a characteristic frequency of our problem, X is locally a radial coordinate and Y is a poloidal coordinate. A phenomenological model [20] is chosen for the potential:

$$V_1(X,Y,T) = \sum_{n,m=1}^{N} \frac{V_0\ \cos\chi_{n,m}}{(n^2+m^2)^{3/2}} \tag{2.3}$$

where V_0 is some amplitude of the potential,

$$\chi_{n,m} \equiv \frac{2\pi}{L}nX + \frac{2\pi}{\ell}mY + \phi_{n,m} - \omega T$$

ω is constant, for simplifying the numerical simulations and $\phi_{n,m}$ are some random phases (uniformly distributed).

We introduce the dimensionless variables

$$(x,y,t) \equiv (2\pi X/L,\ 2\pi Y/\ell,\ \omega T) \tag{2.4}$$

So the equations of motion (2.2) in these variables are:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\partial_y V(x,y,t) \\ \partial_x V(x,y,t) \end{pmatrix} \tag{2.5}$$

where $V = \varepsilon(V_1/V_0)$ is a dimensionless electric potential given by

$$V(x,y,t) = \varepsilon \sum_{n,m=1}^{N} \frac{\cos\left(nx + my + \phi_{n,m} - t\right)}{(n^2+m^2)^{3/2}} \tag{2.6}$$

Here

$$\varepsilon = 4\pi^2 (cV_0/B)/(L\ell\omega) \tag{2.7}$$

is the small dimensionless parameter of our problem. We perturb the model potential (2.6) in order to build a transport barrier. The system modeled by Eqs.(2.5) is a $1\frac{1}{2}$ degrees of freedom system with a chaotic dynamics [16, 20]. The poloidal section of our modeled tokamak is a Poincaré section for this problem and the stroboscopic period will be chosen to be 2π, in term of the dimensionless variable t.

The particular choice (2.3) or (2.6) is not crucial and can be generalized. Generally, ω can be chosen depending on n, m. This would make the numerical computations more involved. In the following section, V is chosen completely arbitrary.

2.3 Localized control theory of hamiltonian systems

2.3.1 The control term

In this section we show how to construct a transport barrier for any electric potential V. The electric potential $V(x,y,t)$ yields a non-autonomous Hamiltonian. We expand the two-dimensional phase space by including the canonically-conjugate variables (w,τ),

$$H = H(x,\tau;y,w) = V(x,y,\tau) - w \tag{2.8}$$

The Hamiltonian of our system thus becomes autonomous. Here τ is a new variable whose dynamics is trivial: $\dot{\tau} = 1$ i.e. $\tau = \tau_0 + t$ and w is the variable (momentum) canonically conjugate to τ. The Poisson bracket operator in the expanded phase space for any $U = U(x,\tau;y,w)$ is given by the expression:

$$\{U\} \equiv (\partial_x U)\partial_y - (\partial_y U)\partial_x + (\partial_\tau U)\partial_w - (\partial_w U)\partial_\tau. \tag{2.9}$$

Hence $\{U\}$ is a linear (differential) operator acting on functions of $(x,\tau;y,w)$. We call $H_0 = w$ the unperturbed Hamiltonian and $V(x,y,\tau)$ its perturbation. We now implement a perturbation theory for H_0. The operator of the Poisson bracket (4.6) for the Hamiltonian H is

$$\{H\} = (\partial_x V)\partial_y - (\partial_y V)\partial_x + \partial_\tau + (\partial_\tau V)\partial_w \tag{2.10}$$

So the equations of motion in the expanded phase space are:

$$\dot{y} = \{H\}y = \partial_x V(x,y,\tau) \tag{2.11}$$
$$\dot{x} = \{H\}x = -\,\partial_y V(x,y,\tau) \tag{2.12}$$
$$\dot{w} = \{H\}w = \partial_\tau V(x,y,\tau) \tag{2.13}$$
$$\dot{\tau} = \{H\}\tau = 1 \tag{2.14}$$

We want to construct a small modification F of the potential V such that

$$\widetilde{H} \equiv V(x,y,\tau) + F(x,y,\tau) - w \equiv \widetilde{V}(x,y,\tau) - w \tag{2.15}$$

has a barrier at some chosen position $x = x_0$. So the control term

$$F = \widetilde{V}(x,y,\tau) - V(x,y,\tau) \tag{2.16}$$

must be much smaller than the perturbation (e.g., quadratic in V). One of the possibilities is:

$$\widetilde{V} \equiv V(x + \partial_y f(y,\tau), y, \tau) \tag{2.17}$$

where

$$f(y,\tau) \equiv \int_0^\tau V(x_0,y,t)dt$$

Indeed we have the following theorem:

Theorem 1 *The Hamiltonian $\widetilde{H}$ has a trajectory $x = x_0 + \partial_y f(y,\tau)$ acting as a barrier in phase space.*

Proof

Let the Hamiltonian $\widehat{H} \equiv \exp(\{f\})\tilde{H}$ be canonically related to $\widetilde{H}$. (Indeed the exponential of any Poisson bracket is a canonical transformation.) We show that $\widehat{H}$ has a simple barrier at $x = x_0$. We start with the computation of the bracket (4.6) for the function f. Since $f = f(y,\tau)$, the expression for this bracket contains only two terms,

$$\{f\} \ \equiv\ -f'\partial_x \ +\ \dot{f}\partial_w \tag{2.18}$$

where

$$f' \equiv \partial_y f \text{ and } \dot{f} \equiv \partial_\tau f \tag{2.19}$$

which commute:

$$[f'\partial_x, \dot{f}\partial_w] = 0 \tag{2.20}$$

Now let us compute the coordinate transformation generated by $\exp(\{f\})$:

$$\exp(\{f\}) \;\equiv\; \exp(-f'\partial_x)\; \exp(\dot{f}\partial_w), \tag{2.21}$$

where we used (2.20) to separate the two exponentials.

Using the fact that $\exp(b\partial_x)$ is the translation operator of the variable x by the quantity b: $[\exp(b\partial_x)W](x) = W(x+b)$, we obtain

$$\begin{aligned}
\widehat{H} &= e^{\{f\}}\tilde{H} \;\equiv\; e^{\{f\}}\tilde{V}(x,y,\tau) \;-\; e^{\{f\}}w \\
&= \widetilde{V}\,(x - f', y, \tau) \;-\; \left(w \;+\; \dot{f}\right) \\
&= V(x + f' - f', y, \tau) - \; V(x_0, y, \tau) - w \\
&= V(x, y, \tau) - V(x_0, y, \tau) - w
\end{aligned} \tag{2.22}$$

This Hamiltonian has a simple trajectory $x = x_0, w = w_0$, i.e. any initial data $x = x_0,\; y = y_0, w = w_0, \tau = \tau_0$ evolves under the flow of $\widehat{H}$ into $x = x_0, y = y_t, w = w_0, \tau = \tau_0 + t$ for some evolution y_t that may be complicated, but not useful for our problem. Hamilton's equations for x and w are now

$$\dot{x} \;=\; \{\widehat{H}\}x = \partial_y\left[V(x_0,y,\tau) - V(x,y,\tau)\right] \tag{2.23}$$

$$\dot{w} \;=\; \{\widehat{H}\}w = \partial_\tau\left[V(x_0,y,\tau) - V(x,y,\tau)\right] \tag{2.24}$$

so that for $x = x_0$, we find $\dot{x} = 0 = \dot{w}$. Then the union of all points (x, y, w, τ) at $x = x_0 \;\; w = w_0$:

$$\mathfrak{B}_0 = \bigcup_{y,\tau,w_0} \begin{pmatrix} x_0 \\ y \\ w_0 \\ \tau \end{pmatrix} \tag{2.25}$$

is a 3-dimensional surface $\mathbb{T}^2 \times \mathbb{R}$, $(\mathbb{T} \equiv \mathbb{R}/2\pi\mathbb{Z})$ preserved by the flow of $\widehat{H}$ in the 4-dimensional phase space. If an initial condition starts on $\mathfrak{B}_0$, its evolution under the flow $\exp(t\{\widehat{H}\})$ will remain on $\mathfrak{B}_0$.

So we can say that $\mathfrak{B}_0$ act as a barrier for the Hamiltonian $\widehat{H}$: the initial conditions starting inside $\mathfrak{B}_0$ can't evolve outside $\mathfrak{B}_0$ and vice-versa.

To obtain the expression for a barrier $\mathfrak{B}$ for $\widetilde{H}$ we deform the barrier for $\widehat{H}$ via the transformation $\exp(\{f\})$. As

$$\widetilde{H} = e^{-\{f\}}\widehat{H} \tag{2.26}$$

and $\exp(\{f\})$ is a canonical transformation, we have

$$\{\widetilde{H}\} = \{e^{-\{f\}}\widehat{H}\} = e^{-\{f\}}\{\widehat{H}\}e^{\{f\}} \tag{2.27}$$

Now let us calculate the flow of $\widetilde{H}$:

$$e^{t\{\widetilde{H}\}} = e^{t\left(e^{-\{f\}}\{\widehat{H}\}e^{\{f\}}\right)} = e^{-\{f\}}e^{t\{\widehat{H}\}}e^{\{f\}} \tag{2.28}$$

Indeed:

$$e^{t\left(e^{-\{f\}}\{\widehat{H}\}e^{\{f\}}\right)} = \sum_{n=0}^{\infty} \frac{t^n(e^{-\{f\}}\{\widehat{H}\}e^{\{f\}})^n}{n!} \tag{2.29}$$

For instance when $n = 2$:

$$\begin{aligned} t^2(e^{-\{f\}}\{\widehat{H}\}e^{\{f\}})^2 &= t^2 e^{-\{f\}}\{\widehat{H}\}e^{\{f\}}e^{-\{f\}}\{\widehat{H}\}e^{\{f\}} \\ &= t^2 e^{-\{f\}}\{\widehat{H}\}^2 e^{\{f\}} \end{aligned} \tag{2.30}$$

and so

$$e^{t\{\widetilde{H}\}} = \sum_{n=0}^{\infty} \frac{t^n e^{-\{f\}}\{\widehat{H}\}^n e^{\{f\}}}{n!} = e^{-\{f\}}e^{t\{\widehat{H}\}}e^{\{f\}} \tag{2.31}$$

As we have seen before:

$$e^{\{f\}} \begin{pmatrix} x \\ y \\ w \\ \tau \end{pmatrix} = \begin{pmatrix} x - f' \\ y \\ w - \dot{f} \\ \tau \end{pmatrix}$$

and

$$e^{t\{\widehat{H}\}} \begin{pmatrix} x_0 \\ y \\ w_0 \\ \tau \end{pmatrix} = \begin{pmatrix} x_0 \\ y_t \\ w_0 \\ \tau + t \end{pmatrix} \tag{2.32}$$

Multiplying (2.28) on the right by $e^{-\{f\}}$ we obtain:

$$\begin{aligned} e^{t\{\widetilde{H}\}}e^{-\{f\}} &= e^{-\{f\}}e^{t\{\widehat{H}\}} \\ e^{t\{\widetilde{H}\}}e^{-\{f\}} \begin{pmatrix} x_0 \\ y \\ w_0 \\ \tau \end{pmatrix} &= e^{t\{\widetilde{H}\}} \begin{pmatrix} x_0 + f'(y,\tau) \\ y \\ w_0 + \dot{f}(y,\tau) \\ \tau \end{pmatrix} \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} e^{-\{f\}}e^{t\{\widehat{H}\}} \begin{pmatrix} x_0 \\ y \\ w_0 \\ \tau \end{pmatrix} &= e^{-\{f\}} \begin{pmatrix} x_0 \\ y_t \\ w_0 \\ \tau + t \end{pmatrix} \\ &= \begin{pmatrix} x_0 + f'(y_t, \tau + t) \\ y_t \\ w_0 + \dot{f}(y_t, \tau + t) \\ \tau + t \end{pmatrix} \end{aligned} \tag{2.34}$$

So the flow $\exp(t\{\widetilde{H}\})$ preserves the set

$$\mathfrak{B} = \bigcup_{y,\tau,w_0} \begin{pmatrix} x_0 + f'(y,\tau) \\ y \\ w_0 + \dot{f}(y,\tau) \\ \tau \end{pmatrix} \tag{2.35}$$

$\mathfrak{B}$ is a 3 dimensional invariant surface, topologically equivalent to $\mathbb{T}^2 \times \mathbb{R}$ into the 4 dimensional phase space. $\mathfrak{B}$ separates the phase space into 2 parts, and is a barrier between its interior and its exterior. $\mathfrak{B}$ is given by the deformation $\exp(\{f\})$ of the simple barrier $\mathfrak{B}_0$.

The section of this barrier on the sub space (x, y, t) is topologically equivalent to a torus $\mathbb{T}^2$.

This method of control has been successfully applied to a real machine: a traveling wave tube to reduce its chaos [21].

2.3.2 Properties of the control term

In this Section, we estimate the size and the regularity of the control term (2.16).

Theorem 2 *For the phenomenological potential (2.6) the control term (2.16) verifies:*

$$\|F\|_{\frac{1}{N},\frac{1}{N}} \leq \varepsilon^2 N^2 \frac{e^3}{4\pi} \tag{2.36}$$

if ε is small enough, i.e. if $|\varepsilon| \leq \frac{\sqrt{\pi}}{2Ne^{3/2}}$ where N is the number of modes in the sum (2.6).

Proof The proof of this estimation is given in [22] and is based on rewriting

$$\begin{aligned} F = V(x+f') - V(x) &= \int_0^1 ds\ \partial_x V(x+sf',y,\tau) f'(y,\tau) \\ &= \mathcal{O}(V^2) \end{aligned} \tag{2.37}$$

and then use Cauchy's Theorem.

2.4 Numerical investigations for the control term

In this Section, we present the results of our numerical investigations for the control term F. The theoretical estimate presented in the previous section shows that its size is quadratic in the perturbation. Figure 2.2 shows the contour plot of $V(x, y, t)$

and $\widetilde{V}(x,y,t)$ ($\widetilde{V} = V + F$) at some fixed time t, for example $t = \frac{\pi}{4}$. One can see that the contours of both potentials are very similar. But the dynamics of the systems with V and $\tilde{V}$ are very different.

For all numerical simulations we choose the number of modes $N = 25$ in (2.6). In all plots the abscissa is x and the ordinate is y.

./Fig1.eps

Figure 2.2: Uncontrolled and controlled potential for $\varepsilon = 0.6$, $t = \frac{\pi}{4}$, $x_0 = 2$.

2.4.1 Phase portrait for the exact control term

To explore the effectiveness of the barrier, we plot (in Fig. 2.3) the phase portraits for the original system (without control term) and for the system with the exact control term F. We choose the same initial conditions. The time of integration is $T = 2000$, the number of trajectories: $N_{traj} = 200$ (number of initial conditions, all taken in the strip $-1 - \pi \leq x \leq -\pi$; $0 \leq y \leq 2\pi$) and the parameter $\varepsilon = 0.9$. We choose the barrier at position $x_0 = 2$. To get a Poincaré section, we plot the poloidal section when $t \in 2\pi\mathbb{Z}$. Then we compare the number of trajectories passing through the barrier during this time of integration for each system. We eliminate the points after the crossing. For the uncontrolled system 68% of the initial conditions cross the barrier at $x_0 = 2$ and for the controlled system only 1% of the trajectories escape from the zone of confinement. The theory announces the existence of an exact barrier for the controlled system: these escaped trajectories (1%) are due to numerical errors in the integration. One can observe that the barrier for the controlled system is a straight line. In fact this barrier moves, its expression depends on time:

$$x = x_0 + f'(y,t) \tag{2.38}$$

But when $t \in 2\pi\mathbb{Z}$ its oscillation around $x = x_0$ vanishes: $f'(y, 2k\pi) = \int_0^{2k\pi} \partial_y V(x_0, y, t)dt = 0$. This is what we see on this phase portrait. In fact we create 2 barriers at position $x = x_0$, and $x = x_0 - 2\pi$ (and also at $x_0 + 2n\pi$) because of the periodicity of the problem. We note that the mixing increases inside the two

./Fig2.eps

Figure 2.3: Phase portraits without control term and with the exact control term, for $\varepsilon = 0.9$, $x_0 = 2$, $N_{traj} = 200$.

barriers. The same phenomenon was also observed in the control of fluids [23], where the same method was applied.

2.4.2 Robustness of the barrier

In a real Tokamak, it is impossible to know an analytical expression for electric potential V. So we can't implement the exact expression for F. Hence we need to test the robustness of the barrier by truncating the Fourier decomposition (for instance in time) of the controlled potential.

Fourier decomposition

Theorem 3 *The potential (2.17) can be decomposed as* $\widetilde{V} = \sum_{k\in\mathbb{Z}} \widetilde{V}_k$, *where*

$$\widetilde{V}_k = \varepsilon \sum_{n,m=1}^{N} \frac{\mathcal{J}_k(n\rho)}{(n^2+m^2)^{3/2}} \cos\left(\eta + k\Theta + (k-1)t\right) \tag{2.39}$$

with

$$\begin{aligned}
\eta_{n,m}(y) &= nx + my + \phi_{n,m} + n\varepsilon F_c && (2.40)\\
F_c(y) &= \sum_{n,m=1}^{N} \frac{m\ \cos(K_{n,m,y})}{(n^2+m^2)^{3/2}} && (2.41)\\
F_s(y) &= \sum_{n,m=1}^{N} \frac{m\ \sin(K_{n,m,y})}{(n^2+m^2)^{3/2}} && (2.42)\\
K_{m,n,y} &= nx_0 + my + \phi_{n,m} && (2.43)
\end{aligned}$$

and $\mathcal{J}_k$ is the Bessel's function

$$\mathcal{J}_k(n\rho) = \frac{1}{\pi}\int_0^{\pi} \cos\left(ku - n\rho \sin u\right) du \tag{2.44}$$

Proof We rewrite explicitly the expression (2.17) for our phenomenological controlled potential $\widetilde{V}(x, y, t)$:

$$\widetilde{V}(x, y, t) = \varepsilon \sum_{n,m=1}^{N} \frac{\cos\Big(n(x + f'(y,t)) + my + \phi_{n,m} - t\Big)}{(n^2 + m^2)^{3/2}} \tag{2.45}$$

with

$$f'(y, t) = \varepsilon \sum_{n,m=1}^{N} \frac{m\,\Big(\cos K_{n,m,y} - \cos(K_{n,m,y} - t)\Big)}{(n^2 + m^2)^{3/2}} \tag{2.46}$$

With the definition (2.41) and (2.42) we have:

$$f'(y, t) = \varepsilon(F_c(y)\ (1 - \cos t) - F_s(y)\ \sin t) \tag{2.47}$$

Let us introduce

$$\rho = \varepsilon(F_c^2 + F_s^2)^{1/2} \tag{2.48}$$

and Θ by

$$\rho \sin\Theta \equiv -\varepsilon F_c(y) \qquad\qquad \rho\cos\Theta \equiv -\varepsilon F_s(y) \tag{2.49}$$

so that

$$\widetilde{V} = \varepsilon \sum_{n,m=1}^{N} \frac{\cos\left(\eta - t + n\rho \sin(\Theta + t)\right)}{(n^2 + m^2)^{3/2}} \tag{2.50}$$

Using Bessel's functions properties [24]

$$\cos(\rho\sin\Theta) = \sum_{k\in\mathbb{Z}} \mathcal{J}_k(\rho)\cos k\Theta \tag{2.51}$$

$$\sin(\rho\sin\Theta) = \sum_{k\in\mathbb{Z}} \mathcal{J}_k(\rho)\sin k\Theta \tag{2.52}$$

we get

$$\cos\left(\eta - t + n\rho\sin(\Theta + t)\right) = \sum_{k\in\mathbb{Z}} \mathcal{J}_k(n\rho)\cos\left(\xi\right) \tag{2.53}$$

where $\xi = \eta + k\Theta + (k-1)t$, and we finally obtain (2.39). The theorem is proved. ■

During numerical simulations we truncate the controlled potential by keeping only its first 3 temporal Fourier harmonics:

$$\widetilde{V}_{tr} = \varepsilon \sum_{n,m=1}^{N} \frac{A_0 + A_1\cos t + B_1 \sin t + A_2\cos 2t + B_2\sin 2t}{(n^2 + m^2)^{3/2}} \tag{2.54}$$

$$
\begin{aligned}
A_0 &= \mathcal{J}_0(n\rho)\cos(\eta+\Theta) \\
A_1 &= \mathcal{J}_0(n\rho)\cos\eta + \mathcal{J}_2(n\rho)\cos(\eta+2\Theta) \\
B_1 &= \mathcal{J}_0(n\rho)\sin\eta - \mathcal{J}_2(n\rho)\sin(\eta+2\Theta) \\
A_2 &= \mathcal{J}_3(n\rho)\cos(\eta+3\Theta) - \mathcal{J}_1(n\rho)\cos(\eta-\Theta) \\
B_2 &= -\mathcal{J}_3(n\rho)\sin(\eta+3\Theta) - \mathcal{J}_1(n\rho)\sin(\eta-\Theta)
\end{aligned}
$$

Figure 2.4 compares the two contour plots for the exact control term and the

./Fig3.eps

Figure 2.4: Exact Control Term and Truncated Control Term with $\varepsilon = 0.6, t = \frac{\pi}{4}$.

truncated control term (2.54). Figure2.5 compares the two phase portraits for the system without control term and for the system with the above truncated control term (2.54). The computation of $\widetilde{V}_{tr}$ on some grid has been performed in Matlab and the numerical integration of the trajectories was done in C. One can see a barrier

./Fig4.eps

Figure 2.5: $\varepsilon = 0.3$, $T = 2000$, $N_{traj} = 50$.

for the system with the truncated control term. As for the system with the exact control term we create two barriers at positions $x = x_0$ and $x = x_0 - 2\pi$ and the phenomenon of increasing the mixing inside the barriers persist.

Table 2.1: Squared ratios of the amplitudes of the control term and the uncontrolled electric potential ζ_{ex}, ζ_{tr}; ratios of electric energy of the control term and the uncontrolled electric potential η_{ex}, η_{tr}; for the system with exact and truncated control term.

ε	ζ_{ex}	ζ_{tr}	η_{ex}	η_{tr}
0.3	0.1105	0.1193	0.6297	0.1431
0.4	0.1466	0.1583	0.7145	0.2393
0.5	0.1822	0.1967	0.8161	0.3550
0.6	0.2345	0.2137	0.9336	0.4883
0.7	0.2518	0.2716	1.0657	0.6375
0.8	0.2858	0.3038	1.2119	0.8014
0.9	0.3191	0.3439	1.3722	0.9796
1.5	0.5052	0.5427	2.6247	2.3037

2.4.3 Energetical cost

As we have seen before, the introduction of the control term into the system can reduce and even stop the diffusion of the particles through the barrier. Now we estimate the energy cost of the control term F and the truncated control term $F_{tr} \equiv \widetilde{V}_{tr} - V$.

Definition 1 *The average of any function $W = W(x, y, \tau)$ is defined by the formula:*

$$< |W| >= \int_0^{2\pi} dx \int_0^{2\pi} dy \int_0^{2\pi} dt \; |W(x, y, t)| \tag{2.55}$$

Now we calculate the ratio between the absolute value of the truncated control (electric potential) or the exact control and the uncontrolled electric potential:

$$\zeta_{ex} =< |F|^2 > / < |V|^2 >$$

and

$$\zeta_{tr} =< |F_{tr}|^2 > / < |V|^2 >$$

We also compute the ratio between the energy of the control electric field and the energy of the uncontrolled system in their exact and truncated version

$$\eta_{ex} =< |\nabla F|^2 > / < |\nabla V|^2 >$$

and

$$\eta_{tr} =< |\nabla F_{tr}|^2 > / < |\nabla V|^2 >$$

for different values of ε. Results are shown in Table 2.1.

Table 2.2: Number of escaping particles without control term $\mathcal{N}_{without}$, and for the system with the exact control term $\mathcal{N}_{exact}$ and the truncated control term $\mathcal{N}_{tr}$.

ε	$\mathcal{N}_{without}$	$\mathcal{N}_{exact}$	$\mathcal{N}_{tr}$
0.4	22%	0%	6%
0.5	26%	0%	18%
0.9	68%	1%	44%
1.5	72%	1%	54%

Table 2.3: Difference $\Delta\mathcal{N}$ of the number of particles passing through the barrier and difference of relative electric energy $\Delta\eta$ for the controlled and uncontrolled system.

ε	$\Delta\mathcal{N}$	$\Delta\eta$
0.3	8%	0.49
0.4	16%	0.47
0.5	8%	0.46
0.9	24%	0.39
1.5	18%	0.32

One can see that the truncated control term needs a smaller energy than the exact control term. In Table 2.2, we present the number of particles passing through the barrier in function of ε, after the same integration time.

Let $\Delta\mathcal{N} = \mathcal{N}_{without} - \mathcal{N}_{tr}$ be the difference between the number of particles passing through the barrier for the system without control and with the truncated control and $\Delta\eta = \eta_{ex} - \eta_{tr}$ the difference between the relative electric energy for the system with the exact control term and the system with the truncated control term. In Table 2.3 we present $\Delta\mathcal{N}$ and $\Delta\eta$ for differents values of ε.

For ε below 0.2 the non controlled system is rather regular, there is no particles stream through the barrier, so we have no need to introduce the control electric field. For ε between 0.3 and 0.9 the truncated control field is quite efficient, it allows to drop the chaotic transport through the barrier by a factor 8% to 24% with respect to the uncontrolled system and it requires less energy than the exact control field. For ε greater than 1 the truncated control field is less efficient than the exact one, because the dynamics of the system is very chaotic. For example when $\varepsilon = 1.5$, there are 72% of the particles crossing the barrier for the uncontrolled system and 54% for the system with the truncated control field. At the same time the energetical cost of the truncated control field is above 70% of the exact one, which allows to stop the transport through the barrier. So for $\varepsilon \geq 1$ we need to use the exact control field rather than the truncated one.

2.5 Discussion and Conclusion

In this article, we studied a possible improvement of the confinement properties of a magnetized fusion plasma. A transport barrier conception method is proposed as an alternative to presently achieved barriers such as the H-mode and the ITB scenarios. One can note, that our method differs from an ITB construction. Indeed, in order to build-up a transport barrier, we do not require a hard modification of the system, such as a change in the q-profile. Rather, we propose a weak change of the system properties that allow a barrier to develop. However, our control scheme requires some knowledge and information relative to the turbulence at work, these having weak or no impact on the ITB scenarios.

2.5.1 Main results

First of all we have proved that the local control theory gives the possibility to construct a transport barrier at any chosen position $x = x_0$ for any electric potential $V(x, y, t)$. Indeed, the proof given in section 2.3 does not depend on the model for the electric potential V. In Subsection 2.3.1, we give a rigorous estimate for the norm of the control term F, for some phenomenological model of the electric potential. The introduction of the exact control term into the system inhibits the particle transport through the barrier for any ε while the implementation of a truncated control term reduces the particle transport significantly for $\varepsilon \in (0.3, 1.0)$.

2.5.2 Discussion, open questions

Comparison with the global control method

Let us now compare our approach with the global control method [16] which aims at globally reducing the transport in every point of the phase space. Our approach aims at implementing a transport barrier. However, one also observes a global modification of the dynamics since the mixing properties seem to increase away from the barriers.

Furthermore, in many cases, only the first few terms of the expansion of the global control term [16] can be computed explicitly. Here we have an explicit exact expression for the local control term.

Effectiveness and properties of the control procedure

In subsection 2.2.2, we have introduced the dimensionless variables (2.4) and defined a dimensionless control parameter $\varepsilon \equiv 4\pi^2(cV_0/B)/(L\ell\omega)$. In the simplifying case where $l = L = 2\pi/k$ is the characteristic length of our problem, we have $\varepsilon = ck^2V_0/(\omega B)$. Let us consider a symmetric vortex, hence with characteristic scale $1/k$. Let us now consider the motion of a particle governed by such a vortex. The order of magnitude of the drift velocity is therefore $v_E = kcV_0/B$ and the associated characteristic time τ_{ETT}, $\tau_{ETT} \equiv 1/(kv_E)$, is the eddy turn over time. Let ω be the characteristic evolution frquency of the turbulent eddies, here of the electric field, then the Kubo number K is $K = 1/\omega\tau_{ETT}$. This parameter is the dimensionless control parameter of this class of problems, and we remark that in our case $K = \varepsilon$. It is also important to remark that the parameter K also characterises the diffusion properties of our system. Indeed, let δ be a step size of our particle in a random walk process and let τ be the associated characteristic time, the diffusion coefficient is then $D = \delta^2/\tau$. Since one can relate the characteristic step and time by the velocity, $\delta = v_E\tau$, on also finds:

$$D = \frac{(v_E\tau)^2}{\tau} = \frac{k^2c^2V_0^2}{B^2}\tau = \frac{1}{k^2\tau_{ETT}^2}\tau = \frac{K^2}{k^2}\omega^2\tau \tag{2.56}$$

We also introduce the reference diffusion coefficient $\bar{D} = k^{-2}\omega$, so that:

$$D/\bar{D} \equiv K^2\omega\tau \tag{2.57}$$

They are two asymptotic regimes for our system. The first one, is the regime of weak turbulence, characterised by $\omega\tau_{ETT} \gg 1$ and therefore $K \ll 1$. In this regime, the electric potential evolution is fast, the particle trajectories only follow the eddy geometry on distances much smaller than the eddy size. The steps δ are small and the characteristic time τ of the random walk such that $\omega\tau \approx 1$. The particle diffusion (2.57) is then such that:

$$D/\bar{D} \approx K^2 \quad for \qquad \omega\tau_{ETT} \gg 1 \tag{2.58}$$

The second asymptotic regime is the regime of strong turbulence, with $\omega\tau_{ETT} \ll 1$ and $K \gg 1$. Particles then explore the eddies before decorrelation and the characteristic time of the random step is typically $\tau \approx \tau_{ETT}$ and:

$$D/\bar{D} \approx K \quad for \qquad \omega\tau_{ETT} \ll 1 \tag{2.59}$$

The first regime corresponds to the weak turbulence limit with weak Kubo number and particle diffusion and the second to strong turbulence and large Kubo number and particle diffusion. The control method developed in this article does not depend on $K \equiv \varepsilon$. There is always a possibility to construct an exact transport barrier. However for the numerical simulations, we have remarked, that for small ε one can observe a stable barrier without escaping particles, and for ε close or more than 1 there is some leaking of particles across the barrier. The barrier is more difficult to enforce. Also when considering the truncated control term, one finds that the control term is ineffective in the strong turbulence limit.

Let us now consider the implementation of our method to turbulent plasmas where the turbulent electric field is consistent with the particle transport. The theoretical proof of an hamiltonian control concept is developped provided the system properties at work are completely known. For example the analytic expression for the electric potential. This is impossible in a real system, since the measurements take place on a finite spatio-temporal grid. This has motivated our investigation of the truncated control term by reducing the actually used information on the system. As pointed out previously, one finds that this approach is ineffective for strong turbulence. Another issue is the evolution of the turbulent electric field following the appearance of a transport barrier. This issue would deserve a specific analysis and very likely updating the control term on a trasnport characteristic time scale. An alternative to such a process would be to use a retroactive Hamiltonian approach (a classical field theory) [10] and to develop the control theory in that framework.

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Chapter 3

Maxwell-Vlasov conservation law

3.1 Introduction and physical motivations

The Maxwell-Vlasov gyrokinetic approach represents a powerful tool for the investigation of turbulent behavior of low-frequency strongly magnetized plasmas. It is well known that one of the possible ways for investigating the properties of a physical system is to derive its conservation laws. Noether's theorem plays a fundamental role in theoretical physics by relating conservation laws and symmetries. For example, the energy conservation law is associated with symmetries under infinitesimal time translation $t \to t + \delta t$ and momentum conservation law is associated with the symmetries under infinitesimal spatial translations $\mathbf{x} \to \mathbf{x} + \delta\mathbf{x}$. Generally Noether method for fluids and plasmas can be presented for Euler-Lagrangian (E-L) and Euler-Poincaré (E-P) variational principles which differ by their treatment of fields variations. In fact, the essential difference between these variational principles is to consider dynamical fields to be varied independently (E-L) or not. In what follows we deal with Euler-Poincaré variational principle for Maxwell-Vlasov system. We remark here that one of the serious advantages of Noether's method for derivation of gyrokinetic Maxwell-Vlasov system conservation laws is that this method permits us to obtain exactly conserved properties even for systems with asymptotically reduced dynamics.

The gyrokinetic energy conservation law was recently obtained in [25]. The goal of our study here is to derive an exact gyrokinetic Vlasov-Poisson momentum conservation law. This investigation can have an important field of applications. First of all an exactly conserved quantity can be implemented as a numerical simulations verification. In the other hand, interpreted like a momentum transport equation, momentum conservation law can also be used for investigation of intrinsic plasma rotation phenomena, which play an important role in fusion plasma stabilization. Further it can also be considered as a potential tool for plasma control by investigation of transport barrier creation.

In fact, transport barrier creation represents the results of one of the self-consistent field-particle interaction. For example energy and momentum exchange between particles and fields in plasma. More precisely, energy exchange leads to plasma heating, and momentum exchange leads to current drive, so both phenomena can be considered as one of the sources for the transport barrier creation. Conservation laws guarantee a proper exchange between particles and fields and then permits us to explore self-consistent mechanisms that govern plasma behavior.

3.2 Maxwell-Vlasov equations and variational principles

Due to their large applicability Maxwell-Vlasov equations of ideal plasma dynamics has a long history and was studied extensively. It was firstly used in their simpler form known as Poisson-Vlasov equations by Jeans [26]for investigation of structure formation on stellar and galactic scales and even before by Poincaré [27] in his work on determination of stability conditions for stellar configurations. On the other hand Poisson-Vlasov equation can be also applied in order to study self-consistent dynamics of electrostatic collisionless plasma whereas Maxwell-Vlasov equations permits us to study self-consistent collisionless dynamics of plasma in electromagnetic field case. In order to prepare the study of stability of plasma equilibrium, Low in 1956 has presented his variational principle for Maxwell-Vlasov system. Low's action is expressed in mixture of Lagrangian particle variables and Eulerian fields variables. Since then a variety of variational formulations for Maxwell-Vlasov equations have appeared. Particular attention was payed to the formulation of the particle part of the action. For example its mixed Eulerian-Lagrangian formulation was used in Hamiltonian-Jacobi action presented in [28, 28–30] and [31]. A purely Eulerian formulation was proposed in [32, 33] through the introduction of two functions known as Clebsch potentials introduced in [34, 35] and appropriate action principle with Clebsch action. The leaf action variational principle introduced by Ye and Morrison in [36] uses a single generating function as the dynamical variable for describing the particle distribution and represents a link between Lagrangian and Eulerian representations for actions. A more systematic derivation for a different Eulerian variational principle was presented by Cendra et al in [37]. It is obtained by following the reduction procedure of Low variational principle, much as one does in the corresponding derivation of non-canonical Poisson bracket in the Hamiltonian formulation for the Maxwell-Vlasov system. Similarly to ideal fluid Eulerian variational principle, constrained variations on six dimensional phase space was introduced in this work. Finally, a new Eulerian variational principle that uses constrained variations on extended eight dimensional phase space was presented by A.J. Brizard in [38]. The transition from the six-dimensional phase space to the eight dimensional phase space permits us to express Vlasov distribution variation in terms of canonical Poisson bracket and a single scalar field $\delta\mathcal{S}$ which generate a virtual displacements on the extended phase space $:Z^\alpha \to Z^\alpha + \delta Z^\alpha$, $\alpha \in \{1, \dots 8\}$, where $\delta Z^\alpha \equiv \{Z^\alpha, \delta\mathcal{S}\}$. In what follows we show how this variational principle can be applied for derivation of conservation laws for perturbed Maxwell-Vlasov system and gyrokinetic Maxwell-Vlasov system in the case of electrostatic fluctuations.

3.3 Variational principle for perturbed Maxwell-Vlasov

This section is dedicated to the derivation of momentum conservation law in the case of the perturbed Maxwell-Vlasov system. In particular we consider that magnetic field is given by $\mathbf{B} = \mathbf{B}_0 + \epsilon\mathbf{B}_1$ where $\mathbf{B}_0 = \nabla \times \mathbf{A}_0$ denotes the background time-independent equilibrium component, and $\mathbf{B}_1 = \nabla \times \mathbf{A}_1$ its fluctuation. At the same time the electric field contains only a fluctuating part $\mathbf{E}_1 = -\nabla\Phi_1 - c^{-1}\partial_t\mathbf{A}_1$.

In order to represent particle part of dynamics in extended eight dimensional phase space, first of all we introduce an extended Hamiltonian $\mathcal{H} = H - w$ where H is a Hamiltonian of a charged particle in an external perturbed electromagnetic field $\mathbf{B}_1, \mathbf{E}_1$:

$$H = \frac{1}{2m}(\mathbf{p} - \frac{e}{c}\mathbf{A})^2 + e\epsilon\,\Phi_1 \tag{3.1}$$

where $\mathbf{A} \equiv \mathbf{A}_0 + \epsilon\mathbf{A}_1$ Then we introduce extended Vlasov distribution function

$$\mathcal{F}(\mathcal{Z}) \equiv c\delta(w - H)F(\mathbf{p}, \mathbf{x}) \tag{3.2}$$

where F is the Vlasov distribution function on 6 dimensional phase space. This definition insures that the extended Hamiltonian $\mathcal{H}$ satisfies the physical constraint $H = w$. Here w is a variable that is canonically conjugate to t and the Poisson bracket is an extended canonical Poisson bracket:

$$\{F, G\}_{ext} = \nabla F \cdot \frac{\partial G}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}} \cdot \nabla G + \frac{\partial F}{\partial t} \cdot \frac{\partial G}{\partial w} - \frac{\partial F}{\partial w} \cdot \frac{\partial G}{\partial t} \tag{3.3}$$

Note that the dynamical variables in this approach are: electromagnetic fluctuating fields $\mathbf{B}_1$, $\mathbf{E}_1$ and extended Vlasov distribution function $\mathcal{F}$. Now we give an expression for action functional corresponding to our system and then we use it order to write corresponding Hamilton's action principle $\delta\mathcal{A} \equiv 0$:

$$\mathcal{A} = -\int d^8\mathcal{Z}\mathcal{F}(\mathcal{Z})\mathcal{H}(\mathcal{Z}; \Phi_1, \mathbf{A}_1) + \int \frac{d^4x}{8\pi}\left(\epsilon^2|\mathbf{E}_1|^2 - |\mathbf{B}_0 + \epsilon\mathbf{B}_1|^2\right) \equiv \int \mathcal{L}d^4x \tag{3.4}$$

Note that the extended phase space integration in the expression below is defined by $d^8\mathcal{Z} \equiv c\ dtd^3xd^4p$ where $d^4p \equiv c^{-1}d^3pdw$. In order to proceed with writing of Hamilton's action principle

$$\delta\mathcal{A} = \int d^4x\,\delta\mathcal{L} = 0 \tag{3.5}$$

we need first to obtain the Eulerian variation of Lagrangian density $\delta\mathcal{L}$.

3.3.1 Eulerian variations

The Eulerian variation of the Lagrangian density given by expression (3.4) is expressed as:

$$\delta\mathcal{L} = -\int (\delta\mathcal{F}\mathcal{H} + \delta\mathcal{H}\mathcal{F})\, d^4p + \frac{\epsilon}{4\pi}\left(\epsilon^2\, \delta\mathbf{E}_1 \cdot \mathbf{E}_1 - \epsilon\, \delta\mathbf{B}_1 \cdot \mathbf{B}\right) \tag{3.6}$$

Here $\mathbf{B}_0$ is excluded as a variational field (since it is time independent). Eulerian electromagnetic field variations are naturally related to the electromagnetic potential variations as follows

$$\delta\mathbf{E}_1 = -\nabla\delta\Phi_1 - c^{-1}\partial_t\delta\mathbf{A}_1 \tag{3.7}$$

$$\delta\mathbf{B}_1 = \nabla\times\delta\mathbf{A}_1 \tag{3.8}$$

they satisfy the constraints given by two of Maxwell's equations

$$\nabla\times\delta\mathbf{E}_1 = \frac{1}{c}\frac{\partial\delta\mathbf{B}_1}{\partial t} \tag{3.9}$$

$$\nabla\cdot\delta\mathbf{B}_1 = 0 \tag{3.10}$$

The Eulerian variation for the extended distribution function (3.2) is obtained by using the fundamental relation between Eulerian ($\delta\mathcal{F}$) and Lagrangian $\Delta\mathcal{F}$ variations:

$$\delta\mathcal{F} \equiv \Delta\mathcal{F} - \delta\mathcal{Z}^a\frac{\partial\mathcal{F}}{\partial\mathcal{Z}^a} = -\{\mathcal{Z}^a, S\}_{ext}\,\frac{\partial\mathcal{F}}{\partial\mathcal{Z}^a} \equiv \{S,\mathcal{F}\}_{ext} \tag{3.11}$$

It preserves the Vlasov constraint $\int\mathcal{F}d^8\mathcal{Z} = 0$ under a virtual canonical transformation $\mathcal{Z}^a \to \mathcal{Z}^a + \delta\mathcal{Z}^a$ in extended phase space (as a result of integration of an exact Poisson bracket over phase space). To obtain the expression (3.11) we use two facts. The first one is that the virtual canonical transformation is generated by the extended scalar field S: $\delta\mathcal{Z}^a \to \mathcal{Z}^a + \delta\mathcal{Z}^a$. The second one is that the Lagrangian variation of extended distribution function $\mathcal{F}$ is equal to zero. This is a direct consequence of the fact that the distribution function is constant along any trajectory in the phase space (Liouville's theorem). Finally the Eulerian variation of the extended Hamiltonian $\delta\mathcal{H}$ is given by:

$$\delta\mathcal{H} = \delta\Phi_1\,\frac{\delta H}{\delta\Phi_1} + \delta\mathbf{A}_1\cdot\frac{\delta H}{\delta\mathbf{A}_1} \tag{3.12}$$

Now our goal is to rewrite the expression for Lagrangian variation density (3.6) so that the variation generators $(S, \delta\Phi_1, \delta\mathbf{A}_1)$ appear explicitly [1]. This will give us the

[1] You can find a detailed calculation that permits us the passage between the general expression for Maxwell-Vlasov Lagrangian density to the equations of motion and Noether's terms in Appendix A

possibility to derive the equations of motion and at the same time to obtain the Noether terms necessary for the derivation of conservation laws.

$$\begin{aligned}\delta\mathcal{L} &= \left(\frac{\partial\Lambda}{\partial t}+\nabla\cdot\mathbf{\Gamma}\right)+\delta\Phi_1\left[\frac{\epsilon^2}{4\pi}\nabla\cdot\mathbf{E}_1-\int d^4p\,\frac{\delta H}{\delta\Phi_1}\,F\right] \\ &+ \delta\mathbf{A}_1\cdot\left[\frac{\epsilon}{4\pi c}\left(\epsilon\frac{\partial\mathbf{E}_1}{\partial t}-c\nabla\times\mathbf{B}\right)-\int d^4p\,\frac{\delta H}{\delta\mathbf{A}_1}\,F\right]-\int S\,\{\mathcal{F},\mathcal{H}\}_{ext}\,d^4p\end{aligned} \tag{3.13}$$

where the Noether fields Λ and $\mathbf{\Gamma}$ are given by

$$\Lambda \equiv \int d^4p\,S\mathcal{F}-\frac{\epsilon^2}{4\pi\,c}\delta\mathbf{A}_1\cdot\mathbf{E}_1 \tag{3.14}$$

$$\mathbf{\Gamma} \equiv \int d^4p\,S\,\mathcal{F}x-\frac{\epsilon^2}{4\pi}\delta\mathbf{A}_1\times\mathbf{B}_1 \tag{3.15}$$

with $\dot{\mathbf{x}}\equiv\{\mathbf{x},H\}$ representing the particle velocity. Note that here the Noether space-time divergence terms $\partial\Lambda/\partial t+\nabla\cdot\mathbf{\Gamma}$ do not contribute to the variational principle.

Now we introduce this expression into Hamilton's action principle (3.5). Here each term that is multiplied by the generators of the variations will give us corresponding equations of motion. All the other terms are expressed as divergence and exact time-derivative, and so do not influence the dynamics of the system. These are the Noether terms, which contribute to the derivation of conservation laws. We remark that this expression is general and gives the possibility to obtain the equations of motion and Noether terms for any system of Maxwell-Vlasov equations (reduced or not).

3.3.2 Perturbed Maxwell-Vlasov equations

In this section we deal with perturbed Maxwell-Vlasov system, so we use (3.1) in order to obtain corresponding equations of motion. The functional derivatives $\frac{\delta H}{\delta\Phi_1}$ and $\frac{\delta H}{\delta\mathbf{A}_1}$ are given by:

$$\frac{\delta H}{\delta\Phi_1} = \epsilon e \tag{3.16}$$

$$\frac{\delta H}{\delta\mathbf{A}_1} = -\frac{\epsilon}{m}\frac{e}{c}\left(\mathbf{p}-\frac{e}{c}\left(\mathbf{A}_0+\epsilon\mathbf{A}_1\right)\right)\equiv\epsilon\frac{e}{c}\,\mathbf{v} \tag{3.17}$$

So finally the perturbed Maxwell equations are given by the following expression:

$$\epsilon\nabla\cdot\mathbf{E}_1 = 4\pi e\int d^4p\,\mathcal{F} \tag{3.18}$$

$$\nabla\times\mathbf{B} = \epsilon\frac{1}{c}\frac{\partial\mathbf{E}_1}{\partial t}+4\pi e\int d^4p\,\mathcal{F}\,\frac{\mathbf{v}}{c} \tag{3.19}$$

Then the extended Vlasov equation is given by:

$$\{\mathcal{F}, \mathcal{H}\}_{ext} = 0 \tag{3.20}$$

In order to obtain the Vlasov equation we perform the integration over the energy coordinate $\int dw$ of the extended Vlasov equation (see for details Appendix A.2.2).

$$\frac{\partial F}{\partial t} + \{F, H\} = \frac{\partial F}{\partial t} + \nabla F \cdot \frac{\partial H}{\partial \mathbf{p}} - \nabla H \cdot \frac{\partial F}{\partial \mathbf{p}} = 0 \tag{3.21}$$

and then the perturbed Maxwell equations of motion become

$$\epsilon \nabla \cdot \mathbf{E}_1 = 4\pi e \int d^3p\, F \tag{3.22}$$

$$\nabla \times \mathbf{B} = \epsilon \frac{1}{c}\frac{\partial \mathbf{E}_1}{\partial t} + 4\pi e \int d^3p\, F\, \frac{\mathbf{v}}{c} \tag{3.23}$$

3.4 Momentum conservation law

In this section we use Noether method in order to derive exact momentum conservation law for perturbed Maxwell-Vlasov system. The Noether's theorem states that for each symmetry of the Lagrangian density $\mathcal{L}$ there corresponds a conservation law (and vice versa). When the Lagrangian is invariant under a time translation, a space translation, or a spatial rotation, the conservation law involves energy, momentum, or angular momentum conservation respectively. The formal proof of this statement can be found in [39].

After substituting the perturbed equations of motion (3.21,3.22,3.23) into the expression for Eulerian variation of the Lagrangian density (3.14), we obtain Noether equation:

$$\delta \mathcal{L} = \frac{\partial \Lambda}{\partial t} + \nabla \cdot \mathbf{\Gamma} \tag{3.24}$$

Now the variations $(S, \delta\Phi_1, \delta\mathbf{A}_1)$ are no longer consider arbitrary but are generated by infinitesimal space-time translations correspondingly to the conservation law that we derive. Before we proceed with the derivation of the conservation laws, we note that the Noether components $(\Lambda, \mathbf{\Gamma})$ are defined up to the following transformations:

$$\bar{\Lambda} \equiv \Lambda + \nabla \cdot \eta \tag{3.25}$$

$$\bar{\mathbf{\Gamma}} \equiv \mathbf{\Gamma} - \frac{\partial \eta}{\partial t} + \nabla \times \sigma \tag{3.26}$$

where η and σ are arbitrary vector fields. These vector fields will be used in order to obtain conservation laws in gauge-independent form. Note that these transformations are obtained naturally. In fact one can add and then sustain to the Noether equation (3.24) the following quantity: $\nabla \partial_t \eta = \partial_t \nabla \eta$. Another vector field that we note σ can be added to $\mathbf{\Gamma}$ component due to the fact that $\nabla \cdot (\nabla \times \sigma) = 0$ for any vector field σ.

3.4.1 Constrained variations

Constrained variations for electromagnetic potentials

The variations of electromagnetic potentials generated by infinitesimal space-time translations can be expressed in terms of Lie-derivative $\pounds_{\delta x}$ where δx represents an infinitesimal translation in the four-dimensional phase space. In general theory the expression for constrained variations of the Eulerian variational principle in terms of Lie-derivative appears when the equivalence between Lagrangian and Eulerian variational principle is discussed. On the other hand, one can interpret this fact only by geometrical considerations, using the fact that the Lie -derivative can be viewed as a simple generalization of directional derivative. In this section we deal with geometrical tools in order to obtain the expression for electromagnetic field constrained variations.

We start with choice of the metric, here we deal with space-like or Minkowski type of metric:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.27}$$

We chose also the following definition of covariant and contravariant components: $A_\mu = (A_0, A_i)$ and $A^\mu = g^{\mu\nu}A_\nu = (-A^0, A^i)$. Infinitesimal space-time variations are represented by the vector: $\delta x^\mu = (-c\delta t, \delta x^i)$ and covariant differentiation is given by: $\partial_\mu = (-c^{-1}\partial_t, \partial_i)$ where $\partial_i \equiv \partial/\partial x^i$. Then we can write an expression for one-form electromagnetic four potential.

$$A = A_\mu dx^\mu = -c\ A_0\ dt + \mathbf{A}\cdot d\mathbf{x} \tag{3.28}$$

Using the Cartan formula for Lie-derivative we have:

$$\pounds_{\delta x} A = i_{\delta x}\cdot dA + d(i_{\delta x}\cdot A) \tag{3.29}$$

where the inner product operator $i_{\delta x}$ acts as follows on one ($A = A_\mu dx^\mu$) and two ($\partial_\mu A_\nu dx^\mu \wedge dx^\nu$) forms:

$$d(i_{\delta x}\cdot A) = dx^\nu \partial_\nu \left(A_\mu \delta x^\mu\right) = dx^\nu \partial_\nu \left[-c\ A_0\ \delta t + A_i \delta x^i\right] \tag{3.30}$$

$$i_{\delta x}\cdot dA = \partial_\mu A_\nu\ \delta x^\mu\ dx^\nu - \partial_\mu A_\nu\ dx^\mu\ \delta x^\nu \equiv \delta x^\mu F_{\mu\nu} dx^\nu \tag{3.31}$$

then

$$\pounds_{\delta x} A = \left(\delta x^\mu F_{\mu\nu} + \partial_\nu (A_\mu \delta x^\mu)\right) dx^\nu \equiv -\delta A_\nu dx^\nu \tag{3.32}$$

Then the variation of electromagnetic potential component:

$$\delta A_0 = -\delta x^i F_{i0} + c^{-1}\partial_t \left(-A_0 c\delta t + A_j \delta x^j\right) \tag{3.33}$$

$$\delta A_i = c\delta t F_{0i} - \delta x^j F_{ji} - \partial_i \left(-A_0\ c\ \delta t + A_j \delta x^j\right) \tag{3.34}$$

By substituting the components of antisymmetric field tensor $F_{0i} = -\partial_i A_0 - c^{-1}\partial_t \mathbf{A}$ and $F_{ji} = \partial_j A_i - \partial_i A_j$ we obtain:

$$\begin{aligned} \delta A_0 &= -\delta \mathbf{x} \cdot \nabla A_0 - \delta t \, \partial_t A_0 && (3.35) \\ \delta \mathbf{A} &= -\delta \mathbf{x} \cdot \nabla \mathbf{A} - \delta t \, \partial_t \mathbf{A} && (3.36) \end{aligned}$$

We note here that $\Phi \equiv A_0$ and $\delta\Phi \equiv \delta A_0$, then

$$\delta\Phi = -\delta x^i \, \partial_i \Phi - \delta t \, \partial_t \Phi \tag{3.37}$$

Another possibility to deal with covariant and contravariant vectors in this place is to suppose that $A^0 = \Phi$ and $A_0 = -\Phi$ then the sign does not appear inside the definition of covariant and contravariant components of vector potential $A_\mu = (A_0, A_i)$ and $A^\mu = (A^0, A^i)$. The infinitesimal space-time variations vector is $\delta x^\mu = (\delta x^0, \delta x^i)$ with $\delta x^0 = c\delta t$ and $\partial_\mu = (\partial_{x_0}, \partial_{x_i})$ with $\partial x_0 = c^{-1}\partial_t$. Then $A = A_\mu dx^\mu = A_0 dx^0 + \mathbf{A} \cdot d\mathbf{x}$ and using the Eq. (3.32) we obtain:

$$\begin{aligned} \delta A_0 &= -\delta x^i F_{i0} - \partial_0 \left(A_0 \delta x^0 + \mathbf{A} \cdot d\mathbf{x} \right) && (3.38) \\ \delta A_i &= -\delta x^0 F_{0i} - \delta x^j F_{ji} - \partial_j \left(A_0 \delta x^0 + \mathbf{A} \cdot d\mathbf{x} \right) && (3.39) \end{aligned}$$

with $F_{0i} = \partial_0 A_i - \partial_i A_0$ and $F_{ij} = \partial_i A_j - \partial_j A_i$,

$$\begin{aligned} \delta A_0 &= -\delta x^i \partial_i A_0 - \delta t \partial_t A_0 && (3.40) \\ \delta A_i &= -\delta \mathbf{x} \cdot \nabla \mathbf{A} - \delta t \partial_t \mathbf{A} && (3.41) \end{aligned}$$

Replacing now $A_0 \equiv -\Phi$ we obtain (3.37)

Constrained variations for Lagrangian density

By analogy with constrained variations for electromagnetic potentials, using the Cartan formula for Lie derivative, we can obtain the variation for Lagrangian density. In fact, let us consider four-form $\mathcal{L}\Omega$ where Ω is the oriented space-time volume element, then

$$\delta\mathcal{L} \, \Omega \equiv -\pounds_{\delta x} \left(\mathcal{L} \, \Omega \right) \tag{3.42}$$

Using the Cartan formula we obtain

$$\pounds_{\delta x} \left(\mathcal{L} \, \Omega \right) = d \left(i_{\delta x} \left(\mathcal{L} \, \Omega \right) \right) + i_{\delta x} \left(d \left(\mathcal{L} \, \Omega \right) \right) \tag{3.43}$$

The second term in this expression is equal to zero because $(d \, (d^4x) \equiv 0)$, the first term can be rearranged as follows:

$$d \left(i_{\delta x} \left(\mathcal{L} \, \Omega \right) \right) = \left(\partial_\alpha \mathcal{L} \, \delta x^\alpha \right) \Omega \tag{3.44}$$

here we use that:

$$i_{\delta x} \left[dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\varsigma \right] = \delta x^\alpha \left(dx^\beta \wedge dx^\gamma \wedge dx^\varsigma \right) \tag{3.45}$$

and

$$d\left(\mathcal{L}\, i_{\delta x}\Omega\right) = \partial_\alpha \mathcal{L} \delta x^\alpha \left(dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\zeta\right) \equiv \left(\partial_\alpha \mathcal{L}\, \delta x^\alpha\right) \Omega \tag{3.46}$$

By substituting the formula (3.44) into the expression (3.42) we obtain:

$$\delta \mathcal{L} = -\partial_\mu \left(\delta x^\mu\, \mathcal{L}\right) \tag{3.47}$$

Due to the fact that in our approach we decompose the initial magnetic field into its dynamical part $\mathbf{B}_1$ and its background part $\mathbf{B}_0$ we should take it into account when defining the Eulerian variation of Lagrangian density. In order to do that, we correct the expression for variation $\delta\mathcal{L}$ by subtracting from $\partial_\mu \mathcal{L}$ the derivative of the Lagrangian density with respect to the space-time variables while all the dynamical fields are held constant. Then only the background fields contribute. We indicate such a derivative by $\partial'_\mu \mathcal{L}$. So finally

$$\delta \mathcal{L} = -\delta x^i \left(\partial_i \mathcal{L} - \partial'_i \mathcal{L}\right) + \delta t \left(\partial_t \mathcal{L} - \partial'_t \mathcal{L}\right) \tag{3.48}$$

3.4.2 Noether method

In order to obtain the momentum conservation law for the perturbed Maxwell-Vlasov system, we use the Noether equation (3.24) and consider infinitesimal space translations $\mathbf{x} \to \mathbf{x} + \delta\mathbf{x}$ generated by:

$$\left.\begin{array}{lcl} S & = & (\mathbf{p}_*) \cdot \delta\mathbf{x} \\ \delta\Phi_1 & = & -\delta\mathbf{x} \cdot \nabla\Phi_1 \equiv \delta\mathbf{x} \cdot (\mathbf{E}_1 + c^{-1}\, \partial_t \mathbf{A}_1) \\ \delta\mathbf{A}_1 & = & -\delta\mathbf{x} \cdot \nabla\mathbf{A}_1 \equiv \delta\mathbf{x} \times \mathbf{B}_1 - \nabla\mathbf{A}_1 \cdot \delta\mathbf{x} \\ \delta\mathcal{L} & = & -\delta\mathbf{x} \cdot (\nabla\mathcal{L} - \nabla'\mathcal{L}) \end{array}\right\} \tag{3.49}$$

where the expression for canonical particle momentum $\mathbf{p}_*$ will be discussed below. The expression for variations of electromagnetic fields $\delta\Phi_1, \delta\mathbf{A}_1, \delta\mathcal{L}$ are obtained from the general theory as the spatial component of the Lie derivative $\delta A = \delta A_\mu dx^\mu \equiv -\pounds_{\delta\mathbf{x}} A$ and $\delta\mathcal{L}\ \Omega = -\pounds_{\delta\mathbf{x}} \left(\mathcal{L}\,\Omega\right)$. Here the notation $\nabla'\mathcal{L}$ in the expression for $\delta\mathcal{L}$ denotes the explicit spatial gradient of the Lagrangian density $\mathcal{L}$ with dynamical fields $\mathbf{E}_1, \mathbf{B}_1, F$ held constant. Since we consider the case of spatially uniform background magnetic field $\mathbf{B}_0$, we have

$$\nabla'\mathcal{L} \equiv \nabla\mathbf{B}_0 \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{B}_0} = -\nabla\mathbf{B}_0 \cdot \left(\frac{\mathbf{B}}{4\pi} + \int F \frac{\partial H}{\partial \mathbf{B}_0} d^3p\right) \tag{3.50}$$

where the first term denotes the contribution from the Maxwell part $\mathcal{L}_M \equiv (\epsilon^2|\mathbf{E}_1|^2 - |\mathbf{B}|^2)/8\pi$ of the Lagrangian density while the second term involves the magnetization contribution associated with the background magnetic field [40]. Then the second term is formally equal to zero while we still work in canonical variables. In fact, in our case the particle Hamiltonian (3.1) is expressed in terms of electromagnetic potential $\mathbf{A} = \mathbf{A}_0 + \epsilon\mathbf{A}_1$, and not magnetic field $\mathbf{B}_0 = \nabla \times \mathbf{A}_0$.

***Primitive* momentum conservation law**

By inserting the variations (3.49) into the Noether equation (3.24), we obtain the *primitive* momentum conservation law

$$\begin{aligned}
-\left(\nabla\mathcal{L}-\nabla'\mathcal{L}\right) = & \frac{\partial}{\partial t}\left[\int \mathcal{F}\left(\mathbf{p}_*\cdot\delta\mathbf{x}\right)d^4p-\frac{\epsilon^2}{4\pi\, c}\left(\delta\mathbf{x}\times\mathbf{B}_1-\nabla\delta\chi_1\right)\cdot\mathbf{E}_1\right] \\
& + \nabla\cdot\left[\int \mathcal{F}\left(\mathbf{p}_*\cdot\delta\mathbf{x}\right)\dot{\mathbf{x}}d^4p-\frac{\epsilon^2}{4\pi}\left(\delta\mathbf{x}\cdot\mathbf{E}_1+\frac{1}{c}\frac{\partial\delta\chi_1}{\partial t}\right)\mathbf{E}_1\right. \\
& \left.-\frac{\epsilon}{4\pi}\left(\delta\mathbf{x}\times\mathbf{B}_1-\nabla\delta\chi_1\right)\times\mathbf{B}\right] \qquad (3.51)
\end{aligned}$$

where we have introduced a gauge-dependent field $\delta\chi_1\equiv\mathbf{A}_1\cdot\delta\mathbf{x}$ and rewrite the variations (3.49) as displayed. Consequently to that this *primitive* form of the momentum conservation law is not gauge invariant.

Gauge-independent momentum conservation law

In order to remove the gauge-dependent term in expression (3.51), we use the transformations (3.26), with $\eta\equiv(\epsilon^2/4\pi\ c)(\mathbf{A}_1\cdot\delta\mathbf{x})\mathbf{E}_1$ and $\sigma\equiv(\epsilon/4\pi)(\mathbf{A}_1\cdot\delta\mathbf{x})\ \mathbf{B}$ (the details of this calculation are given in the Appendix). Finally the gauge-independent momentum conservation law is

$$\frac{\partial\mathbf{P}}{\partial t}+\nabla\cdot\mathbf{\Pi}=\nabla\mathbf{B}_0\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{B}_0} \qquad (3.52)$$

According to the Noether theorem, the component of the gyrokinetic momentum in the direction of the background magnetic field B_0 field spatial symmetry is conserved.

Here the momentum density, after integrating over energy variable $\int dw$, is

$$\mathbf{P}=\int F\ \left(\mathbf{p}_*-\epsilon\frac{e}{c}\mathbf{A}_1\right)d^3p+\frac{\epsilon^2}{4\pi c}\mathbf{E}_1\times\mathbf{B}_1 \qquad (3.53)$$

and the *canonical* momentum-stress tensor is

$$\begin{aligned}
\mathbf{\Pi} \ = & \left[\frac{\epsilon^2}{8\pi}\left(|\mathbf{E}_1|^2+|\mathbf{B}_1|^2\right)-\frac{|\mathbf{B}_0|^2}{8\pi}\right]\mathbf{I}-\frac{1}{4\pi}\left(\epsilon^2\mathbf{E}_1\mathbf{E}_1+\epsilon\mathbf{B}_1\mathbf{B}\right) \\
+ & \int F\left[\dot{\mathbf{x}}\ \mathbf{p}_*-\epsilon\frac{e}{c}\left(\mathbf{v}\ \mathbf{A}_1\right)\right]d^3p \qquad (3.54)
\end{aligned}$$

Note that we take into account the Vlasov condition $\mathcal{FH}\equiv 0$ in extended phase space when evaluating the derivative $\nabla\mathcal{L}$, so only the Maxwell part of Lagrangian density will give the contribution in $\mathbf{\Pi}$.

3.4.3 Proof of Momentum conservation

In this section we give an explicit proof of the momentum conservation law (3.52). We will see that the momentum conservation yields the dynamics of the system. We start by taking partial time derivative of the perturbed momentum density (3.53):

$$\begin{aligned}\frac{\partial \mathbf{P}}{\partial t} &= \int \left[\frac{\partial F}{\partial t}\left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right) + F\frac{\partial}{\partial t}\left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right)\right] d^3p \\ &+ \frac{\epsilon^2}{4\pi\, c}\left(\frac{\partial \mathbf{E}_1}{\partial t} \times \mathbf{B}_1 + \mathbf{E}_1 \times \frac{\partial \mathbf{B}_1}{\partial t}\right) \end{aligned} \tag{3.55}$$

By substituting into the expression below the Maxwell-Vlasov equations (3.22 , 3.23) and the Vlasov equation in the phase-space divergence form:

$$\frac{\partial F}{\partial t} + \nabla \cdot (F\,\dot{\mathbf{x}}) - \frac{\partial}{\partial \mathbf{p}} \cdot (F\,\dot{\mathbf{p}}) = 0 \tag{3.56}$$

we have:

$$\begin{aligned}\frac{\partial \mathbf{P}}{\partial t} &= -\nabla \cdot \left[\int F\,\dot{\mathbf{x}}\left(\mathbf{p}_* - \epsilon\frac{e}{c}\,\mathbf{A}_1\right) d^3p + \frac{\epsilon^2}{8\pi}\left(|\mathbf{E}_1|^2 + |\mathbf{B}_1|^2\right) - \frac{1}{4\pi}\left(\epsilon\,\mathbf{B}_1\mathbf{B} + \epsilon^2\mathbf{E}_1\mathbf{E}_1\right)\right] \\ &- \frac{\epsilon}{4\pi}\,\nabla\mathbf{B}_0 \cdot \mathbf{B}_1 + \int F\left[\frac{d}{dt}\left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right) - \epsilon\, e\left(\mathbf{E}_1 + \frac{\mathbf{v}}{c} \times \mathbf{B}_1\right)\right] d^3p \end{aligned} \tag{3.57}$$

where

$$\frac{d}{dt}\left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right) \equiv \frac{\partial}{\partial t}\left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right) + \left\{\left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right), H\right\} \tag{3.58}$$

The detailed calculation that permits us the transition between the Eq.(3.55) to the Eq. (3.57) is given in the appendix B. Now we add $\nabla \cdot \mathbf{\Pi}$ to the result of the explicit time differentiation of the perturbed momentum density $\partial \mathbf{P}/\partial t$, where $\mathbf{\Pi}$ is defined by the Eq. (3.54). So the momentum conservation law (3.52) becomes:

$$\begin{aligned} 0 &\equiv \frac{\partial \mathbf{P}}{\partial t} + \nabla \cdot \mathbf{\Pi} - \nabla\mathbf{B}_0 \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{B}_0} \\ &= \int F\left[\frac{d}{dt}\left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right) - \epsilon\, e\left(\mathbf{E}_1 + \frac{\mathbf{v}}{c} \times \mathbf{B}_1\right)\right] d^3p \end{aligned} \tag{3.59}$$

Therefore, the last equation yields the perturbed canonical momentum equation:

$$\frac{d}{dt}\left(\mathbf{p}_* - \epsilon\frac{e}{c}\,\mathbf{A}_1\right) \equiv \epsilon\, e\,\left(\mathbf{E}_1 + \frac{\mathbf{v}}{c} \times \mathbf{B}_1\right) \tag{3.60}$$

The equation (3.60) can be used to define the perturbed canonical momentum $\mathbf{p}_*$ that intervenes when defining the generating function of spatial translations (3.49). We remark that we can replace the momentum conservation law for the perturbed Maxwell-Vlasov system (3.52) by the equation for perturbed canonical momentum in the particle phase space (3.60). This connection is analogous to the standard connection between the momentum conservation law and the equation for the particle canonical momentum in the case of the non-perturbed Maxwell-Vlasov system.

3.4.4 Particle canonical momentum

In previous section we have obtained the equation for the perturbed canonical momentum $\mathbf{p}_*$. We now discuss its properties by comparing with the case of the unperturbed Maxwell-Vlasov system.

In general theory [38] for the full Maxwell-Vlasov system when magnetic field is not divided into its equilibrium (non-dynamical part) and perturbed (dynamical) part, the generating function S for the derivation of the momentum conservation law is given by the particle canonical momentum $\mathbf{p} \equiv m\mathbf{v} + \frac{e}{c}\mathbf{A}$.

Let see now what changes for the perturbed Maxwell-Vlasov system, when the electromagnetic fields $\mathbf{E} \equiv \epsilon\mathbf{E}_1 = -\Phi_1 - c^{-1}\partial_t\mathbf{A}_1$ and $\mathbf{B} \equiv \mathbf{B}_0 + \epsilon\mathbf{B}_1 = \nabla \times (\mathbf{A}_0 + \epsilon\mathbf{A}_1)$ are expressed in terms of background fields $(0, \mathbf{B}_0)$ and perturbation fields $(\mathbf{E}_1, \mathbf{B}_1)$. In this case the corresponding particle canonical momentum should be expressed as $\mathbf{p} \equiv m\mathbf{v} + \frac{e}{c}\mathbf{A}_0 + \epsilon\frac{e}{c}\mathbf{A}_1$ and the corresponding equation of motion that can be directly derived from the Hamiltonian (3.1) is (see for details C):

$$\frac{d}{dt}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}_0 - \epsilon\frac{e}{c}\mathbf{A}_1\right) = e\left(\epsilon\ \mathbf{E}_1 + \frac{\mathbf{v}}{c} \times (\mathbf{B}_0 + \epsilon\ \mathbf{B}_1)\right) \tag{3.61}$$

Let now compare this equation to the equation (3.60) for the perturbed canonical momentum $\mathbf{p}_*$. First we can remark that the magnetic part of the Lorentz force in r.h.s of the equation (3.60) does not contain the contribution coming from the equilibrium magnetic field $\mathbf{B}_0$, there is only the contribution coming from the dynamical electromagnetic fields $\mathbf{B}_1$ and $\mathbf{E}_1$. We remark that such an equation can be derived from the Hamiltonian (3.1) after performing on it the gauge transformation $\mathbf{A} \to \mathbf{A}' = \mathbf{A} - \nabla\chi$ where the gauge field is chosen such that $\nabla\chi \equiv \mathbf{A}_0$

$$H = \frac{1}{2m}\left(\mathbf{p} - \epsilon\frac{e}{c}\mathbf{A}_1\right)^2 + \epsilon\ e\ \Phi_1 \tag{3.62}$$

Then in order to be coherent, $\mathbf{p}_*$ should be defined as follows

$$\mathbf{p}_* = m\mathbf{v} + \epsilon\frac{e}{c}\mathbf{A}_1 \tag{3.63}$$

It represents a mixed-canonical momentum (i.e., it is a kinetic momentum in the absence of the magnetic field perturbation generated by the vector field $\mathbf{A}_1$). We can see that the generating function S for the perturbed Maxwell-Vlasov system contains only the dynamical part of the vector potential.

Now we use the expression for the mixed-canonical momentum $\mathbf{p}_*$ (3.63) in order to simplify the expressions for the momentum density (3.53) and the momentum canonical tensor (3.54). Finally we obtain the expressions for the momentum density and the momentum stress tensor with symmetrized Vlasov part:

$$\mathbf{P} = \int F\,\mathbf{v}\,d^3p + \frac{\epsilon^2}{4\pi c}\mathbf{E}_1 \times \mathbf{B}_1 \tag{3.64}$$

$$\begin{aligned}\mathbf{\Pi} &= \left[\frac{\epsilon^2}{8\pi}\left(|\mathbf{E}_1|^2 + |\mathbf{B}_1|^2\right) - \frac{|\mathbf{B}_0|^2}{8\pi}\right]\mathbf{I} - \frac{1}{4\pi}\left(\epsilon^2\mathbf{E}_1\mathbf{E}_1 + \epsilon\mathbf{B}_1\mathbf{B}\right) \\ &+ \int F\,m\mathbf{v}\mathbf{v}\,d^3p \end{aligned} \tag{3.65}$$

3.4.5 Momentum conservation law in background separated form

In this section we show how the momentum conservation law can be used in order to study the momentum exchange between the background field and plasma. For doing this we rewrite the momentum conservation law in its background separated form. We define

$$\bar{\mathbf{P}} \equiv \mathbf{P} = \int F\,\frac{\mathbf{v}}{c}\,d^3p \tag{3.66}$$

$$\begin{aligned}\bar{\mathbf{\Pi}} &= \mathbf{\Pi} + \frac{|\mathbf{B}_0|^2}{8\pi}\,\mathbf{I} + \frac{\epsilon}{4\pi}\mathbf{B}_1\cdot\mathbf{B}_0 = \\ &\quad \frac{\epsilon^2}{8\pi}\left[\left(|\mathbf{E}_1|^2 + |\mathbf{B}_1|^2\right)\right]\mathbf{I} - \frac{1}{4\pi}\left(\epsilon^2\mathbf{E}_1\mathbf{E}_1 + \epsilon\mathbf{B}_1\mathbf{B}_1\right) \\ &+ \int F\,m\mathbf{v}\mathbf{v}\,d^3p \end{aligned} \tag{3.67}$$

Then the momentum conservation law (3.52) became:

$$\frac{\partial\mathbf{P}}{\partial t} + \nabla\cdot\mathbf{\Pi} = \frac{1}{4\pi}\,\mathbf{J}_0 \times \mathbf{B}_1 \tag{3.68}$$

where we make appear the background component of current $\mathbf{J}_0 \equiv (\nabla \times \mathbf{B}_0)$, we use that $\nabla\cdot(\mathbf{B}_1\,\mathbf{B}_0) - \nabla\mathbf{B}_0\cdot\mathbf{B}_1 = (\nabla\times\mathbf{B}_0)\times\mathbf{B}_1$. Let us now consider the equation (3.68), its l.h.s. contains purely plasma contributions into the momentum density $\mathbf{P}$ and the momentum stress tensor $\mathbf{\Pi}$, the r.h.s. contains the coupling between the background magnetic field $\mathbf{B}_0$, represented by background current $\mathbf{J}_0$, and the plasma magnetic field $\mathbf{B}_1$. So we can say that momentum conservation law describes exchange between the background fields and plasma.

3.5 Gyrokinetic variational principle

In sections 3.3 and 3.4 we have considered derivation of momentum conservation law for perturbed Maxwell-Vlasov system. In this section we will deal with derivation of momentum conservation law in the case of reduced (by using the Lie-transform method [41]) Maxwell-Vlasov system. Note that differently from the perturbed Maxwell-Vlasov system case, here we consider only the electrostatic fluctuations with dynamical electric field $\mathbf{E}_1 = -\nabla\Phi_1$ and the non-dynamical background magnetic field $\mathbf{B}_0 = \nabla \times \mathbf{A}_0$. The case of the electromagnetic fluctuations represents a part of the future work.

Previously, several works dealt with variational formulation of reduced Maxwell-Vlasov system. For example Sugama in [42] has presented Lagrangian variational principle, in which an action functional for gyrocenter particles was derived from the Low Lagrangian formalism. Here we still use the Eulerian variational principle proposed by A.J. Brizard in [38] and then adapted by him in [43] for the case of the reduced Maxwell-Vlasov system.

Here we generally follow the same schema for momentum conservation law derivation that in the case of the perturbed Maxwell-Vlasov system.

Gyrokinetic action functional for electrostatic perturbation

In order to prepare the introduction of gyrokinetic electrostatic Maxwell-Vlasov action functional, we first present extended reduced (gyrocenter) Hamiltonian $\mathcal{H}_{gy}$ and Vlasov distribution function $\mathcal{F}_{gy}$. Accordingly to the Lie-transform phase space method for gyrokinetic dynamical reduction [44], the gyrocenter Hamiltonian is given by

$$H_{gy}\left(\mathbf{X}, p_{\,||}, \mu, t, \Phi_1\right) = H_{gc}\left(\mathbf{X}, p_{\,||}, \mu\right) + \langle \epsilon\ e\ \Phi_{1\ gc}\rangle - \frac{\epsilon^2}{2}\ e\ \langle \pounds_{gy}\Phi_{1\ gc}\rangle \tag{3.69}$$

where $\langle \dots \rangle$ denotes the gyroangle-averaging operation, the unperturbed gyrocenter Hamiltonian is defined as the guiding-center Hamiltonian $H_{gc} \equiv \mu B_0 + p_{||}^2/2m$, the effective first-order guiding-center potential in electrostatic turbulence case is

$$\Phi_{1gc} \equiv = \mathsf{T}_{gc}^{-1}\Phi_1 \tag{3.70}$$

where T_{gc}^{-1} denotes push-forward gyrocenter operator. The second order ponderomotive potential in Eq. (3.69) is expressed in terms of the gyrocenter Lie-derivative $\pounds_{gy}$ [25], which is defined for a general function G in electrostatic turbulence case as

$$\pounds_{gy}G \equiv \frac{e}{\Omega}\{\widetilde{\Psi}_{1gc}, G\}_{gc} \tag{3.71}$$

where $\{.\,,.\}_{gc}$ represents the guiding-center Poisson bracket [45] and $\widetilde{\Psi}_{1gc}$ is defined from the following equation:

$$\partial_\theta\widetilde{\Psi}_1 \equiv \widetilde{\Phi}_{1gc} = e\left(\Phi_{1gc} - \langle\Phi_{1gc}\rangle\right) \tag{3.72}$$

$\Omega = eB_0/mc$ denotes the Larmor frequency. We remark that while the gyroangle-averaged potential $\langle\phi_{1gc}\rangle$ contributes to the linear (first order) perturbed gyrocenter

Hamiltonian dynamics, the gyroangle-dependent potential $\widetilde{\Phi}_{1gc}$ contributes to the (second-order) gyrocenter ponderomotive Hamiltonian in Eq.(3.69). The extended gyrocenter Hamiltonian

$$\mathcal{H}_{gy}\left(\mathbf{X}, p_{||}, \mu, t, w; \Phi_1\right) \equiv H_{gy}\left(\mathbf{X}, p_{||}, \mu, t; \Phi_1\right) - w \tag{3.73}$$

is expressed in terms of the time-dependent gyrocenter Hamiltonian given by Eq. (3.69) and the gyrocenter energy coordinate w. The extended Vlasov distribution function

$$\mathcal{F}_{gy}(\mathcal{Z}) \equiv F(\mathbf{X}, p_{||}, \mu, t)\ c\ \delta(w - H_{gy}) \tag{3.74}$$

ensures that the gyrocenter Hamiltonian dynamics satisfies the physical constraint $\mathcal{H}_{gy} \equiv 0$. Now we have all the elements to give the expression for the electrostatic gyrokinetic action functional $\mathcal{A}_{gy}$ and to the corresponding Hamilton's action principle $\delta\mathcal{A}_{gy} = 0$.

$$\mathcal{A}_{gy} = -\int d^8\mathcal{Z}\ \mathcal{F}_{gy}(\mathcal{Z})\ \mathcal{H}_{gy}(\mathcal{Z}; \Phi_1) + \int \frac{d^4x}{8\pi}\left(|\mathbf{E}_1|^2 - |\mathbf{B}_0|^2\right) \equiv \int \mathcal{L}_{gy}\ d^4x \tag{3.75}$$

Note that now the integration is realized over the extended reduced phase space $d^8\mathcal{Z} \equiv dt\ d^3X\ d^4p$, where $d^4p \equiv c^{-1}dw\ d^3p$ and $d^3p = 2\pi\ m\ B^*_{||}\ dp_{||}\ d\mu$. Here $2\pi\ m\ B^*_{||}$ represents the Jacobian of the guiding-center transformation.

3.5.1 Eulerian variations

The general expression for Eulerian variation of Lagrangian density in the case of gyrokinetic Maxwell-Vlasov system $\mathcal{L}_{gy}$ is given by:

$$\delta\mathcal{L}_{gy} = \frac{\epsilon}{4\pi}\left[(\delta\mathbf{E}_1 \cdot \mathbf{E}_1) - \int\ \delta\mathcal{F}_{gy}\ \mathcal{H}_{gy} + \epsilon\left(\delta\Phi_1\ \frac{\delta H_{gy}}{\delta\Phi_1}\right)\mathcal{F}_{gy}\right] \tag{3.76}$$

where B_0 is a non-dynamical field and the constrained Eulerian variation for electric field $\delta\mathbf{E}_1 = -\nabla\Phi_1$ preserves the constraint $c\nabla \times \delta\mathbf{E}_1 = 0$. The Eulerian variation for the extended gyrocenter Vlasov distribution is

$$\delta\mathcal{F}_{gy} \equiv \{S_{gy}, \mathcal{F}_{gy}\}_\epsilon \tag{3.77}$$

where $\{.,.\}_\epsilon$ denotes the extended guiding-center Poisson bracket. Similarly to previous case this Eulerian variation preserves the Vlasov constraint $\int \delta\mathcal{F}_{gy} d^8\mathcal{Z} = 0$ under a virtual canonical transformation $\mathcal{Z} \to \mathcal{Z} + \delta\mathcal{Z}$ in extended phase space. Now the virtual canonical transformation is generated by the extended scalar field S_{gy}: $\delta\mathcal{Z}^\alpha \equiv \{\mathcal{Z}^\alpha, S_{gy}\}_\epsilon$

Gyrocenter Hamiltonian functional derivative

In this subsection we give some details about evaluation of functional derivative $\delta H_{gy}/\delta\Phi(\mathbf{x})$. Before starting this calculation we have to make some remarks.

Due to the gyrokinetic dynamical reduction one have to pay attention to the fact that the electromagnetic fields $\Phi_1 = \Phi_1(\mathbf{x}), \mathbf{A}_0 = \mathbf{A}_0(\mathbf{x})$ and the particle (gyrocenters) $\mathbf{X}$ are now evaluated at different spatial positions. The fields are still evaluated at the full particle position $\mathbf{x}$ while the positions of the gyrocenters are $\mathbf{X} \equiv \mathbf{x} - \rho_{\mathbf{0}}$ where the difference between them is ρ_0 which denotes the Larmor radius. To give the link between the electric field evaluated in the position $\mathbf{x}$ and the gyrocenter electric field evaluated into the reduced (gyrocenter) position, we introduce the guiding-center delta function $\delta^3_{gc} \equiv \delta^3\left(\mathbf{X} + \boldsymbol{\rho}_0 - \mathbf{x}\right)$. It indicates that the gyrocenter contribution at a fixed point $\mathbf{x}$ only comes from gyrocenters located on the ring $\mathbf{X} = \mathbf{x} - \boldsymbol{\rho}_0$. Then

$$\Phi_{1gc} = \int d^3x\ \delta^3_{gc}\ \Phi_1(\mathbf{x}) \tag{3.78}$$

The variation of the gyrocenter Hamiltonian (3.69) is given by:

$$\delta H_{gy} = \epsilon\ e\langle\delta\Phi_{1gc}\rangle - \frac{\epsilon^2\ e^2}{2\Omega}\langle\{\delta\widetilde{\Psi}_{1gc}, \widetilde{\Phi}_{1gc}\}_{gc} + \{\widetilde{\Psi}_{1gc}, \delta\widetilde{\Phi}_{1gc}\}_{gc}\rangle \tag{3.79}$$

Accordingly to the Eq.(3.78), functional derivative of the first order gyrocenter Hamiltonian is

$$\frac{\langle\delta\Phi_{1gc}\rangle}{\delta\Phi_1(\mathbf{x})} = \delta^3_{gc} \tag{3.80}$$

In order to evaluate functional derivative of the second order correction to the gyrocenter Hamiltonian, we integrate it by parts with $\widetilde{\Phi}_{1gc} = \partial_\theta\widetilde{\Psi}_{1gc}$

$$\begin{aligned}\langle\{\delta\widetilde{\Psi}_{1gc}, \widetilde{\Phi}_{1gc}\}_{gc}\rangle &= \langle\{\delta\widetilde{\Psi}_{1gc}, \frac{\partial\widetilde{\Psi}_{1gc}}{\partial\theta}\}_{gc}\rangle = -\langle\{\frac{\partial\delta\widetilde{\Psi}_{1gc}}{\partial\theta}, \widetilde{\Psi}_{1gc}\}_{gc}\rangle \\ &= -\langle\{\delta\widetilde{\Phi}_{1gc}, \widetilde{\Psi}_{1gc}\}_{gc}\rangle = \langle\{\widetilde{\Psi}_{1gc}, \delta\widetilde{\Phi}_{1gc}\}_{gc}\rangle\end{aligned} \tag{3.81}$$

Then

$$\frac{\delta H_{gy}}{\delta\Phi_1(\mathbf{x})} = \epsilon\ e\langle\delta^3_{gc}\rangle - \frac{\epsilon^2\ e^2}{\Omega}\langle\{\widetilde{\Psi}_{1gc}, \delta^3_{gc}\}_{gc}\rangle \equiv \langle\mathsf{T}^{-1}_{gc}\delta^3_{gc}\rangle \tag{3.82}$$

Here we make appear the push-forward gyrocenter operator $\mathsf{T}^{-1}_{gc} \equiv \mathbf{1} - \epsilon\pounds_{\mathbf{gy}}$ (up to the first order).

Following the schema presented for perturbed Maxwell-Vlasov system we rewrite the expression (3.76) for Eulerian variations of $\mathcal{L}_{gy}$ so that the variations generators $(S_{gy}, \delta\Phi_1)$ appears explicitly:

$$\begin{aligned}\delta\mathcal{L}_{gy} &= \left(\frac{\partial\Lambda}{\partial t} + \nabla\cdot\boldsymbol{\Gamma}\right) - \int S_{gy}\{\mathcal{F}_{gy}, \mathcal{H}_{gy}\}_\epsilon\ d^4p \\ &+ \delta\Phi_1\left[\frac{\epsilon^2}{4\pi}\nabla\cdot\mathbf{E}_1 - \epsilon\ e\int \mathcal{F}_{gy}\langle\mathsf{T}^{-1}_{gc}\delta^3_{gc}\rangle d^4p\ d^3X\right]\end{aligned} \tag{3.83}$$

here we have used Eq.(3.82) and Eq.(3.77). The Noether fields Λ and $\mathbf{\Gamma}$ that does not contribute to the variational principle

$$\Lambda \equiv \int S_{gy}\mathcal{F}_{gy}d^4p \tag{3.84}$$

$$\mathbf{\Gamma} \equiv \int S_{gy}\,\mathcal{F}_{gy}\,\dot{\mathbf{X}}\,d^4p - \frac{\epsilon^2}{4\pi}\delta\Phi_1\,\mathbf{E}_1 \tag{3.85}$$

with $\dot{\mathbf{X}} \equiv \{\mathbf{X}, H_{gy}\}_{gc}$ representing the gyroangle-independent gyrocenter velocity.

3.5.2 Gyrokinetic Maxwell-Vlasov equations

After substituting the variation (3.83) into the variational principle $\int\ \delta\mathcal{L}_{gy}d^4x = 0$ for arbitrary variation generators $(S_{gy}, \delta\Phi_1)$, we obtain the gyrokinetic Vlasov equation

$$\{\mathcal{F}_{gy}, \mathcal{H}_{gy}\}_\epsilon = 0 \tag{3.86}$$

and the gyrokinetic Poisson equation:

$$\epsilon\,\nabla\cdot\mathbf{E}_1 = 4\pi\,e\int\ \mathcal{F}_{gy}\langle\mathsf{T}_{gc}^{-1}\,\delta^3_{gc}\rangle d^{\,4}p\,d^{\,3}X \tag{3.87}$$

Performing the integration over the energy coordinate ($\int dw$) on the extended gyrokinetic Vlasov equation (3.86) we obtain the gyrokinetic Vlasov equation (see for details Appendix A.2.2).

$$\frac{\partial F}{\partial t} + \{F, H_{gy}\}_{gc} \equiv \frac{\partial F}{\partial t} + \dot{\mathbf{X}}\cdot\nabla F + \dot{p}_{\,||}\frac{\partial F}{\partial p_{\,||}} = 0 \tag{3.88}$$

and the gyrokinetic Poisson equation

$$\epsilon\,\nabla\cdot\mathbf{E}_1 = 4\pi\,e\int\ F\,\langle\mathsf{T}_{gc}^{-1}\,\delta^3_{gc}\rangle d^{\,3}p\,d^{\,3}X \tag{3.89}$$

3.6 Gyrokinetic momentum conservation law

Following the procedure for deriving the momentum conservation law that was established for the perturbed Maxwell-Vlasov system case, we now substitute the gyrokinetic Vlasov-Poisson equations (3.88) and (3.89) into the variational equation (3.83), we obtain the corresponding Noether equation

$$\delta\mathcal{L}_{gy} = \frac{\partial\Lambda}{\partial t} + \nabla\cdot\mathbf{\Gamma} \tag{3.90}$$

where Λ and $\mathbf{\Gamma}$ are defined up to the transformation (3.26).

3.6.1 Noether Method

Comparing to the expressions (3.49) for the variations generated by infinitesimal space translations $\mathbf{x} \to \mathbf{x} + \delta\mathbf{x}$ in the case of the full perturbed Maxwell-Vlasov system, the variations (S_{gy}, Φ_1) for the gyrokinetic electrostatic Vlasov-Maxwell system are now given by:

$$
\begin{aligned}
S_{gy} &= \mathbf{p}_{gy} \cdot \delta\mathbf{x} && (3.91)\\
\delta\Phi_1 &= -\delta\mathbf{x} \cdot \nabla\Phi_1 \equiv \delta\mathbf{x} \cdot \mathbf{E}_1 && (3.92)\\
\delta\mathcal{L}_{gy} &= -\delta\mathbf{x} \cdot (\nabla\mathcal{L}_{gy} - \nabla'\mathcal{L}_{gy}) && (3.93)
\end{aligned}
$$

where S_{gy} is the gyrocenter generating scalar field for the virtual spatial translation $\delta\mathbf{x}$ contains gyrocenter canonical momentum $\mathbf{p}_{gy}$ its expression will be discussed below. The expressions for electric field variation and Lagrangian density variation are obtained following the same procedure that was presented in Section 3.4.1. Note that comparing to the Eq. (3.50) for the derivative of the Lagrangian density with respect to the background magnetic field $\mathbf{B}_0$ we should replace $\mathcal{L}$ by $\mathcal{L}_{gy}$ and H by H_{gy}:

$$
\nabla'\mathcal{L}_{gy} \equiv \nabla\mathbf{B}_0 \cdot \frac{\partial\mathcal{L}_{gy}}{\partial\mathbf{B}_0} = -\nabla\mathbf{B}_0 \cdot \left(\frac{\mathbf{B}_0}{4\pi} + \int F\,\frac{\partial H_{gy}}{\partial\mathbf{B}_0}\,d^{\,3}p\right) \tag{3.94}
$$

We note also that in the absence of the perturbed magnetic field $\mathbf{B}_1$ the momentum conservation law that we derive is directly gauge independent:

$$
\begin{aligned}
-\delta\mathbf{x} \cdot (\nabla\mathcal{L}_{gy} - \nabla'\mathcal{L}_{gy}) &= \frac{\partial}{\partial t}\left[\int \mathcal{F}_{gy}\left(\langle\mathbf{p}_{gy}\rangle \cdot \delta\mathbf{x}\right) d^4p\right] && (3.95)\\
&+ \nabla \cdot \left[\int \mathcal{F}_{gy}\left(\langle\mathbf{p}_{gy}\rangle \cdot \delta\mathbf{x}\right)\dot{\mathbf{X}}_{gy}\,d^4p - \frac{\epsilon^2}{4\pi}\delta\mathbf{x} \cdot \mathbf{E}_1\mathbf{E}_1\right]
\end{aligned}
$$

Note that while deriving this momentum conservation law we have used only the gyroaveraged part of the generating function $S_{gy} = \langle\mathbf{p}_{gy}\rangle$. In fact it is necessary in order to be coherent with dynamics generated by the gyrocenter gyroangle-independent Hamiltonian (3.69). For example the gyrokinetic Poisson equation (3.89) is driven only by the gyroaveraged part of the gyrokinetic charge density $\rho_{gk} \equiv \int F\,\langle\mathsf{T}_{gy}^{-1}\delta_{gc}^3\rangle d^{\,3}p$. In what concerns the gyrokinetic Vlasov equation (3.88), it is obtained by assuming supplementary that $\partial_\theta F = 0$ (see appendix A.2.2).

Then we rewrite the expression (3.94) as

$$
\frac{\partial\mathbf{P}_{gy}}{\partial t} + \nabla \cdot \mathbf{\Pi}_{gy} = \nabla\mathbf{B}_0 \cdot \frac{\partial\mathcal{L}_{gy}}{\partial\mathbf{B}_0} \equiv \partial'\mathcal{L}_{gy} \tag{3.96}
$$

Then by performing the integration over the energy coordinate $\int dw$, the gyrokinetic momentum density is

$$
\mathbf{P}_{gy} = \int F\,\langle\mathbf{p}_{gy}\rangle\,d^3p \tag{3.97}
$$

and the gyrokinetic momentum stress tensor is

$$\mathbf{\Pi}_{gy} = \frac{1}{8\pi}\left(\epsilon^2|\mathbf{E}_1|^2 - |\mathbf{B}_0|^2\right)\mathbf{I} - \frac{\epsilon^2}{4\pi}\,\mathbf{E}_1\mathbf{E}_1 + \int F\,\dot{\mathbf{X}}_{gy}\,\langle\mathbf{p}_{gy}\rangle\,d^3p \tag{3.98}$$

where $\dot{\mathbf{X}}_{gy} \equiv \{\mathbf{X}, H_{gy}\}_{gc}$

3.6.2 Proof of Gyrokinetic Momentum conservation

As in previous case we give an explicit proof of the gyrokinetic momentum conservation law (3.96). We begin with the partial time derivative of the gyrokinetic momentum density (3.97):

$$\frac{\partial \mathbf{P}_{gy}}{\partial t} = \int \left[\frac{\partial F}{\partial t}\,\langle\mathbf{p}_{gy}\rangle + F\frac{\partial\langle\mathbf{p}_{gy}\rangle}{\partial t}\right] d^3p \tag{3.99}$$

By substituting corresponding gyrokinetic Vlasov equation in its phase-space divergence form (see for details appendix A.2.2)

$$\{F, H_{gy}\}_{gc} = \frac{1}{B^*_{||}}\nabla\cdot\left(B^*_{||}\,\dot{\mathbf{X}}\,F\right) + \frac{1}{B^*_{||}}\,\frac{\partial}{\partial p_{||}}\,\left(B^*_{||}\,\dot{p}_{||}\,F\right) \tag{3.100}$$

after integration by parts we obtain

$$\frac{\partial \mathbf{P}_{gy}}{\partial t} = \int F\,\frac{d_{gy}\,\langle\mathbf{p}_{gy}\rangle}{dt}\,d^3p - \nabla\cdot\left[\int F\,\dot{\mathbf{X}}_{gy}\,\langle\mathbf{p}_{gy}\rangle\right] d^3p \tag{3.101}$$

where

$$\frac{d_{gy}\,\langle\mathbf{p}_{gy}\rangle}{dt} \equiv \frac{\partial\langle\mathbf{p}_{gy}\rangle}{\partial t} + \{\langle\mathbf{p}_{gy}\rangle, H_{gy}\}_{gc} \tag{3.102}$$

Due to the fact that we have taken only the gyroaveraged part of the generating function S_{gy} in order to derive the gyrokinetic momentum conservation law and only $\langle\mathbf{p}_{gy}\rangle$ intervene into our calculations, we have:

$$B^*_{||}\{\langle\mathbf{p}_{gy}\rangle, H_{gy}\}_{gc} = \nabla\cdot\left(B^*_{||}\,\langle\mathbf{p}_{gy}\rangle\,\dot{\mathbf{X}}_{gy}\right) + \frac{\partial}{\partial p_{||}}\,\left(B^*_{||}\,\langle\mathbf{p}_{gy}\rangle\,\dot{p}_{||}\right) \tag{3.103}$$

Now we use the gyrokinetic Poisson equation (3.89) and the electrostatic constraint $\nabla\times\mathbf{E}_1 = 0$ in order to perform the electrostatic Maxwell part of the gyrokinetic momentum stress divergence $\nabla\cdot\mathbf{\Pi}_{gy}$

$$\begin{aligned}\frac{\epsilon^2}{4\pi}\nabla\cdot\left(\frac{1}{2}|\mathbf{E}_1|^2 - \mathbf{E}_1\mathbf{E}_1\right) &= \frac{\epsilon^2}{4\pi}\left(\mathbf{E}_1\times(\nabla\times\mathbf{E}_1) - \nabla\cdot\mathbf{E}_1\,\mathbf{E}_1\right)\\ = \epsilon\,e\int F\,\langle\mathsf{T}^{-1}_{gy}\delta^3_{gc}\,\mathbf{E}_1\rangle d^3p\,d^3x &\equiv \epsilon\,e\int F\,\langle\mathsf{T}^{-1}_{gy}\mathbf{E}_{1gc}\rangle\,d^3p \end{aligned} \tag{3.104}$$

Using the expression below we add and we subtract $\frac{\epsilon^2}{4\pi}\nabla\cdot\left(\frac{1}{2}|\mathbf{E}_1|^2-\mathbf{E}_1\mathbf{E}_1\right)$, and $\nabla\mathbf{B}_0\cdot\frac{\partial\mathcal{L}_{gy}}{\partial\mathbf{B}_0}$ given by the Eq. (3.94)to the r.h.s. of the Eq.(3.101) we obtain

$$\begin{aligned}
0 \;&\equiv\; \frac{\partial\mathbf{P}_{gy}}{\partial t}+\nabla\cdot\mathbf{\Pi}_{gy}=\nabla\mathbf{B}_0\cdot\frac{\partial\mathcal{L}_{gy}}{\partial\mathbf{B}_0} \\
&= \int F\left[\frac{d_{gy}\,\langle\mathbf{p}_{gy}\rangle}{dt}-\epsilon\, e\langle\mathsf{T}_{gy}^{-1}\mathbf{E}_{1gc}\rangle+\nabla\mathbf{B}_0\cdot\frac{\partial H_{gy}}{\partial\mathbf{B}_0}\right]d^{\,3}p
\end{aligned}\tag{3.105}$$

which yields the electrostatic gyrocenter canonical momentum equation:

$$\frac{d_{gy}\,\langle\mathbf{p}_{gy}\rangle}{dt}\equiv\epsilon\, e\langle\mathsf{T}_{gy}^{-1}\mathbf{E}_{1gc}\rangle-\nabla\mathbf{B}_0\cdot\frac{\partial H_{gy}}{\partial\mathbf{B}_0}\tag{3.106}$$

3.6.3 Gyrokinetic particle canonical momentum

In the section 3.4.4 we have discussed the expression of the perturbed particle canonical momentum that generate the momentum conservation law for perturbed Maxwell-Vlasov system. We have seen that the momentum conservation law was generated by a mixed-canonical momentum : $\mathbf{p}_*\equiv m\mathbf{v}+\dfrac{e}{c}\mathbf{A}_1$, which simply becomes in the electrostatic perturbation case the particle kinetic momentum.

Let us now analyze the equation (3.106) for electrostatic canonical momentum. We can deduce by analogy with the non-reduced case, which $\mathbf{p}_{gy}$ now represents kinetic gyrocenter canonical momentum $\mathbf{p}_{gy}=m\dot{\mathbf{X}}_{gy}$.

Due to the fact that H_{gy} given by Eq. (3.69) does not depend on the gyroangle coordinate and supposing that in the first approximation, the Jacobian of the guiding-center transform does not depend on the gyroanlge ($\mathcal{J}=2\pi mB_{||}^*\equiv 2\pi mB_{||}^*(\mathbf{X}_{gy})=2\pi m\mathbf{B}_*(\mathbf{X}_{gy})\cdot\hat{\mathbf{b}}_0(\mathbf{X})$), according to the gyrocenter equations of motion given in the Appendix (A.2.2), we have $\dot{\mathbf{X}}_{gy}\equiv\langle\dot{\mathbf{X}}_{gy}\rangle$. Then the expressions for $\mathbf{P}_{gy}$ and $\mathbf{\Pi}_{gy}$ become

$$\begin{aligned}
\mathbf{P}_{gy}&=\int F\,m\,\dot{\mathbf{X}}_{gy}\,d^3p \\
\mathbf{\Pi}_{gy}&=\frac{1}{8\pi}\left(\epsilon^2|\mathbf{E}_1|^2-|\mathbf{B}_0|^2\right)\mathbf{I}-\frac{\epsilon^2}{4\pi}\,\mathbf{E}_1\mathbf{E}_1+\int F\,\mathbf{p}_{gy}\,\dot{\mathbf{X}}_{gy}\,d^3p
\end{aligned}\tag{3.107}$$

3.7 Applications of the gyrokinetic momentum conservation law

In this section we explore the possible ways for applications of the gyrokinetic momentum conservation law (3.60). In particular here we consider the parallel and toroidal gyrokinetic momentum transport equation, derived in the axisymmetric magnetic geometry from this general gyrokinetic momentum conservation law.

These two equations are often considered as the same assuming that the background magnetic field is mainly in the toroidal direction. This is true only for a simple tokamak geometry, when the background magnetic field is considered as a toroidal magnetic field, but is invalid for spherical tokamak geometries, for example. So the consideration of this both equations can be useful.

3.7.1 Gyrokinetic momentum conservation law in background separated form

In order to study the momentum exchange between plasma and the background magnetic field, let us first rewrite the momentum conservation law (3.60) in its background separated form:

$$\frac{\partial \bar{\mathbf{P}}_{gy}}{\partial t} + \nabla \cdot \bar{\mathbf{\Pi}}_{gy} = -\nabla \mathbf{B}_0 \int d^3p \frac{\partial H_{gy}}{\partial \mathbf{B}_0} \tag{3.108}$$

where

$$\bar{\mathbf{P}}_{gy} \equiv \mathbf{P}_{gy} = \int F\, m\, \dot{\mathbf{X}}_{gy}\, d^3p \tag{3.109}$$

$$\bar{\mathbf{\Pi}}_{gy} \equiv \frac{1}{8\pi}\left(\epsilon^2 |\mathbf{E}_1|^2\, \mathbf{I} - 2\, \mathbf{E}_1 \mathbf{E}_1\right) + \int F\, m\, \dot{\mathbf{X}}_{gy}\, \dot{\mathbf{X}}_{gy}\, d^3p \tag{3.110}$$

Let us now consider the terms in the r.h.s. of the equation (3.108) that represent the exchange between plasma and background field. At the lowest order in ϵ, we obtain the guiding-center magnetization $\partial H_{gy}/\partial \mathbf{B}_0 = \mu \hat{\mathbf{b}}_0$.

For higher-order terms in $\partial H_{gy}/\partial \mathbf{B}_0$, in the first approximation, we take the limit where the background magnetic field is uniform. Details of the calculations for the higher - order gyrocenter contributions can be found in the appendix D. For example the first order correction $\epsilon\, \partial \langle \phi_{1gc} \rangle / \partial \mathbf{B}_0$ involves the divergence of the perpendicular perturbed electric field $\dfrac{\partial}{\partial \mathbf{B}_0} \langle \phi_{1gc} \rangle = \dfrac{\mu}{2m\Omega^2} \hat{\mathbf{b}}_0 \left(\nabla \cdot \mathbf{E}_{1\perp}\right)$ where $\nabla \cdot \mathbf{E}_{1\,\perp} \equiv \mathbf{I}_\perp : \nabla \mathbf{E}_1$.

3.7.2 Parallel momentum conservation law

Let us now project the equation (3.108) on the direction of the background magnetic field $\hat{\mathbf{b}}_0$

$$\begin{aligned} &\frac{\partial P_{\|}}{\partial t} + \hat{\mathbf{b}}_0 \cdot \left(\nabla \cdot \bar{\mathbf{\Pi}}_{gy}\right) = \\ &-\left(\int \mu\; F\; \left(1 + \epsilon \frac{1}{2m\Omega^2} \nabla \cdot \mathbf{E}_{1\,\perp}\right)\, d^3p\right)\, \hat{\mathbf{b}}_0 \cdot \nabla B_0 \end{aligned} \tag{3.111}$$

where $P_{||} \equiv \mathbf{P} \cdot \hat{\mathbf{b}}_0$, and we have used that: with $\mathbf{B}_0 \equiv B_0 \hat{\mathbf{b}}_0$ and $\nabla \mathbf{B}_0 = \nabla B_0 \hat{\mathbf{b}}_0 + B_0\, \nabla \hat{\mathbf{b}}_0$, we have $\hat{\mathbf{b}}_0 \cdot \nabla \mathbf{B}_0 \cdot \hat{\mathbf{b}}_0 = \hat{\mathbf{b}}_0 \cdot \nabla B_0 + B_0 \hat{\mathbf{b}}_0 \cdot \underbrace{\nabla \hat{\mathbf{b}}_0 \cdot \hat{\mathbf{b}}_0}_{=0}$ because of $\nabla (\hat{\mathbf{b}}_0 \cdot \hat{\mathbf{b}}_0) = 0$.

Here the terms in the r.h.s. of this equation represents the polarization due to the background magnetic field. At the lowest order we obtain the guiding-center mirror-force density and at the first order in electrostatic perturbation we have proportional to the perpendicular part of the gyrokinetic charge density $\rho_\perp \equiv \nabla \cdot \mathbf{E}_{1\perp}$ term.

3.7.3 Toroidal gyrokinetic momentum conservation law

According to Noether's theorem, the component of the gyrokinetic momentum in the direction of a spatial symmetry of the unperturbed magnetic field is conserved. In axisymmetric tokamak geometry, for example, where the background magnetic field is independent of the toroidal angle ϕ (i.e. $\frac{\partial \mathbf{x}}{\partial \phi} \cdot \nabla \mathbf{B}_0 = \frac{\partial \mathbf{B}_0}{\partial \phi} = 0$ and then $\frac{\partial' \mathcal{L}_{gy}}{\partial \phi} = 0$), the l.h.s. of the equation (3.96) vanish and the toroidal gyrokinetic momentum density $P_\phi \equiv \mathbf{P}_{gy} \cdot \frac{\partial \mathbf{x}}{\partial \phi}$ satisfies the toroidal gyrokinetic transport equation [2]:

$$\frac{\partial P_\phi}{\partial t} = -\frac{\partial \mathbf{x}}{\partial \phi} \cdot \left(\nabla \cdot \bar{\mathbf{\Pi}}_{gy} \right) \tag{3.112}$$

3.7.4 Intrinsic plasma rotation mechanisms identification

The problem of the identification of intrinsic rotation mechanism represents one of the relevant problems for magnetically confined plasmas. In fact, plasma rotation plays an important role in turbulence stabilization and transport reduction, therefore in improvement of tokamak performance both in stability and confinement. In present-day machines, rotation is usually driven by external sources, such as neutral beam injection. The problem is that such a rotation mechanism generators can become unavailable in a future fusion devices as ITER for example, due to their large size and high plasmas densities. In the same time one possible issue seems to be indicated by the system itself. It was observed in various fusion devices, such as Alcator C-Mod [46], DIII-D [47] and NSTX (National Spherical Torus Experiment)[48] that the plasma rotation has a spontaneous (intrinsic) component. As a consequence of these observations, numerous theoretical works on the establishing physical mechanisms of the non-diffusive momentum transport have been stimulated.

In fact, the turbulent momentum stress tensor $\mathbf{\Pi}$ (called *flux* term in the general theory for transport equation), plays the role of the key physics quantity for

[2]Note here that there is no ambiguity related to the contraction $\frac{\partial \mathbf{x}}{\partial \phi} \cdot \left(\nabla \cdot \bar{\mathbf{\Pi}}_{gy} \right)$ because the gyrokinetic momentum stress tensor $\bar{\mathbf{\Pi}}_{gy}$ is the second rank tensor and then $\nabla \cdot \bar{\mathbf{\Pi}}_{gy}$ is a vector, so there is no difference between the left and right contraction with the basis vector $\frac{\partial \mathbf{x}}{\partial \phi}$

the identification of the plasma rotation mechanism. In general theory it can be decomposed as follows [49]:

$$\mathbf{\Pi}_\phi \equiv \mathbf{\Pi} \cdot \frac{\partial \mathbf{x}}{\partial \phi} = -\chi_\phi \frac{\partial \dot{X}_\phi}{\partial r} + V \dot{X}_\phi + \mathbf{\Pi}^R \tag{3.113}$$

where χ_ϕ is the turbulent diffusivity, V is the convective (pinch) velocity and $\mathbf{\Pi}^R$ represents residual stress tensor. The two last terms both represents a non-diffusive contributions. The turbulent diffusivity is studied now since 20 years [**?**] while pinch and residual stress tensor mechanism were actively studied only in the last 3 years [49, 52, 53]. Special attention was payed to the residual stress investigation [49–51]. In fact the diffusive and convective mechanisms have an analogue in particle transport, while the residual stress tensor has not. As a consequence, the residual stress can be viewed as a candidate for treating the field-particle exchange [49].

The non-diffusive mechanisms listed in the r.h.s. of the equation (3.113) can certainly be detailed and completed. Let see for example the pinch mechanism. Initially the origin of the pinch term was shown to be connected with $\mathbf{E} \times \mathbf{B}$ shear mechanism in [52] and [53]. On the other hand in [54] a novel complementary to the $\mathbf{E} \times \mathbf{B}$ shear, pinch mechanism was identified originated from the symmetry breaking due to the magnetic field curvature. This is why it can be interesting to go into the depth of new non-diffusive momentum transport mechanisms identification.

Let us now analyze the momentum stress tensor (momentum flux) derived from the gyrokinetic variational principle. After a simple projection (on the right) to the parallel or toroidal direction we can identify the pinch term, proportional to the parallel or toroidal velocity, and the residual stress tensor that is simply represented by the Maxwell tensor $E_{1,a} E_{1\,||}$ where a represents the 3 spatial coordinates.

$$\begin{aligned} &\mathbf{\Pi}_{gy} \cdot \hat{\mathbf{b}}_0 \equiv \mathbf{\Pi}_{\,||} \\ &= \frac{\epsilon^2}{8\pi} \left(|\mathbf{E}_1|^2 \hat{\mathbf{b}}_0 - 2\, \mathbf{E}_1 E_{1\,||} \right) + \underbrace{\int d^3p\; F\; \dot{\mathbf{X}}_{gy}}_{=V} \dot{X}_{gy\,||} \end{aligned} \tag{3.114}$$

Comparing with the toroidal momentum conservation equation with vanishing r.h.s. due to the background magnetic field symmetry, the parallel momentum transport equation possesses some source terms which originate from background gyrokinetic magnetic field magnetization. Such a terms show the connection between plasma and the background field and should also be considered as a momentum transport mechanisms. Investigation of such terms represents an opportunity for future work.

3.7.5 Toroidal momentum evolution equation

Let us now consider the toroidal momentum evolution equation. We suppose that the poloidal component of the background magnetic field can be neglected with respect to the toroidal component of the background magnetic field $B_\phi \gg B_\theta$, it permits us in the first approximation to identify the parallel and the generalized unit toroidal direction with $\hat{\mathbf{b}}_0 = \hat{\nabla}\phi$.[3] By consequence we can use the toroidal gyrokinetic momentum transport equation (3.112) rather then parallel gyrokinetic momentum transport equation (3.111).

The details about toroidal projection of gyrokinetic momentum stress tensor in arbitrary axisymmetric geometry are presented in the appendix E. [4] Then the toroidal gyrokinetic momentum conservation law in cylindrical geometry is given by:

$$
\begin{aligned}
&\frac{\partial \mathbf{P}_{||}}{\partial t} + \int d^3p \left[R\,|\dot{\mathbf{X}}|^2 \frac{\partial F}{\partial \phi} - F\,\dot{X}^z \frac{\partial \dot{X}_{||}}{\partial z} - \underbrace{F\,\dot{X}^R \frac{\partial \dot{X}_{||}}{\partial R}}_{\text{anormal diffusion}} \right] + \\
&m \int d^3p \left[F\,\dot{X}_{||} \underbrace{\left(\frac{\partial \dot{X}^z}{\partial z} + \frac{\partial \dot{X}^R}{\partial R} + \frac{1}{R}\dot{X}^R \right)}_{\text{pinch velocity}} \right] = \\
&\frac{\epsilon^2}{4\pi} \left[\left(E_{||}\frac{\partial E^z}{\partial z} - E^z \frac{\partial E_{||}}{\partial z} \right) + \left(E_{||}\frac{\partial E^R}{\partial R} - E^R \frac{\partial E_{||}}{\partial R} \right) + \frac{1}{R}\,E_{||}E^R \right]
\end{aligned}
\tag{3.115}
$$

Here we have defined

$$
\int d^3p\,F\,\left(R\,\dot{X}_\phi \right) = \int d^3p\,F\,\dot{X}_{||}\ \equiv \mathbf{P}_{||}
\tag{3.116}
$$

Let us now analyze the equation (3.116) by comparing it to the equation (3.113) (appearing as Eq.2 in [49]). We remark that in our case we can similarly identify 3 principal groups of mechanisms responsible for intrinsic plasma rotation.

The first one contains terms proportional to the parallel velocity (*the pinch velocity*). The second one contains terms proportional to the gradient of the parallel velocity, such a terms are referred to the *abnormal diffusion* mechanisms[5]. The last group is classified as a group containing the Maxwell tensor components, called *residual stress* terms.

---

[3] Here $\nabla\phi = \frac{1}{R}\hat{\nabla}\phi$ and then the covariant parallel component of any vector $\mathbf{C}$ is related to its toroidal components as $\mathbf{C}_{||} = 1/R\,\mathbf{C}_\phi$

[4] We use the expression (E.63) in the case of the cylindrical geometry, where we identify coordinates of general axisymmetric geometry (ψ, θ, ϕ) to coordinates (R, z, ϕ). Here the metric tensor coordinates are $g_{RR} = 1, g_{\phi\phi} = R^2, g_{zz} = 1$ the Jacobian $\mathcal{J} = R$ and $\frac{\partial \mathcal{J}}{\partial z} = 0$, $\frac{\partial \mathcal{J}}{\partial R} = 1$

[5] Here the term *abnormal* means that the diffusion processus is driven by turbulence (for example, random walk of a particle driven by fluctuations of electromagnetic fields) and not by collisions leading to the dissipation. This type of diffusion is possible in the Hamiltonian framework.

Now we compare the momentum transport equation (3.116) to the momentum transport equation derived from the first moment for the gyrokinetic Vlasov equation given in [50]. In both cases the r.h.s. of these equations contains the gyrokinetic Maxwell stress tensor $\frac{\partial \mathbf{E}_{1a}\mathbf{E}_{1b}}{\partial y^a}$ that can be treated as a part of the residual stress tensor. However the origins of this term are different for each method mentioned here. Following the method presented in this chapter that uses the gyrokinetic Maxwell-Vlasov variational principle, the gyrokinetic Maxwell stress tensor originates directly from the expression for the gyrokinetic Maxwell-Vlasov action functional (3.4). In what concerns the second method used in [50], the same term appears as a result of the violation of the gyrocenter quasi neutrality at the second order of the electrostatic perturbation.

The gyrokinetic variational principle provides an exact momentum conservation equation at the third order, the momentum conservation equation given in [50] contains the highest order corrections to the residual stress tensor.

3.8 Summary

In this chapter the derivation of an exact gyrokinetic momentum conservation law using the gyrokinetic variational principle, presented in [38], is done in the cases of the full perturbed Maxwell-Vlasov and the electrostatic gyrokinetic Maxwell-Vlasov system. This chapter is organized so that the derivation of the momentum conservation law for the full perturbed Maxwell-Vlasov system prepare the derivation of the momentum conservation law in the case of the gyrokinetically reduced Maxwell-Vlasov system. In the first case only the effects resulting from the background magnetic field separation are considered. For example, the adaptation of the Eulerian variations for the Lagrangian density is discussed in the section 3.4.1. Then in Section 3.4.4 its influence on the particle canonical momentum $\mathbf{p}$ and therefore on the generating function $S = \mathbf{p} \cdot \delta\mathbf{x}$ is compared to the full Maxwell-Vlasov system case.

Further in 3.5 electrostatic gyrokinetic Maxwell-Vlasov system with background separated magnetic field is considered. The corresponding expression for the particle canonical momentum is discussed in 3.6.3.

Finally, one of the possible applications of the gyrokinetic momentum conservation law, the investigation of momentum transport phenomena, is considered in Section 3.7, and the toroidal momentum conservation and parallel momentum transport equations are derived. In the latter case the terms related to the exchange between plasma and background magnetic field are presented in (3.111). The identification of the intrinsic plasma rotation mechanisms resulting from the momentum conservation equation is done in the cylindrical geometry case (3.116).

In previous works [51, 55], the derivation of the gyrokinetic momentum transport equation was realized by using moments of the gyrokinetic Vlasov equation, which suffer from the standard gyrokinetic closure problem. In the same time the presented method for the gyrokinetic momentum conservation law derivation provide an exact statement which depends on the nonlinear gyrokinetic physics included in the gyrokinetic action functional.

Exploring the momentum conservation law derivation and its further physical interpretation in the electromagnetically perturbed gyrokinetic Maxwell-Vlasov system represents one of the perspectives of the future research.

Chapter 4

Intrinsic guiding center theory

In this chapter we address the problem of the dynamical reduction for the motion of a charged particle in a non-uniform slowly varying external magnetic field. The general purpose of our work is to face up to the problems related to some geometrical obstructions that were encountered previously.

The main idea for the dynamical reduction arises from consideration of the elementary problem of charged particle motion in a uniform magnetic field, for example $\mathbf{B} = B_0\hat{\mathbf{z}}$. In this case (when the component of initial velocity parallel to the magnetic field line is different from zero), the particle follows a helical orbit (otherwise the motion of the particle is confined to a circle in the plane perpendicular to the magnetic field line). Such a motion is multiscale: it consists of the slow uniform drift along the magnetic field line and the fast uniform rotation (gyration) around the magnetic field line. The frequency of this fast uniform rotation (gyrofrequency) expressed as $\Omega = eB/mc$ is called also the Larmor frequency. The radius of the gyration motion (gyroradius) $\boldsymbol{\rho} = \hat{\mathbf{b}} \times \mathbf{v}_\perp/\Omega$ explicitly depends on the fast variable (gyroangle) ζ. The component of velocity parallel to the magnetic field $v_{||} = \mathbf{v} \cdot \hat{\mathbf{b}}$ is constant. At the same time the kinetic energy and the modulus of velocity $v = (\mathbf{v} \cdot \mathbf{v})^{1/2} = (v_{||}^2 + v_\perp^2)^{1/2}$ are conserved. This yields the invariance of the following quantity $\mu = m\ v_\perp^2/2B$. In other words we can say that μ denotes a dynamical invariant of this system.

The guiding-center theories developed since Northrop [56] provide modifications to this elementary dynamics in order to expand its properties (separation of scales of motion and existence of dynamical invariants) in the case of slowly varying strong magnetic field. Here the slow variance of the magnetic field is defined with respect to the particle motion: we assume that its length scale L_B is large compared to scales of the particle motion: the modulus of the gyroradius $\rho = v_\perp/\Omega$ and the distance $v_{||}/\Omega$ traveled by the particle in one gyroperiod parallel to the magnetic field line. However, in the case of the non-uniform magnetic field for example, one should pay attention to the fact that the gyrofrequency, Ω, is no longer a constant but becomes dependent on the spatial coordinate $\mathbf{r}$. Then some estimate of Larmor frequency, for example $\Omega = \sup_{\mathbf{r}} \Omega(\mathbf{r})$, has to be made when defining the length scales.

The main idea of this approach is based on the physical intuition: in the case of slowly varying magnetic field the particle dynamics should approximately be the same as that in the case of the constant magnetic field.

Let us first consider the kinetic energy of the particle. It will still be conserved because the modulus of the particle velocity is still conserved, but the ratio between parallel and perpendicular velocity components can now vary. This leads to the fact that μ is no more a constant but a slowly varying quantity (an adiabatic invariant).

It was showed by Alfvèn (1940) and recently remind in [5] that the magnetic moment

$$\mu = \frac{e}{m} \int_{-\pi}^{\pi} \frac{d\zeta}{2\pi} \left[m\mathbf{v} + e\,\mathbf{A}(\mathbf{X} + \boldsymbol{\rho}) \right] \cdot \frac{\partial \boldsymbol{\rho}}{\partial \zeta} = \frac{m v_\perp^2}{2B}. \tag{4.1}$$

is an adiabatic invariant associated with the fast gyromotion of a charged particle (with mass m and the charge e) moving in a slowly varying magnetic field. Here the particle position $\mathbf{r}$ is decomposed into its slowly varying component $\mathbf{X}$, called the guiding center, and its rapidly varying component, represented by the gyroradius $\boldsymbol{\rho}$.

The general purpose of finding some adiabatic invariant is the possibility to make an appropriate change of variables that permits us to consider such a slowly varying quantity as one of the variables of the phase space. Then the variable associated to its fast variable will be treated as an ignorable one. Such a procedure permits us to simplify the dynamical description of the initial system.

Finding an adiabatic invariant can be used as a starting point for providing some procedure that yields the series for its high order corrections.

In the first part of this chapter the problem of such a series construction is considered on local particle phase space.

The common point of the guiding center theories developed previously is the introduction of the slowly varying guiding-center position defined by removing the gyroradius vector from the particle position $\mathbf{X} = \mathbf{r} - \boldsymbol{\rho}$. Such a transformation leads to the de-correlation between the positions in which electromagnetic fields (full position ($\mathbf{x}$)) and virtual particles (guiding-centers) (reduced position($\mathbf{X}$)) are evaluated. In the previous chapter, we have considered the gyrokinetic Maxwell-Vlasov equations, derived using reduced phase space coordinates obtained from the Lie-transform perturbation method. We observed that such a decorrelation was expressed through the appearance of the $\delta^3 \equiv \delta(\mathbf{x} - \boldsymbol{\rho})$ function within the reduced equations. One of the principal differences of our method is to not make use of such a reduced position but directly deal with the particle position.

Another important point that will be discussed here concerns the gyrophase definition. This question has been the subject of reflection for a number of plasma physicists since the development of guiding-center theories, from the early work of Hagan and Frieman [57], through that of Littlejohn [58] to the recent work of Sugiyama [59].

In the case of the constant and uniform magnetic field no ambiguity related to the gyrophase appears. It can be defined in the plane perpendicular to the magnetic field line as an angle between some constant perpendicular direction, (that can be conventionally taken as $\hat{\mathbf{x}}$, for example) and the gyroradius vector $\boldsymbol{\rho}$. The situation become more complex in the case of the nonuniform magnetic field. Here it is no longer possible to choose a constant reference direction in the perpendicular plane to represent the gyrophase origin.

The usual procedure proposes to measure this fast angle with respect to some fixed basis in the plane perpendicular to magnetic field, that we will note as $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2)$. Due to the spatial dependence of the magnetic directional vector, $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2)$ must also be dependent on the spatial coordinate. Then at each space point a different basis will be defined. That is not all. Another difficulty lies inside the fact that the vectors $\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2$ are not defined uniquely. The requirement of forming an orthogonal basis with the magnetic field directional vector $\hat{\mathbf{b}}$ leaves open the opportunity for rotation in the perpendicular plane about $\hat{\mathbf{b}}$ by some arbitrary angle, the *gyrogauge* angle. The core of the problematic here lies in a fundamental geometrical effect: anholonomy of a basis field in curved spaces.

The natural question which arises at this point is whether or not a "privileged" choice exists for $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2)$. A suggestion that is often made is to make use of the normal and binormal vectors to the field line, i.e. $\hat{\mathbf{b}}_1 = \dfrac{\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}}{|\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}|}$, $\hat{\mathbf{b}}_2 = \hat{\mathbf{b}} \times \hat{\mathbf{b}}_1$. This choice has some advantages and disadvantages. On the one hand, the vectors $\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2$ are tied to the physical vector $\hat{\mathbf{b}}$. On the other hand such a basis is undefined in the case of straight field lines and becomes discontinuous at field line twisting points. In the first part of this chapter we make use of this natural basis when deriving general expressions for dynamical equations in local coordinates.

At the same time, when implementing these general dynamical equations in the particular case of axisymmetric magnetic field geometry, we will operate with a more practical choice for calculations by taking in place of the curvature vector $\hat{\mathbf{b}}_1$, the magnetic flux coordinate $\nabla\psi$.

In the second part of this chapter an approach that does not involve use of some fixed basis in order to measure the gyrophase angle is presented. Moreover, no gyrophase is used in order to represent rotation in the plane perpendicular to magnetic field. The description of rotations is made on a more abstract level involving intrinsic so(3) Lie structure. A detailed construction for intrinsic gyroaveraging operator is also presented.

In what follows we will refer our results to the guiding-center theory resulting from the Lie-transform perturbation method [2, 3, 5, 45]

4.1 Noncanonical Hamiltonian structure

It is well known that dynamics of a charged particle (of mass m and charge e) in an external non-uniform magnetic field $\mathbf{B} = B\hat{\mathbf{b}}$, (where B is the magnitude of the magnetic field and $\hat{\mathbf{b}}$ is its direction) has a Hamiltonian structure. Such a Hamiltonian structure admits different formulations. The traditional one uses the gauge-dependent electromagnetic potential formalism $\mathbf{B} = \nabla \times \mathbf{A}$; in this case the corresponding Hamiltonian structure is canonical. The phase space consists of the canonically conjugate coordinates: the canonical particle position $\mathbf{q}$ and the canonical particle momentum following from the minimal coupling principle $\mathbf{P} = m\dot{\mathbf{q}} + e\mathbf{A}$.[1]

Another possibility is to use a field formalism that has the advantage to be gauge independent. Here the corresponding Hamiltonian structure is non-canonical because now it contains the coupling between fields and particles. The corresponding phase space consists of non-canonically conjugated variables: a local particle position $\mathbf{r}$ and the particle kinetic momentum $\mathbf{p} = m\dot{\mathbf{r}}$. Such phase space variables represent physical coordinates of the particle and so are better adapted to highlight the underlying physical properties of the system, as for example dynamical scale separation, necessary to realize dynamical reduction. For this purpose, the non-canonical Hamiltonian formulation was used for the first time by R. Littlejohn in [2].

Note that both Hamiltonian representations are related by the mapping:

$$\begin{aligned} \mathbf{r} &= \mathbf{q} && (4.2) \\ \mathbf{p} &= \mathbf{P} - e\mathbf{A} && (4.3) \\ & && (4.4) \end{aligned}$$

In what follows, we consider the free relativistic particle Hamiltonian:

$$H = \left(m^2 + |\mathbf{p}|^2\right)^{\frac{1}{2}} \equiv m\gamma \tag{4.5}$$

and the corresponding non-canonical Poisson bracket given by:

$$\{f, g\} = \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial g}{\partial \mathbf{r}} - \frac{\partial f}{\partial \mathbf{r}} \cdot \frac{\partial g}{\partial \mathbf{p}} - e\mathbf{B} \cdot \frac{\partial f}{\partial \mathbf{p}} \times \frac{\partial g}{\partial \mathbf{p}} \tag{4.6}$$

Then the suitable equations of motion in the non-canonical variables $(\mathbf{r}, \mathbf{p})$ are:

$$\begin{aligned} \dot{\mathbf{r}} &= \{H, \mathbf{r}\} = \frac{\mathbf{p}}{m\gamma} && (4.7) \\ \dot{\mathbf{p}} &= \{H, \mathbf{p}\} = \frac{e}{m\gamma}\left(\mathbf{p} \times \mathbf{B}\right) && (4.8) \end{aligned}$$

where we have used that $|\mathbf{p}|^2 = \mathbf{p} \cdot \mathbf{p}$ and $\partial_{\mathbf{p}} H = \mathbf{p}/m\gamma$.

[1] In this chapter we assume that $c = 1$

4.2 Dynamical reduction

As mentioned above, in order to proceed with dynamical reduction, one should explore the underlying properties of the dynamical system. The first obvious simplification that one can bring into the dynamical description of our system is to remark that the modulus of the particle kinetic momentum $p \equiv \sqrt{\mathbf{p}\cdot\mathbf{p}}$ is a trivial constant of motion. In fact, it immediately arises from the expression (4.5) for particle relativistic Hamiltonian, a trivial function of p. [2] Due to this fact, we make another change of variable:

$$\begin{pmatrix} \mathbf{r} \\ \\ \mathbf{p} \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{r} \\ \\ \hat{\mathbf{p}} = \dfrac{\mathbf{p}}{p} \\ \\ p = (\mathbf{p}\cdot\mathbf{p})^{1/2} \end{pmatrix}, \tag{4.9}$$

With the following decomposition of the particle kinetic momentum $\mathbf{p} \equiv p\,\hat{\mathbf{p}}$, where $\hat{\mathbf{p}}$ denotes the unit momentum vector tangent to the particle orbit. Note that here we do not introduce the guiding center of the particle but work directly with its position. This is one of the principle differences with the earlier work of R. Littlejohn [2].

4.2.1 Rescaled Hamiltonian dynamics

Let us now consider the dynamics generated by the Hamiltonian that is equal to the norm of the particle kinetic momentum p. By using the Poisson bracket defined in (4.6), and the definition of unit vector $\hat{\mathbf{p}} \equiv \mathbf{p}/p$ we obtain:

$$\begin{aligned} \dot{\mathbf{r}} &= \{p, \mathbf{r}\} = \hat{\mathbf{p}} \\ \dot{\mathbf{p}} &= \{p, \mathbf{p}\} = \hat{\mathbf{p}} \times e\mathbf{B}. \end{aligned} \tag{4.10}$$

Note that such a rescaling of the Hamiltonian,

$$m\gamma \equiv \sqrt{p^2 + m^2} \rightarrow p \tag{4.11}$$

is equivalent to the change of the time scale

$$t \ \rightarrow \tau \equiv \ \frac{p}{m\gamma}\, t, \tag{4.12}$$

This dynamics in the new phase space $(\mathbf{r}, \hat{\mathbf{p}}, p)$ is given by:

$$\begin{aligned} \dot{\mathbf{r}} &= \{p\ ,\ \mathbf{r}\} = \hat{\mathbf{p}} \\ \dot{\hat{\mathbf{p}}} &= \{p\ ,\ \hat{\mathbf{p}}\} = \frac{1}{p}\ (\ \hat{\mathbf{p}}\ \times e\ \mathbf{B}) \\ \dot{p} &= 0. \end{aligned} \tag{4.13}$$

[2] Moreover this constant of motion is unstable with respect to the perturbation of the system by an electric field. Our goal here is to obtain a constant of motion that can resist such a perturbation.

Here we have used [3]

$$\frac{\partial \hat{\mathbf{p}}}{\partial \mathbf{p}} = \frac{1}{p}\left(\mathbf{I} - \hat{\mathbf{p}}\hat{\mathbf{p}}\right), \quad \frac{\partial p}{\partial \mathbf{p}} = \hat{\mathbf{p}}. \tag{4.14}$$

The next step in the investigation of this system will depend on the way that we deal with the vector $\hat{\mathbf{p}}$.

The essential point is the choice of the basis for its decomposition and the manner to proceed with the dynamical reduction. In the two following subsections we shortly discuss the difficulties associated with these aspects.

4.2.2 Gyrogauge transformation

One of the possible ways to deal with unit momentum vector $\hat{\mathbf{p}}$ (tangent to the particle trajectory) is to decompose it in the basis associated to magnetic field line by introducing two scalar dynamical variables φ-pitch angle and ζ-gyroangle locally. Such a procedure consists of two steps.

At the first stage we introduce a fixed frame, composed of the tangent to the magnetic field line vector $\hat{\mathbf{b}} = \mathbf{B}/B$ and two vectors $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2)$ in a perpendicular plane. The pitch angle φ measures the projection of the unit momentum parallel to the magnetic field and is defined as $\cos\varphi = \hat{\mathbf{p}}\cdot\hat{\mathbf{b}}$. In order to define the gyrophase ζ we have to introduce two rotating vectors:

$$\hat{\boldsymbol{\rho}}(\mathbf{x},\zeta) = \hat{\mathbf{b}}_1(\mathbf{x})\cos\zeta - \hat{\mathbf{b}}_2(\mathbf{x})\sin\zeta \tag{4.15}$$
$$\hat{\perp}(\mathbf{x},\zeta) = -\hat{\mathbf{b}}_1(\mathbf{x})\sin\zeta - \hat{\mathbf{b}}_2(\mathbf{x})\cos\zeta \tag{4.16}$$

The problem is that the fixed frame vectors $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2)$ are not unique. This frame is defined up to rotations through the angle ξ:

$$\hat{\mathbf{b}}_1' = \hat{\mathbf{b}}_1\cos\xi - \hat{\mathbf{b}}_2\sin\xi \tag{4.17}$$
$$\hat{\mathbf{b}}_2' = -\hat{\mathbf{b}}_1\sin\xi - \hat{\mathbf{b}}_2\cos\xi \tag{4.18}$$

Such a transformation does not affect the spatial variable $\mathbf{x}' = \mathbf{x}$. In order to keep the rotating frame vectors invariant, the translation of the gyrophase ζ on some angle $\xi = \xi(\mathbf{x})$ should be taken into account:

$$\zeta \to \zeta' \equiv \zeta + \xi \tag{4.19}$$

Finally the result of two transformations (4.18) and (4.19), named *the gyrogauge transformations* do not change the rotating vectors. After simple substitution and some trigonometry we obtain $\hat{\boldsymbol{\rho}}'(\mathbf{x}',\zeta') = \hat{\boldsymbol{\rho}}(\mathbf{x},\zeta), \hat{\perp}'(\mathbf{x}',\zeta') = \hat{\perp}(\mathbf{x},\zeta)$.

[3] In writing these expressions, we make use the fundamental identities of vector calculus: $a\cdot(b\times c) = c\cdot(b\times c)$, then $\dot{\hat{\mathbf{p}}} = \{p, \hat{\mathbf{p}}\} = -\frac{\partial \hat{\mathbf{p}}}{\partial \mathbf{p}}\cdot\left(e\,\mathbf{B}\times\frac{\partial p}{\partial \hat{\mathbf{p}}}\right) = \hat{\mathbf{p}}\times\,\mathbf{B}$

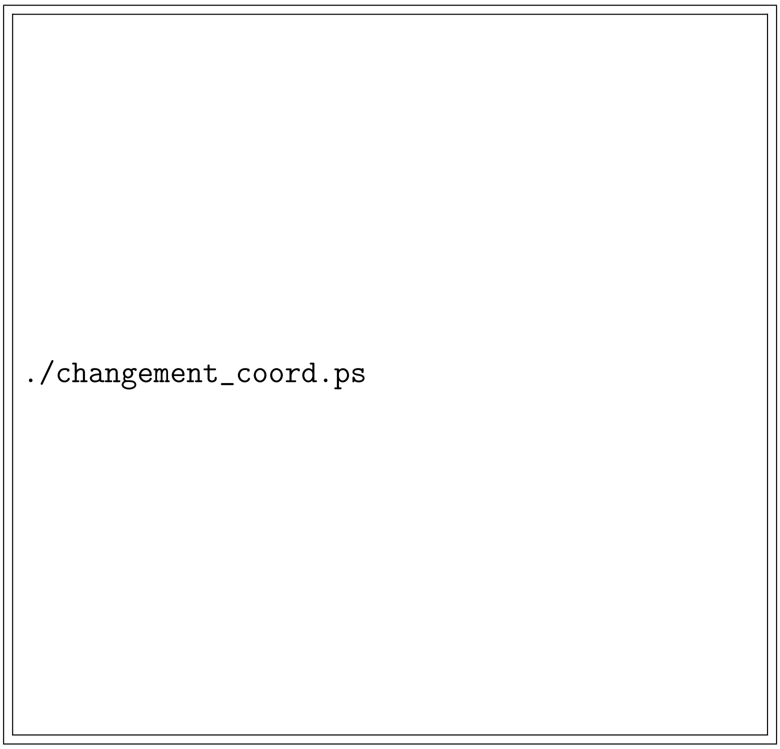

Figure 4.1: Gyrogauge transformation.

However, in the same time, dynamics of the system is not invariant with respect to the transformations (4.18) and (4.19). It was shown by Littlejohn [3] that in order to guarantee the gyrogauge invariance of the theory, the rotation of the fixed frame must be included inside the dynamical equations. This is why the gyrogauge vector $\mathbf{R} = \nabla\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{b}}_2$, which designs such a rotation, should be encountered during the derivation of the equations of motion.

The Lie-transform method presented by R. Littlejohn (1981) [3] and developed by A.J. Brizard in [5, 45, 60] offers an iterative procedure that guarantees the independence of the reduced parallel dynamics on the gyrogauge function $\xi = \xi(\mathbf{x})$ at any order. In the same time the reduced gyrophase variable will still to be dependent on the vector $\mathbf{R}$. In what follows we will see that the same situation produces for obtained by our method exact and gyroaveraged dynamical equations of motion.

The problem of the gyrogauge independence for guiding center theories indicated and resolved by Littlejohn was recently re-evoked in the work of L. Sugiyama [59] and provoked an active discussion inside the gyrokinetics community [61, 62].

It was repeated [59] that the crux point of the guiding center theories is hidden inside the definition of the gyrophase. Moreover it was emphasized that the anholonomy of the fixed basis vectors that occurs in the general 3 dimensional magnetic field configuration, due to its non-zero torsion, can lead to breaking down of the gyrogauge vector $\mathbf{R}$ and failure of perturbative series expansion at the second order. To assure globally consistent definition of the vector $\mathbf{R}$ a serious restriction on the magnetic configuration to be torsion free was indicated. However Krommes [61] indicated that in any case the guiding center theories does not explore the global properties of the magnetic geometry because of the failure of the adiabatic invariant before the moment when the particle starts to be affected by global magnetic field properties.

4.2.3 Constant of motion and Hamiltonian normal form

The phase space variables, canonical or non-canonical are never completely independent. In fact, they are always related by the Hamiltonian stationarity condition $\dot{H} = \dot{H}(\mathbf{r}, \mathbf{p}) = 0$. In our case this condition is expressed as $\dot{p} = 0$.

Let us now consider the dynamics generated by the rescaled Hamiltonian p (4.10). Then after the change of variables (4.9), the rescaled Hamiltonian p represents one of the independent phase space variables $(\mathbf{r}, \hat{\mathbf{p}}, p)$ and the equations of motion become (4.13). If we now suppose that our system possesses a constant of motion $\mathcal{A}$, it can then be viewed as $\mathcal{A} = \mathcal{A}(\mathbf{r}, \hat{\mathbf{p}}, p)$, or by inverting the functional dependencies, the rescaled Hamiltonian p can be viewed as a function of the constant of motion $p = p(\mathbf{r}, \hat{\mathbf{p}}, \mathcal{A})$. This idea leads inside two approaches that we implement in this chapter in order to deal with the dynamical reduction of our Hamiltonian system (4.5),(4.6).

The first one, composed of two principal steps, starts with construction of a constant of motion. It consists to use the corresponding stationarity condition $0 \equiv \dot{\mathcal{A}} = \{\mathcal{A}, H\}$ that yields a partial differential equation for $\mathcal{A}$. Due to the separation of dynamical scales suitable for our system, there will be an opportunity to find one of its physical solutions in a small parameter series decomposition $\mathcal{A}(\mathbf{r}, \hat{\mathbf{p}}, p) = \sum_{i=0}^{n} \mathcal{A}_i(\mathbf{r}, \hat{\mathbf{p}}, p)\, \varepsilon^i$ where $\mathcal{A}_i$ satisfy the stationarity condition at the i-th order.

At the first step, dynamics of the system will be reduced on the hyperplane defined by the functional phase-space dependence of the constant of motion $\mathcal{A} = \mathcal{A}(\mathbf{r}, \hat{\mathbf{p}}, p)$. Then at each order of its series decomposition, the constant of motion can be used to control the precision of the dynamical reduction by considering its time variation. This opportunity will be exploited in the section 4.4 while exploring trapped particle trajectories.

The next step consists in inverting the functional dependence between the new constant of motion $\mathcal{A}$ and the initial phase space variables $(\mathbf{r}, \hat{\mathbf{p}}, p)$ in order to include it in the set of the new phase space variables $(\mathbf{r}, \hat{\mathbf{p}}, \mathcal{A})$. Finally, by rewriting the system dynamics (Hamiltonian and the Poisson bracket) as the functions of $\mathcal{A}$, we obtain the system for which one of their equations of motion is $\dot{\mathcal{A}} = 0$. The procedure of expressing the Hamiltonian as a function of the constant of motion $\mathcal{A}$ leads to construction of its *Hamiltonian normal form*. Due to the fact that the constant of motion $\mathcal{A}$ in our case, can be decomposed in small parameter series, we can built the corresponding Hamiltonian series in the form: $H(\mathbf{r}, \hat{\mathbf{p}}, \mathcal{A}) = \sum_{i=0}^{n} H_i(\mathbf{r}, \hat{\mathbf{p}})\ \mathcal{A}^i\ \varepsilon^i$.

In fact, to know the reduced dynamics of a Hamiltonian system we need to know the Hamiltonian normal form series decomposition.

This is why the second approach deals directly with the *Hamiltonian normal form* series, without passing through the first stage of the construction of the constant of motion. From the beginning we work on the phase space $(\mathbf{r}, \hat{\mathbf{p}}, \mathcal{A})$ and we consider the rescaled Hamiltonian as a function on this phase space $p = p(\mathbf{r}, \hat{\mathbf{p}}, \mathcal{A})$ that satisfies the stationarity condition $\dot{p} = 0$. In fact this condition has to be satisfied independently of the choice of the phase space variables. Moreover it gives the partial differential equation for p that leads to its series decomposition in the new phase space variables.

In the following we consider the problem of dynamical reduction by applying these two approaches. In the first part of this chapter, we pass through the constant of motion construction in order to derive reduced dynamics in the local coordinates. Such a coordinates will be dependent on the choice of the fixed basis $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2)$ associated to the magnetic field line. Our goal here is to present and to illustrate, in a particular case of axisymmetric magnetic geometry, the local dynamical reduction without introducing the guiding-center position.

In the second part, we proceed the intrinsic (independent of the fixed basis $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2)$) dynamical reduction. Here we will pass by the direct way of the Hamiltonian normal form construction.

4.3 Local dynamical reduction

4.3.1 Fixed and dynamical basis

Here we start with introduction of the right-handed set of the fixed vectors $(\hat{\mathbf{b}}_0, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2)$ where $\hat{\mathbf{b}}_0 \equiv \mathbf{B}/B$ is the unit tangent to the magnetic field line vector at some space point $\mathbf{r}$, $\hat{\mathbf{b}}_1 \equiv k^{-1}\hat{\mathbf{b}}_0 \cdot \nabla\hat{\mathbf{b}}_0$ denotes the unit vector in the direction of the magnetic curvature (with $k \equiv |\hat{\mathbf{b}}_0 \cdot \nabla\hat{\mathbf{b}}_0|$), and $\hat{\mathbf{b}}_2 \equiv \hat{\mathbf{b}}_0 \times \hat{\mathbf{b}}_1$. Then we introduce the momentum-space coordinates (p, φ, ζ), where p is the norm of particle momentum defined in (4.2), the pitch angle φ and the gyroangle ζ are given by:

$$\varphi = \arccos \frac{\mathbf{p} \cdot \hat{\mathbf{b}}_0}{\sqrt{\mathbf{p} \cdot \mathbf{p}}}, \quad \zeta = \arctan \frac{\mathbf{p} \cdot \hat{\mathbf{b}}_1}{\mathbf{p} \cdot \hat{\mathbf{b}}_2} \tag{4.20}$$

It permits us to decompose the unit momentum vector $\hat{\mathbf{p}}$ tangent to the particle orbit as follows:

$$\hat{\mathbf{p}} = \frac{\mathbf{p}}{p} = \cos\varphi\ \hat{\mathbf{b}}_0 - \sin\varphi\ \left(\sin\zeta\ \hat{\mathbf{b}}_1 + \cos\zeta\ \hat{\mathbf{b}}_2\right) \tag{4.21}$$

and its associated orthogonal vectors

$$\hat{\mathbf{p}}_1 = \frac{\partial \hat{\mathbf{p}}}{\partial \varphi} = -\sin\varphi\, \hat{\mathbf{b}}_0 - \cos\varphi\, \left(\sin\zeta\, \hat{\mathbf{b}}_1 + \cos\zeta\, \hat{\mathbf{b}}_2\right) \tag{4.22}$$

$$\hat{\mathbf{p}}_2 = -\frac{1}{\sin\varphi}\frac{\partial \hat{\mathbf{p}}}{\partial \zeta} = \cos\zeta\, \hat{\mathbf{b}}_1 - \sin\zeta\, \hat{\mathbf{b}}_2 \tag{4.23}$$

According to expressions above the dynamical set of the vectors $(\hat{\mathbf{p}}, \hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2)$ can be obtained from the fixed set $(\hat{\mathbf{b}}, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2)$ by rotation through the angles φ and ζ. This relation can be expressed with multiplication by matrix:

$$\mathcal{U} = \begin{pmatrix} \cos\varphi & -\sin\varphi\sin\zeta & -\sin\varphi\cos\zeta \\ -\sin\varphi & -\cos\varphi\sin\zeta & -\cos\varphi\cos\zeta \\ 0 & \cos\zeta & -\sin\zeta \end{pmatrix} \tag{4.24}$$

Some remarks about dependence and independence of the phase space variables

We have to emphasize here that the new phase - space variables $(\mathbf{x}, p, \varphi, \zeta)$, where $\mathbf{x}$ denotes the new particle position, have to be considered as independent from each other. An ambiguity can appear due to the fact that the coordinate transformation $(\hat{\mathbf{p}} \to (\varphi, \zeta))$ does not affect the particle position. However the main difference between new and old phase space coordinates is their spatial dependence. To avoid this inconvenience, in what follows we will distinguish two particle positions: $\mathbf{r}$ for the old variables and $\mathbf{x}$ for the new variables.

Note that for the fixed basis vectors the spatial dependence is considered to be the same in old and new variables:

$$\hat{\mathbf{b}}_\alpha = \hat{\mathbf{b}}_\alpha(\mathbf{r}) \equiv \hat{\mathbf{b}}_\alpha(\mathbf{x}) \tag{4.25}$$

where $\alpha \in \{0, 1, 2\}$. This can lead to two different situations: the variables that were initially defined as independent become spatially dependent after passing from $\mathbf{r}$ to $\mathbf{x}$ and vice versa.

For example, the initial phase space variables $(\mathbf{p}, \mathbf{r})$ are considered to be independent of each other, then the particle kinetic momentum $\mathbf{p}$ is independent of the particle position $\mathbf{r}$. After change of variables kinetic particle momentum $\mathbf{p} = p\, \hat{\mathbf{p}}$ is now decomposed accordingly to the eq.(4.21) and became dependent on the new space variable $\mathbf{x}$ through the spatial dependence of the vectors $(\hat{\mathbf{b}}_0(\mathbf{x}), \hat{\mathbf{b}}_1(\mathbf{x}), \hat{\mathbf{b}}_2(\mathbf{x}))$.

On the other hand, the pitch angle variable φ and the gyrophase variable ζ are independent on the new phase space, accordingly to the eq.(4.21) they become spatially dependent when returning to the phase space $(\mathbf{p}, \mathbf{r})$ through basis vectors $(\hat{\mathbf{b}}_0(\mathbf{r}), \hat{\mathbf{b}}_1(\mathbf{r}), \hat{\mathbf{b}}_2(\mathbf{r}))$

We will need to carefully use this information when obtaining equations of motion in the new variables.

4.3.2 Local Poisson bracket

In order to proceed with the derivation of the equations of motion on the new 5-dimensional phase space, $(\mathbf{x}, p, \varphi, \zeta)$ we need to find the corresponding expression for the Poisson bracket (4.6).

There are two possibilities to proceed. The first one is to make the change of variables inside the 2-form that corresponds to the non-canonical bracket (4.6)

$$\sigma = \mathbf{dx} \wedge \mathbf{dp} - e\, B\, \mathbf{dx} \otimes \mathcal{B}\mathbf{dx} \tag{4.26}$$

where $\mathcal{B} \equiv \epsilon_{ijk}\hat{\mathbf{b}}_j$. There are two stages: the first one consists to make the change of variables for 1-forms $\mathbf{dx}$, $\mathbf{dp}$. The second one consists to inverse the corresponding symplectic matrix. Such a procedure is similar to one used inside the Lie-transform perturbation method [5]. Here we will exploit another possibility by making the change of variables directly inside the Poisson bracket. To realize it we use the chain rule:

$$\{f, g\}_{new} = \sum_{i,j} \frac{\partial f}{\partial z_i} \{z_i, z_j\}_{old} \frac{\partial g}{\partial z_j} \tag{4.27}$$

where $z_i = (\mathbf{x}, p, \varphi, \zeta)$ represent the new phase space variables and $\{...\}_{old}$ is the Poisson bracket expressed in initial variables $(\mathbf{r}, \mathbf{p})$.

Note that this formula appears naturally when applying the chain rule:

$$\frac{\partial}{\partial \mathbf{r}} = \frac{\partial \mathbf{x}}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{x}} + \frac{\partial \phi}{\partial \mathbf{r}} \frac{\partial}{\partial \phi} + \frac{\partial \zeta}{\partial \mathbf{r}} \frac{\partial}{\partial \zeta} + \frac{\partial p}{\partial \mathbf{r}} \frac{\partial}{\partial p} \tag{4.28}$$

and

$$\frac{\partial}{\partial \mathbf{p}} = \frac{\partial \mathbf{x}}{\partial \mathbf{p}} \frac{\partial}{\partial \mathbf{x}} + \frac{\partial \phi}{\partial \mathbf{p}} \frac{\partial}{\partial \phi} + \frac{\partial \zeta}{\partial \mathbf{p}} \frac{\partial}{\partial \zeta} + \frac{\partial p}{\partial \mathbf{p}} \frac{\partial}{\partial p} \tag{4.29}$$

Then the expression for the canonical part of the Poisson bracket in new variables appears when developing the expression:

$$\begin{aligned}
&\frac{\partial}{\partial \mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{r}} - \frac{\partial}{\partial \mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{p}} = \\
&\{\phi, \mathbf{x}\} \cdot \left(\frac{\partial}{\partial \phi} \frac{\partial}{\partial \mathbf{x}} - \frac{\partial}{\partial \mathbf{x}} \frac{\partial}{\partial \phi} \right) + \{\zeta, \mathbf{x}\} \cdot \left(\frac{\partial}{\partial \zeta} \frac{\partial}{\partial \mathbf{x}} - \frac{\partial}{\partial \mathbf{x}} \frac{\partial}{\partial \phi} \right) + \{p, \mathbf{x}\} \cdot \left(\frac{\partial}{\partial p} \frac{\partial}{\partial \mathbf{x}} - \frac{\partial}{\partial \mathbf{x}} \frac{\partial}{\partial p} \right) + \\
&\{p, \phi\} \left(\frac{\partial}{\partial p} \frac{\partial}{\partial \phi} - \frac{\partial}{\partial p} \frac{\partial}{\partial \phi} \right) + \{p, \zeta\} \left(\frac{\partial}{\partial p} \frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \zeta} \frac{\partial}{\partial p} \right) + \{\zeta, \phi\} \left(\frac{\partial}{\partial \zeta} \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \phi} \frac{\partial}{\partial \zeta} \right)
\end{aligned} \tag{4.30}$$

Generalized Frenet-Serret equations

During the derivation of the Poisson bracket we will need to deal with spatial derivatives of the fixed basis vectors $(\hat{\mathbf{b}}_0, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2)$. Such a derivatives can be expressed in the following compact bi-vector form [4]

[4]In what follows we use the Greek indices to indicate the different basis vectors $(\hat{\mathbf{b}}_0, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2)$ and the Latin indices in order to indicate its coordinates

$$\nabla\,\hat{\mathbf{b}}_\alpha = M\cdot\mathcal{B}_\alpha \tag{4.31}$$

Note that this relation does not change when passing from the old space variable $\mathbf{r}$ to the new space variable $\mathbf{x}$ because the assumption (4.25) on the invariance of the spatial dependence of the basis vectors.

The bi-vector M is defined as follows[5]:

$$M = (\nabla\hat{\mathbf{b}}_0\cdot\hat{\mathbf{b}}_1)\,\hat{\mathbf{b}}_2 + (\nabla\hat{\mathbf{b}}_2\cdot\hat{\mathbf{b}}_0)\,\hat{\mathbf{b}}_1 + (\nabla\hat{\mathbf{b}}_1\cdot\hat{\mathbf{b}}_2)\,\hat{\mathbf{b}}_0 \tag{4.32}$$

and $\mathcal{B}_\alpha = \epsilon_{ijk}\;\hat{\mathbf{b}}_{\alpha\,j}$ denotes the bi-vector which components are given by the operator "the vector product with the basis vector $\hat{\mathbf{b}}_\alpha$".

In order to prove (4.31) we have to use the fact that $\nabla\hat{\mathbf{b}}_\alpha\cdot\hat{\mathbf{b}}_\beta = -\nabla\hat{\mathbf{b}}_\beta\cdot\hat{\mathbf{b}}_\alpha$ and $\nabla\hat{\mathbf{b}}_\alpha\cdot\hat{\mathbf{b}}_\alpha = 0$.

The expression for basis vector derivatives can be also interpreted as generalized Frenet-Serret equations.

Curvature-torsion

The generalized Frenet-Serret equation (4.31) yields an expression for the curvature-torsion of the fixed basis vectors:

$$\nabla\times\hat{\mathbf{b}}_\alpha = \hat{\mathbf{b}}_\alpha\cdot(M - \mathbf{I}:M) \tag{4.33}$$

where $\mathbf{I} = \hat{\mathbf{b}}_0\hat{\mathbf{b}}_0 + \hat{\mathbf{b}}_1\hat{\mathbf{b}}_1 + \hat{\mathbf{b}}_2\hat{\mathbf{b}}_2$ denotes the identity tensor in the fixed basis. Then the curvature-torsion coefficients are expressed as:

$$\begin{aligned}
\nabla\times\hat{\mathbf{b}}_0 &= -\left(M_{11}+M_{22}\right)\hat{\mathbf{b}}_0 + M_{01}\;\hat{\mathbf{b}}_1 + M_{02}\;\hat{\mathbf{b}}_2 && (4.34)\\
\nabla\times\hat{\mathbf{b}}_1 &= M_{10}\;\hat{\mathbf{b}}_0 - \left(M_{00}+M_{22}\right)\hat{\mathbf{b}}_1 + M_{12}\;\hat{\mathbf{b}}_2 && (4.35)\\
\nabla\times\hat{\mathbf{b}}_2 &= M_{20}\;\hat{\mathbf{b}}_0 + M_{21}\hat{\mathbf{b}}_1 - \left(M_{00}+M_{11}\right)\hat{\mathbf{b}}_2 && (4.36)
\end{aligned}$$

where $M_{ij} = \hat{\mathbf{b}}_i\cdot M\cdot\hat{\mathbf{b}}_j$ the coefficients of the bi-vector M in the fixed basis.

Note that some of the curvature-torsion coefficients may be equal to zero accordingly to the choice of the vectors $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$. For example, in our case the coefficient $M_{01} = -\hat{\mathbf{b}}_0\cdot\nabla\hat{\mathbf{b}}_0\cdot\hat{\mathbf{b}}_2 = k\;\hat{\mathbf{b}}_1\cdot\hat{\mathbf{b}}_2 = 0$

Then we obtain:

$$\nabla\times\hat{\mathbf{b}}_0 = \tau\;\hat{\mathbf{b}}_0 + k\;\hat{\mathbf{b}}_2 \tag{4.37}$$

with the torsion coefficient $\tau \equiv \hat{\mathbf{b}}_0\cdot\nabla\times\hat{\mathbf{b}}_0 = -M_{11}-M_{22} = -\hat{\mathbf{b}}_1\cdot\nabla\hat{\mathbf{b}}_2\cdot\hat{\mathbf{b}}_0 - \hat{\mathbf{b}}_2\cdot\nabla\hat{\mathbf{b}}_0\cdot\hat{\mathbf{b}}_1$ and the curvature coefficient, that we have defined before as the norm of the vector $\hat{\mathbf{b}}_1$: $|\hat{\mathbf{b}}_0\cdot\nabla\hat{\mathbf{b}}_0| = \hat{\mathbf{b}}_0\cdot\nabla\hat{\mathbf{b}}_0\cdot\hat{\mathbf{b}}_1 = \hat{\mathbf{b}}_2\cdot\nabla\times\hat{\mathbf{b}}_0 = M_{02}$.

[5]Note that the bi-vector M explicitly depends on the gyrogauge vector $\mathbf{R}\equiv\nabla\hat{\mathbf{b}}_1\cdot\hat{\mathbf{b}}_2$

Phase space variables derivatives

To follow the calculation of the Poisson bracket in the new variables we have first find the coefficients of the Jacobian matrix that corresponds to our change of variables

$$\mathcal{J} = \frac{\partial\left(\mathbf{x}, p, \varphi, \zeta\right)}{\partial\left(\mathbf{r}, \mathbf{p}\right)}:$$

$$\begin{aligned}
&\frac{\partial \mathbf{x}}{\partial \mathbf{r}} = 1 && \frac{\partial \mathbf{x}}{\partial \mathbf{p}} = 0 \\
&\frac{\partial p}{\partial \mathbf{r}} = 0 && \frac{\partial p}{\partial \mathbf{p}} = \hat{\mathbf{p}}_0 \\
&\frac{\partial \varphi}{\partial \mathbf{r}} = -M\cdot\hat{\mathbf{p}}_2 && \frac{\partial \varphi}{\partial \mathbf{p}} = \frac{\hat{\mathbf{p}}_1}{p} \\
&\frac{\partial \zeta}{\partial \mathbf{r}} = -\frac{M\cdot\hat{\mathbf{p}}_1}{\sin\varphi} && \frac{\partial \zeta}{\partial \mathbf{p}} = -\frac{\hat{\mathbf{p}}_2}{p\sin\varphi}
\end{aligned} \tag{4.38}$$

where we have used the generalized Frenet-Serret equations.

Then by using (4.38) and (4.36) we can obtain the old brackets (4.6) between the new phase space variables: $\{z_i, z_j\}_{old}$:

$$\begin{aligned}
\{p, \mathbf{x}\} &= \hat{\mathbf{p}}_0, \quad \{\varphi, \mathbf{x}\} = \frac{\hat{\mathbf{p}}_1}{p}, \quad \{\zeta, \mathbf{x}\} = -\frac{\hat{\mathbf{p}}_2}{p\sin\varphi} \\
\{p, \varphi\} &= -\hat{\mathbf{p}}_0\cdot M\cdot\hat{\mathbf{p}}_2 = -\hat{\mathbf{p}}_2\cdot\nabla\times\hat{\mathbf{p}}_0 \\
\{p, \zeta\} &= -\frac{1}{\sin\varphi}\left(\hat{\mathbf{p}}_0\cdot M + \frac{e\mathbf{B}}{p}\right)\cdot\hat{\mathbf{p}}_1 = -\frac{1}{\sin\varphi}\left(\hat{\mathbf{p}}_1\cdot\nabla\times\hat{\mathbf{p}}_0 + \frac{e\mathbf{B}}{p}\cdot\hat{\mathbf{p}}_1\right) \\
\{\zeta, \varphi\} &= \frac{1}{p\sin\phi}\left(\hat{\mathbf{p}}_1\cdot M\cdot\hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2\cdot M\cdot\hat{\mathbf{p}}_2 - \frac{e\mathbf{B}}{p}\cdot\hat{\mathbf{p}}_0\right) = \\
&\quad \frac{1}{p\sin\phi}\left(-\hat{\mathbf{p}}_0\cdot\nabla\times\hat{\mathbf{p}}_0 - \frac{e\mathbf{B}}{p}\cdot\hat{\mathbf{p}}_0\right)
\end{aligned} \tag{4.39}$$

Here we should consider that the coefficients of the bi-vector M in the dynamical basis $\hat{\mathbf{p}}_i\cdot M\cdot\hat{\mathbf{p}}_j$ are the functions of the new phase space variables, then the momentum variables (p, φ, ζ) are considered to be independent of the spatial variable $\mathbf{x}$. Accordingly to the definitions (4.23) and the expression for the curvature-torsion for the fixed basis vectors (4.33) the curvature-torsion for the dynamical basis vectors is[6]

$$\nabla\times\hat{\mathbf{p}}_\alpha = \hat{\mathbf{p}}_\alpha\cdot\left(M - \mathbf{I} : M\right) \tag{4.40}$$

[6]Note that the coefficients of the bi-vector M in the dynamical basis $(\hat{\mathbf{p}}_0, \hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2)$ and in the fixed basis $(\hat{\mathbf{b}}_0, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2)$ are related by rotation transformation $\widetilde{M} = \mathcal{U}^T M\mathcal{U}$

where the unit tensor is now represented in dynamical basis $\mathbf{I} = \hat{\mathbf{p}}_0\hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1\hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2\hat{\mathbf{p}}_2$

Then for the coefficients $\hat{\mathbf{p}}_i \cdot M \cdot \hat{\mathbf{p}}_j$ we have:

$$\begin{aligned}
\hat{\mathbf{p}}_0 \cdot M \cdot \hat{\mathbf{p}}_1 = &\quad \hat{\mathbf{p}}_1 \cdot \nabla \times \hat{\mathbf{p}}_0 \\
\hat{\mathbf{p}}_0 \cdot M \cdot \hat{\mathbf{p}}_2 = &\quad \hat{\mathbf{p}}_2 \cdot \nabla \times \hat{\mathbf{p}}_0 \\
\hat{\mathbf{p}}_1 \cdot M \cdot \hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2 \cdot M \cdot \hat{\mathbf{p}}_2 = & \quad -\hat{\mathbf{p}}_0 \cdot \nabla \times \hat{\mathbf{p}}_0
\end{aligned} \tag{4.41}$$

Then the Poisson bracket in new variables is given by

$$\begin{aligned}
\{f, g\} \quad = \quad & \hat{\mathbf{p}}_0 \left(\frac{\partial f}{\partial p} \cdot \frac{\partial g}{\partial \mathbf{x}} - \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\partial g}{\partial p} \right) + \frac{\hat{\mathbf{p}}_1}{p} \left(\frac{\partial f}{\partial \varphi} \cdot \frac{\partial g}{\partial \mathbf{x}} - \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\partial g}{\partial \varphi} \right) + \\
& \left(-\frac{\hat{\mathbf{p}}_2}{p \sin\varphi} \right) \left(\frac{\partial f}{\partial \zeta} \cdot \frac{\partial g}{\partial \mathbf{x}} - \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\partial g}{\partial \zeta} \right) + \\
& \frac{1}{\sin\varphi} \left(-\hat{\mathbf{p}}_1 \cdot \nabla \times \hat{\mathbf{p}}_0 - \frac{e}{p} \mathbf{B} \cdot \hat{\mathbf{p}}_1 \right) \left(\frac{\partial f}{\partial p} \cdot \frac{\partial g}{\partial \zeta} - \frac{\partial f}{\partial \zeta} \cdot \frac{\partial g}{\partial p} \right) + \\
& \left(-\hat{\mathbf{p}}_2 \cdot \nabla \times \hat{\mathbf{p}}_0 \right) \left(\frac{\partial f}{\partial p} \cdot \frac{\partial g}{\partial \varphi} - \frac{\partial f}{\partial \varphi} \cdot \frac{\partial g}{\partial p} \right) + \\
& \frac{1}{p \sin\varphi} \left(-\hat{\mathbf{p}}_0 \cdot \nabla \times \hat{\mathbf{p}}_0 - \frac{e}{p} \mathbf{B} \cdot \hat{\mathbf{p}}_0 \right) \left(\frac{\partial f}{\partial \zeta} \cdot \frac{\partial g}{\partial \varphi} - \frac{\partial f}{\partial \zeta} \cdot \frac{\partial g}{\partial \varphi} \right)
\end{aligned} \tag{4.42}$$

By implementing the momentum gradient, defined by using the chain rule and the expressions for phase space variables derivatives (4.38):

$$\frac{\partial}{\partial \mathbf{p}} = \frac{\partial p}{\partial \mathbf{p}} \frac{\partial}{\partial p} + \frac{\partial \varphi}{\partial \mathbf{p}} \frac{\partial}{\partial \varphi} - \frac{\partial \zeta}{\partial \mathbf{p}} \frac{\partial}{\partial \zeta} = \hat{\mathbf{p}}_0 \frac{\partial}{\partial p} + \frac{\hat{\mathbf{p}}_1}{p} \frac{\partial}{\partial \varphi} - \frac{\hat{\mathbf{p}}_2}{p \sin\varphi} \frac{\partial}{\partial \zeta} \tag{4.43}$$

We realize that the 3 first terms in the expression (4.42) represent the canonical part of the local Poisson bracket in the new variables, while the 3 last terms give the non-canonical part:

$$e\, \mathbf{B}_* \cdot \left(\frac{\partial f}{\partial \mathbf{p}} \times \frac{\partial g}{\partial \mathbf{p}} \right) = e\, \mathbf{B} \cdot \left(\frac{\partial f}{\partial \mathbf{p}} \times \frac{\partial g}{\partial \mathbf{p}} \right) + p\, \nabla \times \hat{\mathbf{p}}_0 \cdot \left(\frac{\partial f}{\partial \mathbf{p}} \times \frac{\partial g}{\partial \mathbf{p}} \right) \tag{4.44}$$

where

$$\mathbf{B}^* = \mathbf{B} + (p/e) \nabla \times \hat{\mathbf{p}}_0 \tag{4.45}$$

In fact, with

$$\begin{aligned}
p \nabla \times \hat{\mathbf{p}}_0 \cdot \left(\frac{\partial f}{\partial \mathbf{p}} \times \frac{\partial g}{\partial \mathbf{p}} \right) \quad = \quad & \frac{\hat{\mathbf{p}}_1 \cdot \nabla \times \hat{\mathbf{p}}_0}{\sin\varphi} \left(\frac{\partial f}{\partial p} \cdot \frac{\partial g}{\partial \zeta} - \frac{\partial f}{\partial \zeta} \cdot \frac{\partial g}{\partial p} \right) + \\
& \hat{\mathbf{p}}_2 \cdot \nabla \times \hat{\mathbf{p}}_0 \left(\frac{\partial f}{\partial p} \cdot \frac{\partial g}{\partial \varphi} - \frac{\partial f}{\partial \varphi} \cdot \frac{\partial g}{\partial p} \right) + \\
& \frac{\hat{\mathbf{p}}_0 \cdot \nabla \times \hat{\mathbf{p}}_0}{p \sin\varphi} \left(\frac{\partial f}{\partial \zeta} \cdot \frac{\partial g}{\partial \varphi} - \frac{\partial f}{\partial \zeta} \cdot \frac{\partial g}{\partial \varphi} \right)
\end{aligned} \tag{4.46}$$

and

$$e\mathbf{B}\cdot\frac{\partial f}{\partial \mathbf{p}}\times\frac{\partial g}{\partial \mathbf{p}} \;=\; \frac{e}{p\,\sin\varphi}\Big(\mathbf{B}\cdot\hat{\mathbf{p}}_1\Big)\left(\frac{\partial f}{\partial p}\cdot\frac{\partial g}{\partial \zeta}-\frac{\partial f}{\partial \zeta}\cdot\frac{\partial g}{\partial p}\right)$$
$$+\;\frac{e}{p^2\,\sin\varphi}\Big(\mathbf{B}\cdot\hat{\mathbf{p}}_0\Big)\left(\frac{\partial f}{\partial \zeta}\cdot\frac{\partial g}{\partial \varphi}-\frac{\partial f}{\partial \zeta}\cdot\frac{\partial g}{\partial \varphi}\right) \tag{4.47}$$

we obtain the expression for the noncanonical part of the Poisson bracket expressed in the new variables (4.44).

Finally, in the new phase space variables $(\mathbf{x}; p, \varphi, \zeta)$ the local Poisson bracket (4.6) has the following expression:

$$\{f,g\}=\frac{\partial f}{\partial \mathbf{p}}\cdot\frac{\partial g}{\partial \mathbf{x}}-\frac{\partial f}{\partial \mathbf{p}}\cdot\frac{\partial g}{\partial \mathbf{x}}-e\mathbf{B}^*\cdot\frac{\partial f}{\partial \mathbf{p}}\times\frac{\partial g}{\partial \mathbf{p}} \tag{4.48}$$

where the modified magnetic field is given by (4.45) and the momentum gradient is defined in (4.43).

4.3.3 Local equations of motion

Using the expression (4.48) for Poisson bracket in the new local phase space variables, and the physical Hamiltonian H (4.5) we obtain the corresponding equations of motion.

$$\dot{\mathbf{x}} \;=\; \{H,\mathbf{x}\}=\frac{p}{m\gamma}\,\hat{\mathbf{p}}_0 \tag{4.49}$$
$$\dot{p} \;=\; \{H,p\}=0 \tag{4.50}$$
$$\dot{\varphi} \;=\; \{H,\varphi\}=-\frac{p}{m\gamma}\hat{\mathbf{p}}_2\cdot(\nabla\times\hat{\mathbf{p}}_0)=-\frac{p}{m\gamma}\,\Big((\hat{\mathbf{p}}_0\cdot\nabla\hat{\mathbf{p}}_0)\cdot\hat{\mathbf{p}}_1\Big) \tag{4.51}$$
$$\dot{\zeta} \;=\; \{H,\zeta\}=\frac{e\,B_0}{m\gamma}-\frac{p}{m\gamma\sin\varphi}\,\Big(\hat{\mathbf{p}}_1\cdot(\nabla\times\hat{\mathbf{p}}_0)\Big)$$
$$=\;\frac{e\,B_0}{m\gamma}+\frac{p}{m\gamma\sin\varphi}\Big((\hat{\mathbf{p}}_0\cdot\nabla\hat{\mathbf{p}}_0)\cdot\hat{\mathbf{p}}_2\Big) \tag{4.52}$$

where we have used that $\hat{\mathbf{p}}_2=\hat{\mathbf{p}}_0\times\hat{\mathbf{p}}_1$ and $\nabla\hat{\mathbf{p}}_0\cdot\hat{\mathbf{p}}_0=0$.

First of all we realize that the introduction of the new phase space variables $(\mathbf{x}, p, \varphi, \zeta)$ is well suitable for the dynamical description of the considered system because it reveals its underlying separation of scales of motion. We emphasize that here $\mathbf{x}$ denotes an exact particle position and not the guiding-center position, as was used previously in [2] and [44]. Then we can conclude that the introduction of the guiding center (reduced particle position) is not obligatory for show the scales of motion separation.

To represent the system in a more suitable way for numerical simulations, we rewrite the system (4.52) in the dimensionless variables by making appear the small parameter:

$$\varepsilon\equiv\rho_L/L_B\ll 1 \tag{4.53}$$

that represents the ratio between the particle Larmor radius (named also "gyroradius") $\rho_L \equiv p/eB_0$ and the nonuniformity length scale of the magnetic field L_B[7]. For fusion plasma this ratio satisfies the condition (4.53) [8]. Then the dimensionless equations of motion are [9]:

$$\begin{aligned}
\dot{\mathbf{x}} &= \varepsilon\, \hat{\mathbf{p}}_0 \\
\dot{p} &= 0 \\
\dot{\varphi} &= -\varepsilon\, \hat{\mathbf{p}}_2 \cdot (\nabla \times \hat{\mathbf{p}}_0) = -\varepsilon\, \left((\hat{\mathbf{p}}_0 \cdot \nabla \hat{\mathbf{p}}_0) \cdot \hat{\mathbf{p}}_1\right) \\
\dot{\zeta} &= 1 - \frac{\varepsilon}{\sin\varphi}\, \hat{\mathbf{p}}_1 \cdot (\nabla \times \hat{\mathbf{p}}_0) = 1 + \frac{\varepsilon}{\sin\varphi}\, \left((\hat{\mathbf{p}}_0 \cdot \nabla \hat{\mathbf{p}}_0) \cdot \hat{\mathbf{p}}_2\right)
\end{aligned} \tag{4.54}$$

where the spatial coordinate is now dimensionless $\mathbf{x} \to \mathbf{x}/L_B$. Now we can easily remark that the gyrophase angle ζ is the fast variable of our system and the others variables of the local particle phases space correspond to its slow dynamics.

The second remark that we have to make here is about the gyrogauge dependence of these non-reduced equations of motion projected on the natural magnetic field basis.

Such a dependence is essentially geometrical in origin. It was shown in [3] when applying the Lie-transform perturbation method, that it is possible to remove the gyrogauge vector $\mathbf{R}$ dependence from the averaged guiding-center variables $(\bar{\mathbf{X}}, \bar{p_{||}}, \bar{\mu})$. The exception is the gyroangle ζ that still depends on gyrogauge because its modification would bring back rapid oscillation into the reduced system.

This procedure is explicitly shown in [45]. There, the method of bringing the parallel dynamics gyrogauge invariant is based on the fact that the curl of the gyrogauge vector $\nabla \times \mathbf{R}$ is invariant:

$$\nabla \times \mathbf{R}' = \nabla \times (\mathbf{R} + \nabla\psi) = \nabla \times \mathbf{R} \tag{4.55}$$

By consequence the curl of the modified gyrogauge vector $\mathbf{R}^* = \mathbf{R} + \tau/2\hat{\mathbf{b}}$, with $\tau = \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}$, is also gyrogauge invariant, because of the correction $\tau/2\hat{\mathbf{b}}$ is independent of the derivatives of the fixed basis. Such an invariant quantity appears in the corresponding reduced equations of motion for parallel dynamics through its dependence on the modified magnetic field $\mathbf{B}^*$ that contains the correction $\nabla \times \mathbf{R}^*$ at the second order of perturbative expansion. Then its guarantee the gyrogauge

[7]For example in numerical simulations we take the nonuniformity length scale $L_B = 1m$ that is comparable with the small tokamak radius in the case of the tokamak Tore Supra

[8]Note that here, oppositely to the Lie-transform perturbation analysis, presented in [45] we deal with *macroscopic* limit, when the macroscopic length L_B is finite and the Larmor radius $\rho_L \ll L_B$ is small. The *microscopic* limit consists to interpret the inequality (4.53) inversely, by considering that the Larmor radius ρ_L is finite and the macroscopic length is large length $L_B \gg \rho_L$ is large. In the microscopic limit $\epsilon \sim e$ while in the macroscopic limit $\epsilon \sim 1/L_B$

[9]Note that here it is more convenient to use the physical Hamiltonian H rather then the rescaled Hamiltonian p to derive the equations of motion because it permits us to highlight physical small parameter ε

invariance of the parallel reduced dynamics. As was announced in [3] the only variable that still has gyrogauge dependent dynamics is the gyroangle, which contains the vector $\mathbf{R}^*$ explicitly.

If now we reintroduce the vectors $\hat{\boldsymbol{\rho}} = \hat{\mathbf{b}}_1 \cos\zeta - \hat{\mathbf{b}}_2 \sin\zeta$ and $\hat{\perp} = -\hat{\mathbf{b}}_1 \sin\zeta - \hat{\mathbf{b}}_2 \cos\zeta$ inside (4.55), we find:

$$\hat{\mathbf{p}}_0 = \hat{\mathbf{b}}_0 \cos\varphi + \hat{\perp}\ \sin\varphi, \quad \hat{\mathbf{p}}_1 = -\hat{\mathbf{b}}_0 \sin\varphi + \hat{\perp}\ \cos\varphi, \quad \hat{\mathbf{p}}_2 = \hat{\boldsymbol{\rho}} \tag{4.56}$$

Then the gyrogauge vector $\mathbf{R} = \nabla\hat{\perp} \cdot \hat{\boldsymbol{\rho}} = \nabla\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{b}}_2$ appears inside the equation of motion for gyrophase:

$$\dot{\varphi} = -\varepsilon\, \hat{\mathbf{p}}_0 \cdot \left(\nabla\hat{\perp} \cdot \hat{\mathbf{b}}_0\right) \tag{4.57}$$

$$\dot{\zeta} = 1 + \frac{\varepsilon}{\sin\varphi}\left(\hat{\mathbf{p}}_0 \cdot \nabla\hat{\mathbf{b}}_0 \cdot \hat{\boldsymbol{\rho}}\ \cos\varphi + \hat{\mathbf{p}}_0 \cdot \mathbf{R}\ \sin\varphi\right) \tag{4.58}$$

It means that the rotation of the fixed vector basis $\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2$ is involved inside the non-reduced dynamical equations. Similarly to the Lie-transform method, here the parallel averaged and even non-reduced dynamics $(\mathbf{x}, \varphi, p)$ still be gyrogauge invariant. The gyrophase dynamics still naturally be dependent on the gyrogauge because of the gyrophase itself is not gyrogauge invariant(4.19).

This result, one more time emphasizes, that the gyrogauge dependence is not a consequence of the dynamical reduction but of the choice of local basis on which the dynamics is projected.

4.3.4 Iterative construction of the constant of motion

Now we have all the elements to start the iterative procedure for construction of the constant of motion for our system.

First of all, the constant of motion, which we denote $\mathcal{A}$ here, has to satisfy the condition $\dot{\mathcal{A}} \equiv \{\mathcal{A}, H\} = 0$, which can be made explicit as:

$$(\partial_\zeta\, \mathcal{A})\ \dot{\zeta} = -\left(\partial_{\mathbf{x}}\, \mathcal{A}\right)\ \dot{\mathbf{x}} - \left(\partial_\varphi\, \mathcal{A}\right)\dot{\varphi} \tag{4.59}$$

The separation of scales of motion permits us to obtain an iterative procedure for solution of this equation with following series decomposition:

$$\mathcal{A}_0 = \sum_{n=0}^{\infty} \langle \mathcal{A}_n \rangle(\mathbf{x}, p, \varphi)\ \varepsilon^n + \sum_{n=0}^{\infty} \widetilde{\mathcal{A}}_n(\mathbf{x}, p, \varphi, \zeta)\ \varepsilon^{n+1} \tag{4.60}$$

here $\langle \mathcal{A} \rangle$ denotes the gyroaveraged part of the function, and correspondingly $\widetilde{\mathcal{A}}$ denotes its gyroangle-dependent part. To start the iterative procedure, we suppose that at the lowest order the constant of motion is independent of the fast gyrophase variable ζ. Here are the three first steps of the iterative procedure:

$$\varepsilon^0 \quad : \ \partial_\zeta \mathcal{A} \quad = 0 \tag{4.61}$$

$$\varepsilon^1 \quad : \ \partial_\zeta \ \mathcal{A}_1 \quad = -\hat{\mathbf{p}}_0 \ (\partial_{\mathbf{x}} \ \mathcal{A}_0) + (\hat{\mathbf{p}}_0 \cdot \nabla \hat{\mathbf{p}}_0) \cdot \hat{\mathbf{p}}_1 \ (\partial_\varphi \ \mathcal{A}_0) \tag{4.62}$$

$$\varepsilon^2 \quad : \ \partial_\zeta \mathcal{A}_2 \quad = -\frac{1}{\sin\varphi} (\hat{\mathbf{p}}_0 \cdot \nabla \hat{\mathbf{p}}_0) \cdot \hat{\mathbf{p}}_2 \ (\partial_\zeta \ \mathcal{A}_1) \tag{4.63}$$

$$-\hat{\mathbf{p}}_0 \ (\partial_{\mathbf{x}} \ \mathcal{A}_1) + \ (\hat{\mathbf{p}}_0 \cdot \nabla \hat{\mathbf{p}}_0) \cdot \hat{\mathbf{p}}_1 \ (\partial_\varphi \ \mathcal{A}_1) \tag{4.64}$$

where we have used the dimensionless equations of motion (4.55).

Operators of gyroaveraging, gyrofluctuation

As follows from the system (4.64), at each step of the iterative procedure that leads to the construction of the constant of motion $\mathcal{A}$ we have to invert the operator ∂_ζ. In order to construct the corresponding inverse operator, that we will call $\mathcal{G}$, we first introduce the complementary operators of the gyroaveraging $\mathcal{R}$ and gyrofluctuation $\mathcal{N}$:

$$\mathcal{R} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\zeta, \ \ \mathcal{N} = \mathbf{I} - \mathcal{R} \tag{4.65}$$

Applying these operators, each observable $f = f(\mathbf{x}, p, \varphi, \zeta)$ can be decomposed into $f = \mathcal{R}\ f + \mathcal{N}\ f \equiv \langle f \rangle + \widetilde{f}$, such a decomposition is similar to the Fourier series decomposition, where the zero-harmonic is given by application of the operator $\mathcal{R}$ to the observable $f_0 = \mathcal{R}\ f$. In the following we indicate the action of the gyroaverage operator $\mathcal{R}$ with $\langle ... \rangle$.

Then the left-inverse operator $\mathcal{G}$ can be defined as follows: $\mathcal{G}\ \partial_\zeta = \mathcal{N}$ and $\mathcal{G}\ \mathcal{R} = 0$. In fact the operator $\mathcal{G}$ acts as an indefinite integral operator over the variable ζ on the observables that depend on ζ and vanish the observables that do not depend on gyroangle ζ. Then the operator $\mathcal{G}$ has a kernel composed by the gyroaveraged part of the observables. It can be represented as:

$$\mathcal{G} = \chi(n) \int^{\zeta} d\zeta \ \ \text{with} \ \ \chi(n) = \begin{cases} 0, \ n = 0 \\ 1, \ n \neq 0 \end{cases} \tag{4.66}$$

where n designs harmonics in ζ. For example

$$\mathcal{G}(1) = 0, \ \text{and} \ \mathcal{G}(\cos n\zeta) = \frac{\sin n\zeta}{n} \tag{4.67}$$

We apply the operators $\mathcal{R}$ and $\mathcal{G}$ at each step of the iterative procedure. At each stage the application of the operator $\mathcal{R}$ permits us to obtain the gyroaveraged part of the coefficient $\langle \mathcal{A}_{i-1} \rangle$ and the application of the operator $\mathcal{G}$ leads to the expression for the fluctuating part of the next order coefficient $\widetilde{\mathcal{A}}_i$.

For example at the first order of the intrinsic procedure we have:

$$\mathcal{R} \ : \ 0 = -\partial_{\mathbf{x}}\, \mathcal{A}_0 \langle\, \hat{\mathbf{p}}_0 \rangle + \partial_\varphi\, \mathcal{A}_0 \Big\langle (\hat{\mathbf{p}}_0 \cdot \nabla \hat{\mathbf{p}}_0) \cdot \hat{\mathbf{p}}_1 \Big\rangle \tag{4.68}$$

$$\mathcal{G} \ : \ \widetilde{\mathcal{A}}_1 \equiv \mathcal{N}\, \mathcal{A}_2 = -\partial_{\mathbf{x}}\, \mathcal{A}_0\, \mathcal{G}\,(\hat{\mathbf{p}}_0) + \partial_\varphi\, \mathcal{A}_0\, \mathcal{G}\,((\hat{\mathbf{p}}_0 \cdot \nabla \hat{\mathbf{p}}_0) \cdot \hat{\mathbf{p}}_2) \tag{4.69}$$

The first equation yields the differential equation for gyroaveraged part of the zeroth-order coefficient $\langle \mathcal{A}_1 \rangle$. Then the second equation gives the fluctuating part of the first-order coefficient $\widetilde{\mathcal{A}}_1$.

By applying the operator of the gyroaverage to the second order equation (4.64), we will find the first order partial differential equation for the gyroaveraged part of the coefficient $\mathcal{A}_1$:

$$\begin{aligned} &\Big\langle \partial_{\mathbf{x}} \langle \mathcal{A}_1 \rangle\, \hat{\mathbf{p}}_0 \Big\rangle + \Big\langle \partial_\varphi \langle \mathcal{A}_1 \rangle\, \big((\hat{\mathbf{p}}_0 \cdot \nabla \hat{\mathbf{p}}_0) \cdot \hat{\mathbf{p}}_1\big) \Big\rangle = \\ &\Big\langle \partial_{\mathbf{x}} \widetilde{\mathcal{A}}_1\, \hat{\mathbf{p}}_0 \Big\rangle + \Big\langle \partial_\varphi\, \widetilde{\mathcal{A}}_1\, \big((\hat{\mathbf{p}}_0 \cdot \nabla \hat{\mathbf{p}}_0) \cdot \hat{\mathbf{p}}_1\big) \Big\rangle - \Big\langle \partial_\zeta \widetilde{\mathcal{A}}_1 \left(\frac{1}{\sin\varphi} (\hat{\mathbf{p}}_0 \cdot \nabla \hat{\mathbf{p}}_0) \cdot \hat{\mathbf{p}}_2 \right) \Big\rangle \end{aligned} \tag{4.70}$$

Note that this equation will occur at all the following stages of the iterative procedure, such that at the stage number $i+1$ it will permit us to obtain the gyroaveraged part of the coefficient $\mathcal{A}_i$. The same procedure will be implemented in the following section when constructing the Hamiltonian normal form in intrinsic basis.

Zeroth order constant of motion $\mathcal{A}_0$

Here we deal with the solution of the first order partial differential equation (4.68) that leads to the first order of the constant of motion $\mathcal{A}_0$. After explicit evaluation of the gyroaverage $\langle \hat{\mathbf{p}}_0 \rangle = \hat{\mathbf{b}}_0 \ \cos\varphi$ and $\langle \hat{\mathbf{p}}_2 \cdot (\nabla \times \hat{\mathbf{p}}_0) \rangle = 1/2 \ \sin\varphi\ (\nabla \cdot \hat{\mathbf{b}}_0)$ this equation becomes:

$$\frac{1}{2} \tan\varphi\, \partial_\varphi\, \mathcal{A}_0 = -\frac{\hat{\mathbf{b}}_0 \cdot \nabla \mathcal{A}_0}{\hat{\mathbf{b}}_0 \cdot \nabla B} \tag{4.71}$$

here we have also used the condition $\nabla \cdot \hat{\mathbf{b}}_0 = -\hat{\mathbf{b}}_0 \cdot \nabla B$ resulting from the electromagnetic constraint : $\nabla \cdot \mathbf{B} = 0$. Then we apply the method of separation of variables, by supposing that $\mathcal{A}_0(\mathbf{x}, \varphi) = g(\mathbf{x})\ h(\varphi)$ we have

$$\frac{1}{2h} \frac{dh}{d\varphi}\ \tan\varphi = -\frac{B}{\hat{\mathbf{b}}_0 \cdot \nabla B} \cdot \frac{\hat{\mathbf{b}}_0 \cdot \nabla g}{g} = C \tag{4.72}$$

where C is a constant. By integrating each equation separately we obtain

$$h(\varphi) = \left(\frac{\sin\varphi}{\sin\varphi_0} \right)^{2C}, \quad g(\mathbf{x}) = B(\mathbf{x})^{-C} \tag{4.73}$$

Then following the physical intuition that led to the expression for the adiabatic invariant μ_0 (4.1), we set the constant $C = 1$. Then by choosing the constant of integration equal to the modulus of the kinetic momentum p, we have:

$$\mathcal{A}_0 = \frac{p^2 \sin^2 \varphi}{B} = 2m\, \mu_0 \tag{4.74}$$

Here we do not proceed with the evaluation of the higher order corrections for the constant of motion $\mathcal{A}$. The explicit calculation up to the second order is realized in Section 4.5 when applying the intrinsic formalism for Hamiltonian normal form construction. In what follows we explore some possible applications of the local particle dynamics.

4.4 Investigation of trapped particles trajectories.

In this section we present one of the possible applications for the derivation of particle dynamics in local phase-space variables $(\mathbf{x}, p, \varphi, \zeta)$. In particular we deal with investigation of trapped particles trajectories in the axisymmetric magnetic field geometry. Such an investigation is of interest for understanding fast ions ($v \geq v_{th}$) confinement.

In previous studies [63, 64] the constant-of-motion (COM) 3 dimensional phase space was used. The invariance of its variables: the kinetic energy $E = mv^2/2$, zeroth-order magnetic moment $\mu_0 = mv_\perp/2B$ and toroidal canonical momentum $P_\phi = (e/c)\psi - mRv_{||}B_\phi/B$ was supposed. It was also mentioned that such an approximation is suitable only for low $\beta \leq 10\%$ [10], while for high β a correct description can be obtained using guiding-center equations integration.

Here the exact dynamical equations will be integrated. In the same time the variation of the adiabatic invariant μ_0 will be presented for different values of the small parameter $\varepsilon = p/eB$. It will give a possibility to make an estimation of error that can be produced when using the adiabatic invariant as one of the phase space variables.

[10] The quantity β is the normalized plasma pressure defined as follows

$$\beta \equiv \frac{p}{B^2/\boldsymbol{\mu}_0},$$

where $\boldsymbol{\mu}_0 = 4\pi * 10^{-7} Hm^{-1}$ denotes the permittivity of free space. The β represents a ratio of the plasma pressure to magnetic pressure. It is a measure of the efficiency with which the magnetic field confines the plasma. The high β is of interest for an economic power balance in the reactor, but difficult to achieve because of plasmas instabilities. A combination of engineering and nuclear physics constraints has shown that a fusion plasma must achieve a temperature $T \sim 15$ keV, a pressure $p \sim 7$ atm, a plasma $\beta \sim 8\%$ and an energy confinement time $\tau_E \sim 1$ s.

This work is organized as follows. The subsection 4.4.1 is devoted to derivation of the exact dynamical equations in the general axisymmetric magnetic field geometry. In particular, the equations of motion for bi-cylindrical coordinate case will be explicitly obtained in H.1. its numerical integration will be realized by using the package Mathematica. Then in the subsection 4.4.2 particle trajectories analysis will be exposed.

4.4.1 Dynamics in axisymmetric magnetic field

Magnetic surfaces

To describe a magnetic field configuration it is convenient to use coordinates defined by the field itself. The definition of a magnetic configuration corresponding to plasma confinement device is closely related to existence of the magnetic surfaces.

"A two dimensional surface defined by a function $f(\mathbf{x}) = const$ is said to be magnetic surface if at any point the magnetic field lies within the surface, i.e. $\mathbf{B} \cdot \nabla f = 0''$

R.B. White "The Theory of Toroidally Confined Plasmas"

For example in the magnetohydrodynamical approach (MHD) fusion plasma can be considered as magnetized fluid characterized by its kinetic pressure p and current density $\mathbf{j}$. Then the plasma equilibrium is defined by the condition $\mathbf{j} \times \mathbf{B} = \nabla \mathsf{p}$ its implies that the magnetic surfaces are the isobars $\mathbf{B} \cdot \nabla \mathsf{p} = 0$.

A magnetic field, possessing an axial symmetry, suitable for a tokamak, represents one of the 3 possible types of plasma equilibria for which the magnetic surfaces are globally known. Intuitively the existence of closed magnetic surfaces should be one of the conditions for a good plasma confinement. It is well known that in this case their topology consists of nested tubes (tori) of flux. Then it is natural to associate to them a system of general curvilinear coordinates (ψ, θ, ϕ). Where the condition ψ=const defines one of the magnetic surfaces, θ=const corresponds to the general poloidal angle and ϕ=const introduce generalized toroidal direction. Consequently, it is more natural to use the contravariant representation for basis vectors$(\nabla\phi, \nabla\theta, \nabla\phi)$ that are defined as a normal vectors to the corresponding surfaces.

General axisymmetric coordinates

Coordinate definition Here we deal with a general axisymmetric coordinates construction. We start by considering the cylindrical coordinates (R, ϕ, Z), where the radial coordinate R measure the distance from the general axis to the center of a tokamak, the angle θ represents toroidal angle and the coordinate Z permits us to complete the definition of the position in a poloidal machine section. In the second step we pass from the coordinates (R, Z) that define the position in the poloidal

./tore.ps

Figure 4.2: General axisymmetric geometry

plane of a tokamak to the coordinates associated with magnetic surfaces (ψ, θ). We suppose that there exist the functions $R = R(\psi, \theta), Z = Z(\psi, \theta)$ that are invertible $\psi = \psi(R, Z), \theta = \theta(R, Z)$. Note that the toroidal direction $\nabla\phi$ is not affected by this change of variables.

The total transformation can be expressed in Cartesian coordinates as

$$\mathbf{x} = R(\psi, \theta)\ \sin\phi\ \hat{\mathbf{x}} + R(\psi, \theta)\ \cos\phi\ \hat{\mathbf{y}} + Z(\psi, \theta)\ \hat{\mathbf{z}} \tag{4.75}$$

In order to be sure that such a transformation is well defined, we need to suppose that the Jacobian of the direct transformation $\mathcal{J}$ cannot be infinite, or that the Jacobian $\mathcal{J}^{-1}$ of the inverse transformation can not be equal to zero. Because our transformation consist of two stages, we can write

$$\frac{1}{\mathcal{J}} = \det \frac{\partial(\psi, \theta, \phi)}{\partial(x, y, z)} = \det \frac{\partial(\psi, \theta, \phi)}{\partial(R, \theta, Z)} \det \frac{\partial(R, \theta, Z)}{\partial(x, y, z)} \equiv \frac{1}{\mathcal{J}_a} \frac{1}{\mathcal{J}_c} \tag{4.76}$$

It is well known that the Jacobian of the transformation from cylindrical to Cartesian coordinates $\mathcal{J}_c^{-1}$ is different from zero, then to be sure that the total transformation from general to Cartesian coordinates is well defined, it is sufficient to consider the Jacobian of the second part of the transformation:

$$\frac{1}{\mathcal{J}_a} = \det \frac{\partial(\psi, \theta, \phi)}{\partial(R, Z, \phi)} = \det \begin{pmatrix} \partial_R\psi & \partial_Z\psi & 0 \\ \partial_R\theta & \partial_Z\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = |\nabla\psi \times \nabla\theta| = |\nabla\psi||\nabla\theta| \tag{4.77}$$

Then we need to suppose that the product of the vector norms $|\nabla\psi||\nabla\theta| > 0$.

The second supposition on transformation $(R, Z, \phi) \to (\psi, \theta, \phi)$ is that we make here is the metric tensor is diagonal, with:

$$
\begin{aligned}
|\nabla\psi| &= \sqrt{g^{\psi\psi}} = \frac{1}{\sqrt{g_{\psi\psi}}} = \frac{1}{\sqrt{(\partial_\psi R)^2 + (\partial_\psi Z)^2}} && (4.78)\\
|\nabla\theta| &= \sqrt{g^{\theta\theta}} = \frac{1}{\sqrt{g_{\theta\theta}}} = \frac{1}{\sqrt{(\partial_\theta R)^2 + (\partial_\theta Z)^2}} && (4.79)\\
|\nabla\phi| &= \sqrt{g^{\phi\phi}} = \frac{1}{\sqrt{g_{\phi\phi}}} = \frac{1}{R} && (4.80)
\end{aligned}
$$

Note that accordingly to the (4.76) the Jacobian of the total transformation is equal to the product of the basis vectors norms $\dfrac{1}{\mathcal{J}} = \dfrac{1}{R}\,|\nabla\psi||\nabla\theta|$.

Due to the assumption of basis vector orthogonality $\nabla\psi \cdot \nabla\theta = 0$, the variable χ, that permits us to define the transition between the basis vectors in a form of a rotation, can be defined as follows[11]:

$$
\frac{\partial_\psi Z}{\partial_\theta Z} = -\frac{\partial_\theta R}{\partial_\psi R} \equiv \tan\chi \tag{4.81}
$$

Then the relation between the basis vectors $(\nabla R, \nabla Z)$ and $(\nabla\psi, \nabla\theta)$ can be written as follows:

$$
\begin{aligned}
\nabla R &= \cos\chi\widehat{\nabla\psi} - \sin\chi\widehat{\nabla\theta}\\
\nabla Z &= \sin\chi\widehat{\nabla\psi} + \cos\chi\widehat{\nabla\theta}
\end{aligned} \tag{4.82}
$$

and conversely

$$
\begin{aligned}
\widehat{\nabla\psi} &= \sin\chi\nabla Z + \cos\chi\nabla R\\
\widehat{\nabla\theta} &= \cos\chi\nabla Z - \sin\chi\nabla R
\end{aligned} \tag{4.83}
$$

where $\widehat{\nabla}\psi \equiv \dfrac{\nabla\psi}{|\nabla\psi|}$ and $\widehat{\nabla}\theta \equiv \dfrac{\nabla\theta}{|\nabla\theta|}$ define the unit vectors in generalized radial and poloidal directions.

To obtain the relations (4.82), first we have to express the basis vectors $(\nabla R, \nabla Z)$ in new coordinate:

$$
\nabla R = \begin{pmatrix} \partial_\psi R \\ \partial_\theta R \end{pmatrix}, \nabla Z = \begin{pmatrix} \partial_\psi Z \\ \partial_\theta Z \end{pmatrix} \tag{4.84}
$$

[11] Note that in the case of bi-cylindrical coordinates when the magnetic surfaces are defined as the set of concentric circles $R = R_0 + \psi\cos\theta, Z = \psi\sin\theta$, the variable χ coincide with generalized poloidal coordinate θ

Then by using the normalized vectors $\widehat{\nabla\psi}, \widehat{\nabla\theta}$ defined in (4.80) we have:

$$\nabla R = \partial_\psi R\ \nabla\psi + \partial_\theta R\ \nabla\theta = \frac{\partial_\psi R}{\sqrt{g_{\psi\psi}}}\widehat{\nabla\psi} + \frac{\partial_\theta R}{\sqrt{g_{\theta\theta}}}\widehat{\nabla\theta}$$
$$\nabla Z = \partial_\psi Z\ \nabla\psi + \partial_\theta Z\ \nabla\theta = \frac{\partial_\psi Z}{\sqrt{g_{\psi\psi}}}\widehat{\nabla\psi} + \frac{\partial_\theta Z}{\sqrt{g_{\theta\theta}}}\widehat{\nabla\theta}$$

then with (4.81) and due to the trigonometry identities $\cos^2\chi = 1/(1+\tan\chi^2)$ and $\sin^2\chi = \tan^2\chi/(1+\tan\chi^2)$ we can define:

$$\cos\chi = \frac{\partial_\theta Z}{\sqrt{g_{\theta\theta}}} = \frac{\partial_\psi R}{\sqrt{g_{\psi\psi}}} \tag{4.85}$$
$$\sin\chi = -\frac{\partial_\theta R}{\sqrt{g_{\theta\theta}}} = \frac{\partial_\psi Z}{\sqrt{g_{\psi\psi}}} \tag{4.86}$$

Then finally we obtain the relation (4.82) between the basis vectors.

Note here that the expressions for derivatives $\partial_\psi\chi$ and $\partial_\theta\chi$ that will be useful for curvature tensor definition, follow from (4.85),(4.86) by using the symmetry of the second derivatives $\partial_\psi\partial_\theta R$ and $\partial_\psi\partial_\theta Z$.

$$\partial_\psi\chi = -\frac{\partial_\theta\sqrt{g_{\psi\psi}}}{\sqrt{g_{\theta\theta}}},\quad \partial_\theta\chi = \frac{\partial_\psi\sqrt{g_{\theta\theta}}}{\sqrt{g_{\psi\psi}}} \tag{4.87}$$

Curvature tensor In order to derive the equations of motion for axisymmetric magnetic field configuration with basis $y^i = (\psi,\theta,\phi)$, we need to know its curvature tensor $\nabla\nabla y^i$.

We start with calculation of the curvature tensor for cylindrical coordinates.

Cylindrical coordinates The transformation from Cartesian coordinates to cylindrical is given by

$$\mathbf{x}(R,\theta,Z) = R\ \sin\phi\ \hat{\mathbf{x}} + R\ \cos\phi\ \hat{\mathbf{y}} + Z\ \hat{\mathbf{z}} \tag{4.88}$$

Then the metric tensor is diagonal $g^{aa} = 1/g_{aa}$, with

$$g_{aa} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.89}$$

We express the contravariant vectors $(\nabla R, \nabla\phi, \nabla Z)$ as the functions of cylindrical coordinates:

$$\nabla R = \begin{pmatrix} \sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \nabla\phi = \frac{1}{R}\begin{pmatrix} -\cos\phi \\ \sin\phi \\ 0 \end{pmatrix}, \nabla Z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.90}$$

The gradient in cylindrical coordinates is given by:

$$\nabla = \nabla R \frac{\partial}{\partial R} + \nabla \phi \frac{\partial}{\partial \phi} + \nabla Z \frac{\partial}{\partial Z} \tag{4.91}$$

with $\frac{\partial}{\partial y^i} = g_{ii} \nabla y^i \cdot \nabla \equiv \sqrt{g_{ii}}\ \widehat{\nabla y^i} \cdot \nabla$.

Then the curvature tensor for cylindrical coordinates is:

$$\nabla \nabla R = R \nabla \phi \nabla \phi \equiv \frac{1}{R} \widehat{\nabla \phi} \widehat{\nabla \phi} \tag{4.92}$$

$$\nabla \widehat{\nabla \phi} = -\nabla \phi \nabla R = -\frac{1}{R} \widehat{\nabla \phi} \nabla R \tag{4.93}$$

$$\nabla \nabla Z = 0 \tag{4.94}$$

General axisymmetric coordinates The gradient in axisymmetric coordinates is given by:

$$\nabla = \nabla \psi \frac{\partial}{\partial \psi} + \nabla \theta \frac{\partial}{\partial \theta} + \nabla \phi \frac{\partial}{\partial \phi} \tag{4.95}$$

By applying operator ∇ to the equation (4.82) that gives the relation between axisymmetric basis vectors and cylindrical basis vectors , then by using the expressions (4.87) for derivatives of the variable χ, we obtain the curvature tensor for $\widehat{\nabla \psi}$, $\widehat{\nabla \theta}$ [12]:

$$\nabla \widehat{\nabla \psi} = -\frac{\partial_\theta \sqrt{g_{\psi\psi}}}{\sqrt{g_{\theta\theta}}} \nabla \psi \widehat{\nabla \theta} + \frac{\partial_\psi \sqrt{g_{\theta\theta}}}{\sqrt{g_{\psi\psi}}} \nabla \theta \widehat{\nabla \theta} + \frac{\partial_\psi \sqrt{g_{\phi\phi}}}{\sqrt{g_{\psi\psi}}} \nabla \phi \widehat{\nabla \phi} \tag{4.96}$$

$$\nabla \widehat{\nabla \theta} = \frac{\partial_\theta \sqrt{g_{\psi\psi}}}{\sqrt{g_{\theta\theta}}} \nabla \psi \widehat{\nabla \psi} - \frac{\partial_\psi \sqrt{g_{\theta\theta}}}{\sqrt{g_{\psi\psi}}} \nabla \theta \widehat{\nabla \psi} + \frac{\partial_\theta \sqrt{g_{\phi\phi}}}{\sqrt{g_{\theta\theta}}} \nabla \phi \widehat{\nabla \phi} \tag{4.97}$$

Finally for normalized toroidal vector $\widehat{\nabla \phi}$, the corresponding curvature tensor is obtained by transforming the equation (4.93) for its curvature tensor $\nabla \widehat{\nabla \phi}$ in cylindrical coordinates to axisymmetric coordinates using the relation (4.93).

$$\nabla \widehat{\nabla \phi} = -\frac{\partial_\theta \sqrt{g_{\phi\phi}}}{\sqrt{g_{\theta\theta}}} \nabla \phi \widehat{\nabla \theta} - \frac{\partial_\psi \sqrt{g_{\phi\phi}}}{\sqrt{g_{\psi\psi}}} \nabla \phi \widehat{\nabla \psi} \tag{4.98}$$

The expression for curvature tensor can be rewritten in more suitable form as:

$$\nabla \widehat{\nabla \psi} = \nabla \psi \frac{\partial}{\partial \psi} \widehat{\nabla \psi} + \nabla \theta \frac{\partial}{\partial \theta} \widehat{\nabla \psi} + \nabla \phi \frac{\partial}{\partial \phi} \widehat{\nabla \psi} \tag{4.99}$$

[12]To be coherent in notations here we write $\sqrt{g_{\phi\phi}}$ rather then R^2

where the coefficients for basis $(\nabla\psi, \nabla\theta, \nabla\phi)$ decomposition are

$$\frac{\partial}{\partial\psi}\widehat{\nabla\psi} = -\frac{\partial_\theta\sqrt{g_{\psi\psi}}}{\sqrt{g_{\theta\theta}}}\widehat{\nabla\theta},\ \frac{\partial}{\partial\theta}\widehat{\nabla\psi} = \frac{\partial_\psi\sqrt{g_{\theta\theta}}}{\sqrt{g_{\psi\psi}}}\widehat{\nabla\theta},\ \frac{\partial}{\partial\phi}\widehat{\nabla\psi} = \frac{\partial_\psi\sqrt{g_{\phi\phi}}}{\sqrt{g_{\psi\psi}}}\widehat{\nabla\phi} \tag{4.100}$$

Similarly, for $\widehat{\nabla\theta}$ and $\widehat{\nabla\phi}$ we find:

$$\nabla\widehat{\nabla\theta} = \nabla\psi\frac{\partial}{\partial\psi}\widehat{\nabla\theta} + \nabla\theta\frac{\partial}{\partial\theta}\widehat{\nabla\theta} + \nabla\phi\frac{\partial}{\partial\phi}\widehat{\nabla\phi} \tag{4.101}$$

$$\frac{\partial}{\partial\theta}\widehat{\nabla\theta} = \frac{\partial_\theta\sqrt{g_{\psi\psi}}}{\sqrt{g_{\theta\theta}}}\widehat{\nabla\psi},\ \frac{\partial}{\partial\theta}\widehat{\nabla\theta} = -\frac{\partial_\psi\sqrt{g_{\theta\theta}}}{\sqrt{g_{\psi\psi}}}\widehat{\nabla\psi},\ \frac{\partial}{\partial\phi}\widehat{\nabla\theta} = \frac{\partial_\theta\sqrt{g_{\phi\phi}}}{\sqrt{g_{\theta\theta}}}\widehat{\nabla\phi} \tag{4.102}$$

$$\nabla\widehat{\nabla\phi} = \nabla\psi\frac{\partial}{\partial\psi}\widehat{\nabla\phi} + \nabla\theta\frac{\partial}{\partial\theta}\widehat{\nabla\phi} + \nabla\phi\frac{\partial}{\partial\phi}\widehat{\nabla\phi} \tag{4.103}$$

$$\frac{\partial}{\partial\psi}\widehat{\nabla\phi} = \frac{\partial}{\partial\theta}\widehat{\nabla\phi} = 0,\ \frac{\partial}{\partial\phi}\widehat{\nabla\phi} = -\frac{\partial_\theta\sqrt{g_{\phi\phi}}}{\sqrt{g_{\theta\theta}}}\widehat{\nabla\theta} - \frac{\partial_\psi\sqrt{g_{\phi\phi}}}{\sqrt{g_{\psi\psi}}}\widehat{\nabla\psi} \tag{4.104}$$

Bi-cylindrical coordinates Let us now consider the situation when the magnetic surfaces are given by the set of concentric circles. Here transition from cylindrical to the magnetic coordinates is given by:

$$R = R_0 + \psi\cos\theta,\ Z = \psi\sin\theta \tag{4.105}$$

It generate the orthogonal set of vectors $\nabla\psi, \nabla\theta$ and the corresponding diagonal covariant metric tensor is given by:

$$g_{\psi\psi} = 1,\ g_{\theta\theta} = \psi,\ g_{\phi\phi} = R_0 + \psi\cos\theta \tag{4.106}$$

So the coefficient of the curvature tensor are given by:

$$\frac{\partial}{\partial\psi}\widehat{\nabla\psi} = 0,\ \frac{\partial}{\partial\theta}\widehat{\nabla\psi} = \widehat{\nabla\theta},\ \frac{\partial}{\partial\phi}\widehat{\nabla\psi} = \cos\theta\widehat{\nabla\phi} \tag{4.107}$$

$$\frac{\partial}{\partial\psi}\widehat{\nabla\theta} = 0,\ \frac{\partial}{\partial\theta}\widehat{\nabla\theta} = -\widehat{\nabla\psi},\ \frac{\partial}{\partial\phi}\widehat{\nabla\theta} = -\sin\theta\widehat{\nabla\phi} \tag{4.108}$$

$$\frac{\partial}{\partial\psi}\widehat{\nabla\phi} = \frac{\partial}{\partial\theta}\widehat{\nabla\phi} = 0,\ \frac{\partial}{\partial\phi}\widehat{\nabla\phi} = -\cos\theta\widehat{\nabla\psi} + \sin\theta\widehat{\nabla\theta} \tag{4.109}$$

Magnetic field

Now we consider a configuration with axisymmetric magnetic field geometry. We start with definition of the direction of magnetic field with unit vector:

$$\hat{\mathbf{b}} = \cos\eta(\psi,\theta)\widehat{\nabla\phi} + \sin\eta(\psi,\theta)\widehat{\nabla\theta} \tag{4.110}$$

where the function $\eta(\psi,\theta)$ defines the angle between its toroidal and poloidal components:

$$\text{cotan}\,\eta(\psi,\theta) = \frac{\widehat{\nabla\phi}\cdot\hat{\mathbf{b}}}{\widehat{\nabla\theta}\cdot\hat{\mathbf{b}}} \tag{4.111}$$

In the particular case, when $\eta = \eta(\psi)$, the expression (4.111) defines the function often called the "q-profile" or "safety factor profile". The plasma "q" denotes the number of times a magnetic field line turns around a torus in the toroidal direction for each time it comes around its short (poloidal) direction. In a typical tokamak q ranges from near unity in the center of plasma to $2-8$ at the edge. This function is named the *safety factor* because larger values are associated with higher ratios of toroidal field to poloidal field generated by plasma current. Consequently the risk of current-driven plasma instabilities is less for higher values of q.

Here we consider some characteristic for a tokamak "q-profile", quadratic with respect to the magnetic(radial) coordinate ψ:

$$q(\psi) = q_0 + \frac{s_0}{2}\psi^2 \tag{4.112}$$

For example in Semi-Lagrangian Gyrokinetic code GYSELA, $s_0 = 0.854$ and $q_0 = 2.184$

Note that the direction $\hat{\mathbf{b}}$ and the norm B of magnetic field $\mathbf{B}$ cannot be chosen totally independently of each other because they are related through the magnetic constraint:

$$\nabla\cdot\mathbf{B} = 0 \tag{4.113}$$

So if we start the definition of magnetic field by introducing its direction $\hat{\mathbf{b}}$, we should pay attention to the condition (4.113) when choosing its norm. In the case of general axisymmetric geometry, when the norm of magnetic field is supposed to be independent of toroidal coordinate ϕ, it can be found from the differential equation:

$$\frac{\partial}{\partial\theta}\Big(B(\psi,\theta)\ \sqrt{g_{\psi\psi}(\psi,\theta)}\sqrt{g_{\phi\phi}(\psi,\theta)}\sin\eta(\psi,\theta)\Big) = 0 \tag{4.114}$$

Then one of its possible solutions, obtained from the separation of variables method, is:

$$B(\psi,\theta) = \frac{F(\psi)}{\sqrt{g_{\psi\psi}(\psi,\theta)}\sqrt{g_{\phi\phi}(\psi,\theta)}\sin\eta(\psi,\theta)} \tag{4.115}$$

where $F(\psi)$ represents an arbitrary function of radial coordinate ψ, that for simplicity we will take equal to unity. Moreover, in what follows, we consider the case, in which the ratio function η depends only on the radial coordinate ψ and then coincide with the q-profile: cotan $\eta(\psi) \equiv q(\psi)$. Finally, the magnetic field in the bi-cylindrical coordinates case ($\sqrt{g_{\phi\phi}} = R(\psi, \theta) = R_0 + \psi\ \cos\theta$, $\sqrt{g_{\psi\psi}} = 1$) is given by

$$\mathbf{B} = \frac{B_0}{R(\psi,\theta)\sin\eta(\psi)}\left(\cos\eta(\psi)\widehat{\nabla\phi} + \sin\eta(\psi)\widehat{\nabla\theta}\right) \equiv B\ \hat{\mathbf{b}} \tag{4.116}$$

where B_0 is value of magnetic field, measured in Teslas.

Equations of motion

In order to study trapped particles trajectories in axisymmetric magnetic configurations (bi-cylindrical coordinates), we decompose equations of motion in the corresponding basis $(\nabla\psi, \nabla\theta, \nabla\phi)$. Then we integrate this equations by using Mathematica package. The calculation leading to equations of motion in bi-cylindrical geometry is presented in the Appendix H.

4.4.2 Trajectories

Within the standard approach the particles inside a tokamak are divided into two groups

- *passing* particles whose trajectories follow the magnetic field lines

- *trapped* particles bouncing between two local magnetic mirrors (defined by maxima of magnetic field intensity)

The shape of the latter ones is such that when projected on a poloidal cross section, resembles a *banana* with width $\delta_b \sim \epsilon_b q(\psi)\rho_L$ where $\epsilon_b = \psi/R$ is the local aspect ratio of the toroidal magnetic surface with radius ψ and the major radius $R = R_0 + \psi\cos\phi$, $\rho_L = v_\perp/\Omega$ is the ion Larmor radius. Note that such an approximation fails near the magnetic axis when passing into the potato regime [64].

The fast trapped ions appears in tokamaks as a results of auxiliary plasma heating, such as neutral beam injection and radio frequency heating, and production of alpha particles.

In the neoclassical transport theory, which studies the transport due to the Coulomb collisions and takes into account the effects of toroidal geometry, the transport that arises from the small population of the trapped particles dominates the transport resulting from the majority of passing (i.e. untrapped) particles.

Moreover, it was remarked in [63] (1984) and then in [64] (2001) that additionally to the the standard approximation, that divides the particle into the passing and trapped, there exist some special orbits that give rise to new interesting effects, among them orbits for which the times of precessional and bounce motion became comparable. As was mentioned in both of these works, it can have an important consequences on the plasma stability.

Our further investigation here consists of several parts. The goal here is to explore the possible trapping process characteristics: the conditions, region, orbit topology modification during the trapping/untrapping process, curvature magnetic field effects.

First of all we compare the behavior of the particle for different values of the small dimensionless parameter ε that presents the ratio between the modulus of the kinetic particle momentum and the magnitude of magnetic field B.

On the figure 4.3 is presented an overview of the trapped/untrapped particles according to the position of the magnetic surface in the poloidal section of our virtual machine. The value of the small parameter in the left figure is larger then its value in the right figure: $\varepsilon = 3.5 * 10^{-2}$ and $\varepsilon = 2.1 * 10^{-2}$ correspondingly. The left

./surf_weak_trapping_0035.eps ./surf_strong_trapping.eps

Figure 4.3: Weak and strong trapping

figure illustrate the case of weak particle trapping with $\varepsilon = 3.5 * 10^{-2}$ and the right one of the strong particle trapping with $\varepsilon = 2.1 * 10^{-2}$. The interpretation of such a particle behavior is straightforwardly related to the interpretation of variation of the small parameter ε.

The small parameter ε can increase in two situations: when the intensity of magnetic field grows and when the particle slows down. In both cases the trapping is strong: the particles are more tied up to the field lines. If now the velocity of the particle grows or the intensity of the magnetic field decrease, then the particle possesses more freedom to derive between the magnetic surfaces or to become passing. This is the case of weaker trapping.

Another observation that could be made here is about the distribution of trapped/untrapped particles as a function of the radii of the magnetic surface. For instance we can just remark that the passing particles occurs in the center rather then in the edge of machine. Such a behavior is exploited and confirmed within the following study.

The particle is trapped when its pitch angle passes through the value $\pi/2$. This corresponds to the moment when the parallel component of the particle velocity vanishes.

The next question that we address here is about the ratio of the trapped/untrapped trajectories as the function of initial values of the pitch angle φ_0 in different regions of the poloidal section.

From the left to the right: we pass from the center to the edge of the machine with different values for initial pitch angle $\varphi_0 \in [0, \pi]$. Here we use the red color to mark the passing trajectories and we color in blue the trapped trajectories. In order to follow the evaluation of the number of trapped/untrapped trajectories in different regions of the poloidal section. On the figure 4.4 we color in cyan the first trajectory that is trapped on the middle magnetic surface $\psi = 0.5$ (for $\psi \in [0, 1]$). We remark that the cyan trajectory belongs to the passing region in the center of

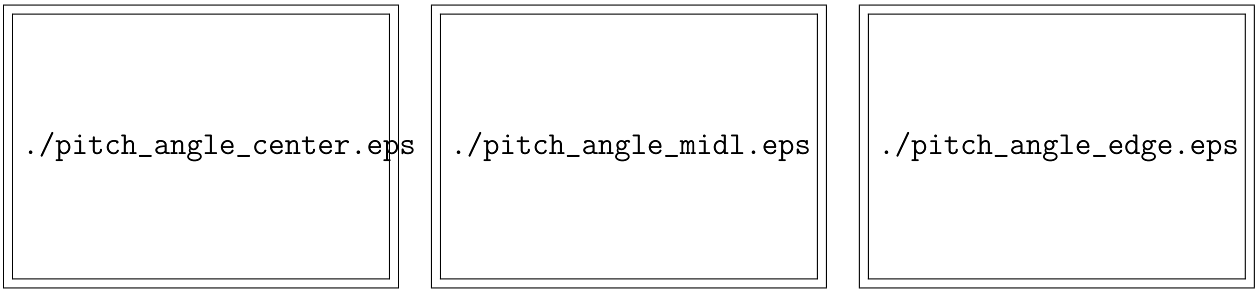

Figure 4.4: Pitch angle for different regions of tokamak: center $\psi_0 = 0.2$,middle $\psi_0 = 0.5$, edge $\psi_0 = 0.9$

machine and the same trajectory lies within and no longer limits the trapped region in the edge. Then there are more trapped particles in the edge than in the center of the machine. Dynamics of the pitch angle ϕ can be used as a criterium of a particle behavior. In these plots, captured particles (pitch angle passes through $\pi/2$) are in red, passing $\phi \in (\pi/2, \pi]$ and co-passing particles $\phi \in [0, \pi/2)$ are in blue.

These two studies confirm that, there are less captured particles at the center that near the edge of machine because of the diameter of magnetic surfaces: smaller is the diameter, more difficult it is for the particle to bounce between its two points, more natural become to turn around following the passing trajectory.

On the figures 4.5,4.6 below we focuss on the trajectories topology changing when trapping and untrapping process and give an overview of *barely* trapped particles or *limiting* orbits. Such a transition process can be observed while changing different parameters of the system:

- The position of the magnetic surface for given initial values of the pitch angle

./trapping.eps

Figure 4.5: Barely trapped particles: trapping.

./untrapping.eps

Figure 4.6: Barely trapped particles: untrapping.

$\phi_0 = \pi/6$, poloidal angle $\theta_0 = \pi/3$, small parameter $\varepsilon = 3.5*10^{-2}$ and magnetic parameters $q_0 = 1$ and $s_0 = 4$. Here the trapping process is obtained when varying $\psi \in [0.34, 0.4]$.

- Magnetic field configuration characteristics; q-profile parameters: q_0 and s_0, here in order to observe the trapping process we have modulate the shear parameter s_0 in interval $[4, 6]$ for $q_0 = 2$. We can remark that for smaller values of s_0 the trapping of the particle was deeper. Note that here a large value of the parameter is considered $\varepsilon = 10.2 * 10^{-2}$ in order to span the transition zone in a relatively short integration time. Here we can also remark that the trajectories are shifted with respect to the field lines. This effect is due to the drift velocity, that in our case possesses only the contributions related to the magnetic curvature. In the limit $\beta \ll 1$ it can be approximated by the expression $v_D = \dfrac{mv_{||}^2 + \mu_0 B}{B^3}\mathbf{B} \times \nabla\mathbf{B}$.
- Another possibility to span the transition region is to change the values for initial condition of the pitch angle φ_0(as in the previous study) or the initial poloidal angle θ_0 (that will be considered in the next step).

We observe that there are 4 main different shapes of trajectory that can occur through the trapping process. It can be classified as follows: tied to the magnetic surface passing trajectory, possessing a cumulative (stagnation) point near the axe, cusp orbit; possessing inner and outer loops: pinch orbit and finally banana orbit. The untrapping process passes through the same stages in the opposite way. The more particular among the mentioned orbits, is the cusp orbit. As was noticed in [64] the cusp orbits are characterized by algebraical divergency of the bounce time. It appears when the inner loop of the pinch orbit degenerates into a pinch point. In the other words, when the trapped particle became almost untrapped. The same situation was observed in our case while integrating the exact particle equations of motion. The integration time considerably increases when passing between the shape 2 and the shape 3.

Another interesting observation that can be made here concerns an asymmetry that occurs in trapping processes for the particles with positive and negative initial parallel velocity (pitch angle φ_0) condition[13]. In the case of the trajectories that passes around magnetic surfaces the terms of *co-passing* ($v_{||} < 0$) and *counter-passing* ($v_{||} > 0$) are employed for it designation.

Accordingly to the sign of φ_0 particle trajectory will be positioned with respect to the magnetic surface. The inner manner when $\varphi_0 < 0$ and in outer manner when $\varphi_0 > 0$. On the two figure below the co-passing (co-trapped) particle is colored in blue and the counter-passing (counter-trapped) in green. Here we consider the different particle trajectories as a function of initial poloidal angle θ_0: On 4.7 the

[13] Assuming the all others initial conditions coincides

./PassCopassNoTitle.eps

Figure 4.7: Asymmetry in trapping process for passing and co-passing particles: as a function of initial poloidal angle θ_0.

figures in the left column represent inner and outer trajectories with initial poloidal angle $\theta_0 \in [0, \pi - \delta]$ and the figure in the right column represents inner and outer trajectories with $\theta_0 \in [\pi + \delta, 2\pi - \theta_0]$. We remark that in the first case the outer trajectories (green curve) are naturally less trapped then the co-passing ones (blue curve). The situation is inverse for initial poloidal angles $\theta_0 = 2\pi - \theta_0$. The exchange happens in the region of $\theta \in (\pi - \delta, \pi + \delta)$. This region is zoomed on the figure 4.8. One can remark that there exists initial condition for which the passing particle still be trapped and the co-passing is untrapped ($\theta_0 = 0.9\pi$), then the both trajectories become untrapped and separated by the magnetic surface ($\theta_0 = 0.97\pi$). In the position $\theta_0 = \pi$ the mixing of inner and outer trajectories takes place. Finally for the initial condition $\theta_0 = 1.1\pi$ the exchange in accomplished: now the inner particle becomes less trapped then the outer. Such an asymmetry can be explained as one of the effects of the magnetic field curvature.

Moreover we remark that a similar transition for inner and outer trajectories occurs when changing magnetic configuration parameters. On the figures 4.9 and 4.10 an example with larger Larmor radius is considered in order to observe in details the evaluation of orbit topology (here the inner trajectory is colored in cyan and and the outer trajectory in blue):

./PassCopassTransNoTitle.eps

Figure 4.8: Asymmetry in trapping process for passing and co-passing particles: transition region.

The transition of the inner trajectory from trapped to passing occurs in the interval of shear parameter $s_0 \in [2.8, 2.95]$; while the same transition for the outer trajectory takes place for $s_0 \in [6, 6.2]$.

The last study that was realized here concerns the fluctuation of the adiabatic invariant for different values of the small parameter ε. The results of such an investigation are summarized in the following table.

ε	$\delta\mu_0/\mu_0$
$10.29 * 10^{-2}$	41.03%
$3.5 * 10^{-2}$	20.7%
$2.1 * 10^{-2}$	12.4%

Table 4.1: Variation of the adiabatic invariant μ_0 as a function of small gyrokinetic parameter $\varepsilon = \frac{\rho_L}{L_B}$ in our simulations with $q_0 = 1$ and $s_0 = 4$

./LargeLarmorInPassBlue.eps

Figure 4.9: Asymmetry in trapping process for passing and co-passing particles. Large Larmor radius case I.

ε	$\delta\mu_0/\mu_0$
1/64	6.04%
1/128	2.98%
1/256	1.48%
1/1024	0.36%

Table 4.2: Variation of the adiabatic invariant μ_0 as a function of small gyrokinetic parameter $\varepsilon = \frac{\rho_L}{L_B}$ in GYSELA with $q_0 = 0.854$, $s_0 = 2.184$

Here we compute the variation of adiabatic invariant for two groups of values of small parameter ε. The first group, including the 3 first values, was used in our numerical simulations. The second group of parameters represents the values of ε that are usually taken in gyrokinetic numerical simulations produced by GYSELA. We are close to the first of them with $\varepsilon = 2.1 * 10^{-2}$, but we can not deeply explore the particle behavior for smallest ones because the integration time becomes too long in the vicinity of the *barely* trapped trajectories. We remark that in the case

./LargeLarmorOutPassBlue.eps

Figure 4.10: Asymmetry in trapping process for passing and co-passing particles. Large Larmor radius case II.

of large value of $\varepsilon = 10.29 * 10^{-2}$, that we have chose for zoom the effects of particle trajectory transitions, the variation of μ_0 is quite important $\delta\mu_0 = 59.4\%$. Therefore such a calculation could be imprecise in the case of the COM phase space.

On the other hand, in the case of values of ε used in the GYSELA code, the fluctuation of the adiabatic invariant μ_0 lies between 0.36% and 6.04%.

In further work it will be interesting to proceed with exploration of particle trajectories in a more realistic magnetic geometry. Moreover, understanding the topology of particle trajectories is not only the subject of interest for laboratory fusion plasmas, but also in the case of astrophysical plasmas. This can open new opportunities for consideration of different magnetic configurations.

4.5 Intrinsic dynamical reduction

In the previous section the problem of local dynamical reduction for charged particle motion in an external non-uniform magnetic field was considered. In this section a more abstract approach, which does not involve the use of the fixed basis vectors and therefore the problems related to the gyrogauge dependence of the dynamics, is presented. Previously, in the local coordinate case an iterative procedure for the constant of motion $\mathcal{A}$ was obtained. Now we will directly proceed with a *Hamiltonian normal form* construction. As was explained in 4.2.3, from the beginning we consider the constant of motion $\mathcal{A}$ as an independent variable of the new phase space, therefore we pass from $(p, \hat{\mathbf{p}}, \mathbf{r})$ to $(\mathcal{A}, \hat{\mathbf{p}}, \mathbf{x})$. As in the local dynamical reduction case, here $\mathbf{r} = \mathbf{x}$.

Note that it suffices to obtain the expression for the rescaled Hamiltonian p in the new phase space variables $(\mathbf{x}, \hat{\mathbf{p}}, \mathcal{A})$. Consequently, it is more convenient to deal here with the rescaled Hamiltonian dynamics (4.13).

At this stage an explicit expression for $\mathcal{A} = \mathcal{A}(\mathbf{r}, p, \hat{\mathbf{p}})$ is not known. By changing the functional dependence $p = p(p, \hat{\mathbf{p}}, \mathbf{r}) \to p = p(\mathcal{A}, \hat{\mathbf{p}}, \mathbf{x})$ directly inside the equations of motion (4.13), we will obtain an implicit expression for dynamics in the new phase space:

$$\begin{pmatrix} \dot{\mathbf{r}} = \hat{\mathbf{p}} \\ \\ \dot{\hat{\mathbf{p}}} = \dfrac{\hat{\mathbf{p}} \times e\,\mathbf{B}(\mathbf{r})}{p\,(\mathbf{r},\ \hat{\mathbf{p}},\ p)} \\ \\ \dot{p} = 0 \end{pmatrix} \to \begin{pmatrix} \dot{\mathbf{x}} = \hat{\mathbf{p}} \\ \\ \dot{\hat{\mathbf{p}}} = \dfrac{\hat{\mathbf{p}} \times e\,\mathbf{B}(\mathbf{x})}{p\,(\mathbf{x}, \hat{\mathbf{p}}, \mathcal{A}\,)} \\ \\ \dot{\mathcal{A}} = 0 \end{pmatrix} \tag{4.117}$$

4.5.1 Hamiltonian normal form

As was mentioned above, in order to obtain a partial differential equation that leads to the expression of the rescaled Hamiltonian p as a function of the new phase space variables, we should use the stationarity condition[14] $\dot{p} = 0$ for $p = p\,(\mathbf{x}, \hat{\mathbf{p}}, \mathcal{A})$:

$$\dot{p} = \dot{\mathbf{x}} \cdot \partial_{\mathbf{x}} p + \dot{\hat{\mathbf{p}}} \cdot \partial_{\hat{\mathbf{p}}}\, p + \dot{\mathcal{A}}\, \partial_{\mathcal{A}} p = 0 \tag{4.118}$$

Then by substituting into (4.118) the equations of motion in the new variables (4.117), the general equation for Hamiltonian normal form becomes:

$$\frac{p}{eB}\, \hat{\mathbf{p}} \cdot \partial_{\mathbf{x}} p = -\left[\, (\hat{\mathbf{p}} \times \hat{\mathbf{b}}) \cdot \partial_{\hat{\mathbf{p}}} \,\right] p \tag{4.119}$$

[14]Tensor calculus: in what follows we deal with vectors (covariant objects) and we use the canonical Euclidian basis in order to express coordinates, for example the gradient $\nabla \equiv \partial_{\mathbf{x}}$ is viewed as a vector with following coordinates: $\nabla_i \equiv \partial_{\mathbf{x}^i}$. Note that use of such a canonical basis allows our description still to be intrinsic. In fact $\mathbf{e}_i \cdot \nabla \mathbf{e}_j = 0$ because the vectors $\mathbf{e}_i$ are independent of particle position $\mathbf{x}$. Here $\cdot$ denotes the tensor contraction (scalar product in the case of the vectors).

The goal of our further work is to solve this differential equation.

4.5.2 Intrinsic basis

To start the solution of the equation (4.119) we need to introduce some basis in order to make a decomposition of the vector $\hat{\mathbf{p}}$. As it has mentioned above, one of this work is to not use the fixed basis associated to the magnetic field line.

As in the case of local dynamical reduction, we start by projecting onto the to the parallel to magnetic field direction $\hat{\mathbf{b}} \equiv \mathbf{B}/B$ of the unit momentum vector $\hat{\mathbf{p}}$ by introduction of the pitch angle φ:

$$\hat{\mathbf{p}} \cdot \hat{\mathbf{b}} = \cos\varphi \tag{4.120}$$

This relation couples spatial dependence and momentum dependence, so that $\varphi = \varphi(\mathbf{x}, \hat{\mathbf{p}})$.

Then in order to project $\hat{\mathbf{p}}$ into the plane perpendicular to the magnetic field direction, we proceed with the direct construction of the dynamical basis obtained by using only physical vectors $\hat{\mathbf{b}}$ and $\hat{\mathbf{p}}$ as follows:

$$\hat{\boldsymbol{\rho}} = \frac{\hat{\mathbf{b}} \times \hat{\mathbf{p}}}{|\hat{\mathbf{b}} \times \hat{\mathbf{p}}|} = \frac{\hat{\mathbf{b}} \times \hat{\mathbf{p}}}{\sqrt{1 - (\hat{\mathbf{b}} \cdot \hat{\mathbf{p}})^2}} \tag{4.121}$$

$$\hat{\perp} = -\hat{\mathbf{b}} \times \frac{\hat{\mathbf{b}} \times \hat{\mathbf{p}}}{|\hat{\mathbf{b}} \times \hat{\mathbf{p}}|} = -\hat{\mathbf{b}} \times \frac{\hat{\mathbf{b}} \times \hat{\mathbf{p}}}{\sqrt{1 - (\hat{\mathbf{b}} \cdot \hat{\mathbf{p}})^2}} \tag{4.122}$$

where we have used the corollary of the (4.120):

$$|\hat{\mathbf{b}} \times \hat{\mathbf{p}}| = \sqrt{1 - (\hat{\mathbf{b}} \cdot \hat{\mathbf{p}})^2} \tag{4.123}$$

The essential difference of this method from the previous one is that here we do not introduce an explicit definition of the gyrophase angle ζ. It is hidden inside the rotating vectors $\hat{\perp}$ and $\hat{\boldsymbol{\rho}}$. In fact an explicit definition of the gyrophase angle ζ inevitably involves the introduction of the fixed basis vectors $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$ (4.21).

The orientation of the intrinsic dynamical basis is organized so that:

$$\hat{\mathbf{b}} \times \hat{\perp} = \hat{\boldsymbol{\rho}}, \quad \hat{\mathbf{b}} \times \hat{\boldsymbol{\rho}} = -\hat{\perp} \tag{4.124}$$

Finally, the unit momentum vector $\hat{\mathbf{p}}$ can be represented as follows:

$$\hat{\mathbf{p}} = \hat{\mathbf{b}} \cos\varphi + \hat{\perp} \sin\varphi \tag{4.125}$$

To elucidate functional dependence, let us consider the whole phase space $(\mathbf{x}, \hat{\mathbf{p}}, \mathcal{A})$ change of variables that arises from the decomposition of unit momentum vector $\hat{\mathbf{p}}$

into the rotating frame (4.125):

$$\begin{pmatrix} \mathbf{x} \\ \hat{\mathbf{p}} \\ \mathcal{A} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{x}' \\ \phi \\ \hat{\perp} \\ \mathcal{A}' \end{pmatrix} \tag{4.126}$$

with $\mathbf{x} = \mathbf{x}'$ and $\mathcal{A} = \mathcal{A}'$.

As in the local approach we assume that the spatial dependence of the unit magnetic field vector $\hat{\mathbf{b}}$ is invariant under this transformation: $\hat{\mathbf{b}}(\mathbf{x}) = \hat{\mathbf{b}}(\mathbf{x}')$.

The table below resumes the functional dependencies of variables before and after the introduction of the rotating frame:

$(\mathbf{x}, \hat{\mathbf{p}}, \mathcal{A})$	$(\mathbf{x}', \hat{\mathbf{p}}, \mathcal{A}')$
$\varphi = \varphi(\mathbf{x}, \hat{\mathbf{p}})$	φ-independent
$\hat{\perp} = \hat{\perp}(\mathbf{x}, \hat{\mathbf{p}})$	$\hat{\perp}$-independent
$\nabla \equiv \partial_{\mathbf{x}}$	$\nabla' \equiv \partial_{\mathbf{x}'} + \partial_{\mathbf{x}}\varphi\partial_{\varphi} + \partial_{\mathbf{x}}\hat{\perp} \cdot \partial_{\hat{\perp}}$

Table 4.3: Comparison of the functional phase space variables dependence before and after introducing the intrinsic rotating frame

Jacobian: space part

The Jacobian matrix of the corresponding transformation was involved when deriving the Poisson bracket in local coordinates. Here only the spatial derivatives will be involved in further calculations, so we need to find $\partial_{\mathbf{x}}\varphi$ and $\partial_{\mathbf{x}}\hat{\perp}$.

Proposition 1

$$\nabla\varphi = -\nabla\hat{\mathbf{b}} \cdot \hat{\perp} \quad (\partial_i\varphi = -\partial_i\, b_k\hat{\perp}_k) \tag{4.127}$$

Proof 1

$$\varphi = \arccos(\hat{\mathbf{b}} \cdot \hat{\mathbf{p}}) \Longrightarrow \nabla\varphi = -\frac{1}{\sqrt{1 - (\hat{\mathbf{b}} \cdot \hat{\mathbf{p}})^2}} \left(\nabla\hat{\mathbf{b}} \cdot \hat{\mathbf{p}} + \nabla\hat{\mathbf{p}} \cdot \hat{\mathbf{b}}\right)$$

with $\nabla\hat{\mathbf{p}} = 0$, $\hat{\mathbf{p}} = \hat{\mathbf{b}}\cos\varphi + \hat{\perp}\sin\varphi$ *and* $\sqrt{1 - (\hat{\mathbf{b}} \cdot \hat{\mathbf{p}})^2} = \sin\varphi$

$$\nabla\varphi = -\nabla\hat{\mathbf{b}} \cdot \hat{\perp}$$

■

Proposition 2

$$\begin{aligned} \nabla\hat{\perp} &= -(\nabla\hat{\mathbf{b}}\cdot\hat{\perp})\,\hat{\mathbf{b}} - \Phi\,(\nabla\hat{\mathbf{b}}\cdot\hat{\boldsymbol{\rho}})\,\hat{\boldsymbol{\rho}} && (4.128)\\ \nabla\hat{\boldsymbol{\rho}} &= -(\nabla\hat{\mathbf{b}}\cdot\rho)\,\hat{\mathbf{b}} + \Phi\,(\nabla\hat{\mathbf{b}}\cdot\hat{\boldsymbol{\rho}})\,\hat{\perp} && (4.129)\end{aligned}$$

where $\Phi \equiv \text{cotan}\,\varphi$

Proof 2 *with* $\hat{\mathbf{p}} = \hat{\mathbf{b}}\ \cos\varphi + \hat{\perp}\ \sin\varphi$, $\nabla\hat{\mathbf{p}} = 0$ *and* $\nabla\varphi = -\nabla\hat{\mathbf{b}}\cdot\hat{\perp}$ *we have:*

$$\nabla\hat{\perp} = (\nabla\hat{\mathbf{b}}\cdot\hat{\perp})\,(\Phi\,\hat{\perp} - \hat{\mathbf{b}}) - \Phi\,\nabla\hat{\mathbf{b}} \tag{4.130}$$

Now we project this expression at the right on the basis vectors $(\hat{\mathbf{b}}, \hat{\perp}, \hat{\boldsymbol{\rho}})$*, we use the following properties:* $\nabla\hat{\mathbf{e}}\cdot\hat{\mathbf{e}} = 0$ *because* $\nabla(\hat{\mathbf{e}}\cdot\hat{\mathbf{e}}) = 0$ *for any unit vector* $\hat{\mathbf{e}}$ *and* $\nabla\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_k = -\nabla\hat{\mathbf{e}}_k\cdot\hat{\mathbf{e}}_i$*, for two different basis vectors, this property is the consequence of the fact that the basis vectors are perpendicular.*

- $\nabla\hat{\perp}\cdot\hat{\mathbf{b}} = -(\nabla\hat{\mathbf{b}}\cdot\hat{\perp})$
- $\nabla\hat{\perp}\cdot\hat{\perp} = \Phi\,\nabla\hat{\mathbf{b}}\cdot\hat{\perp} - \Phi\,\nabla\hat{\mathbf{b}}\cdot\hat{\perp} = 0$ (trivial)
- $\nabla\hat{\perp}\cdot\hat{\boldsymbol{\rho}} = -\Phi\,\nabla\hat{\mathbf{b}}\cdot\hat{\boldsymbol{\rho}}$

Now we contract matrix $\nabla\hat{\perp}$ *by its right with unit dyadic matrix and we use the equations obtained below:*

$$\begin{aligned}&\nabla\hat{\perp} = \nabla\hat{\perp}\cdot(\hat{\mathbf{b}}\hat{\mathbf{b}} + \hat{\perp}\hat{\perp} + \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}}) =\\ &(\nabla\hat{\perp}\cdot\hat{\mathbf{b}})\,\hat{\mathbf{b}} + (\nabla\hat{\perp}\cdot\hat{\perp})\,\hat{\perp} + (\nabla\hat{\perp}\cdot\hat{\boldsymbol{\rho}})\,\hat{\boldsymbol{\rho}} = -(\nabla\hat{\mathbf{b}}\cdot\hat{\perp})\,\hat{\mathbf{b}} - \Phi\,(\nabla\hat{\mathbf{b}}\cdot\hat{\boldsymbol{\rho}})\,\hat{\boldsymbol{\rho}}\end{aligned}$$

So we have (4.128). In order to obtain (4.129), we will proceed similarly. We contract the bi-vector $\nabla\hat{\boldsymbol{\rho}}$ *with unit dyadic matrix and we use* $\nabla\hat{\boldsymbol{\rho}}\cdot\hat{\mathbf{b}} = -\nabla\hat{\mathbf{b}}\cdot\hat{\boldsymbol{\rho}}$ *and* $\nabla\hat{\perp}\cdot\hat{\boldsymbol{\rho}} = -\nabla\hat{\boldsymbol{\rho}}\cdot\hat{\perp} = \Phi\,\nabla\hat{\mathbf{b}}\cdot\hat{\boldsymbol{\rho}}$*, so*

$$\nabla\hat{\boldsymbol{\rho}} = -(\nabla\hat{\mathbf{b}}\cdot\hat{\boldsymbol{\rho}})\,\hat{\mathbf{b}} + \Phi\,(\nabla\hat{\mathbf{b}}\cdot\hat{\boldsymbol{\rho}})\,\hat{\perp} \tag{4.131}$$

The expression (4.129) is also obtained.

■

4.5.3 Intrinsic gyroaveraging

The next step in the procedure of solution of the general normal form equation (4.119), is to make use of a separation into natural scales of motion which permits us to treat the fast dynamics separately from the slow one. For this purpose in an earlier section, when constructing an iterative procedure for the constant of motion series, the operation of gyroaveraging $\mathcal{R} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\zeta$ has been introduced. [15] Here we will proceed with introduction of a suitable gyroaveraging operator in an intrinsic basis.

Fixed-basis-independent rotations

Until now we have used the most natural way to represent rotation of some vector $\hat{\mathbf{p}} \in \mathbb{R}^3$ around some other vector noncollinear with it, $\hat{\mathbf{b}} \in \mathbb{R}^3$. We passed through the definition of two angles: $\varphi = \arccos(\hat{\mathbf{p}} \cdot \hat{\mathbf{b}})$ that represent rotation in the plane that contains the vector $\hat{\mathbf{b}}$ and $\zeta = \arctan(\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{p}} / \hat{\mathbf{b}}_2 \cdot \hat{\mathbf{p}})$ denoting the angle of rotation in the plane perpendicular to $\hat{\mathbf{b}}$. The definition of the first one involves only the initial rotating momentum vector $\hat{\mathbf{p}}$ and directional magnetic field unit vector $\hat{\mathbf{b}}$. At the same time, the second angle definition needs the introduction of some basis in the plane perpendicular to the directional vector $\hat{\mathbf{b}}$. In a general magnetic geometry case such a basis cannot be defined uniquely and leads to the problem of gyrogauge dependence.

To avoid the use of these arbitrarily chosen vectors, we should now consider the definition of rotation at a more abstract level. In the two following subsections we recall the definition of the operation of rotation around some direction in $\hat{\mathbf{b}} \in \mathbb{R}^3$ on some angle α, defined only by the choice of this direction.

Rotations in $\mathbb{R}^3$ It is well known that rotations generators form the Lie algebra $so(3)$. Its representation on $\mathbb{R}^3$ can be given by skew-symmetric matrices. In this case the corresponding Lie bracket is the matrix commutator.

The basis of rotation generators can be presented in matrix form with:

$$A_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad A_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad A_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.132}$$

The commutation relations of the basis elements possesses the following property:

$$[A_x, A_y] = A_z \quad [A_z, A_x] = A_y \quad [A_y, A_z] = A_x \tag{4.133}$$

By using this basis, a generator of rotation about some unit vector $\hat{\mathbf{u}} \in \mathbb{R}^3$ can be composed as follows:

$$\mathcal{U} = \hat{\mathbf{u}}_x\, A_x + \hat{\mathbf{u}}_y\, A_y + \hat{\mathbf{u}}_z\, A_z = \begin{pmatrix} 0 & -\hat{\mathbf{u}}_z & \hat{\mathbf{u}}_y \\ \hat{\mathbf{u}}_z & 0 & -\hat{\mathbf{u}}_x \\ -\hat{\mathbf{u}}_y & \hat{\mathbf{u}}_x & 0 \end{pmatrix} \tag{4.134}$$

[15] One of the L. Sugiyama remarks in [59] is about the problems related to the existence of a globally consistent definition for the standard gyroaveraging operator (4.65). She states that in the case of a non-trivial 3-dimensional magnetic geometry, due to the presence of non-closed magnetic surfaces, the application of the (4.65) can lead to the integration over a cumulative angle.

At the same time this expression can also be rewritten using the Levi-Civita tensor

$$\mathcal{U}_{ik} = \epsilon_{ijk}\ \hat{\mathbf{u}}_j \tag{4.135}$$

If now we equate the expressions (4.134) with (4.135) , we will build an isomorphism between the skew-symmetric matrix representation of $so(3)$ and $\mathbb{R}^3$. Two things will happen:

- Any skew-symmetric matrix will be conveniently identified with a vector

$$\mathcal{U} = \begin{pmatrix} 0 & -\hat{\mathbf{u}}_z & \hat{\mathbf{u}}_y \\ \hat{\mathbf{u}}_z & 0 & -\hat{\mathbf{u}}_x \\ -\hat{\mathbf{u}}_y & \hat{\mathbf{u}}_x & 0 \end{pmatrix} = \epsilon_{ijk}\hat{\mathbf{u}}_j \ \longleftrightarrow\ \hat{\mathbf{u}} = \begin{pmatrix} \hat{\mathbf{u}}_x \\ \hat{\mathbf{u}}_y \\ \hat{\mathbf{u}}_z \end{pmatrix} \tag{4.136}$$

- The skew-symmetric matrix commutator will be identified with vector product[16]

$$[\mathcal{U},\mathcal{V}] = \mathcal{U}\,\mathcal{V} - \mathcal{V}\,\mathcal{U} = \epsilon_{ijk}(\hat{\mathbf{u}}\times\hat{\mathbf{v}}) \ \longleftrightarrow\ \hat{\mathbf{u}}\times\hat{\mathbf{v}} \tag{4.137}$$

As a consequence $\mathbb{R}^3$ will be endowed with a Lie structure represented by cross product.

In the same time we remark that an operator $\mathcal{U}$ can be construe also as an operator of "cross product with the unit vector $\hat{\mathbf{u}}$" acting on $\mathbb{R}^3$.

$$\mathcal{U}\ \hat{\mathbf{v}} = \hat{\mathbf{u}}\times\hat{\mathbf{v}} \tag{4.138}$$

such a notation will be often used.

Operator $\mathcal{B}$ Let us now consider the action of the operator $\mathcal{B} = \epsilon_{ijk}\hat{\mathbf{b}}_j$, on the rotating basis vectors $\hat{\perp}$ and $\hat{\boldsymbol{\rho}}$.

Its action is cyclic:

$$\begin{aligned} \mathcal{B}\hat{\perp} &\equiv \epsilon_{ijk}\ \hat{\mathbf{b}}_j\ \hat{\perp}_k = \hat{\mathbf{b}}\times\hat{\perp} = \hat{\boldsymbol{\rho}} & (4.139)\\ \mathcal{B}\hat{\boldsymbol{\rho}} &\equiv \hat{\mathbf{b}}\times\hat{\boldsymbol{\rho}} = -\hat{\perp} & (4.140)\end{aligned}$$

[16] Here $\mathcal{U} = \epsilon_{ijk}\hat{\mathbf{u}}_j$ and $\mathcal{V} = \epsilon_{\mu\nu\rho}\hat{\mathbf{v}}_\rho$ we have

$$(\mathcal{U}\mathcal{V})_{i\rho} = \epsilon_{ijk}\epsilon_{k\nu\rho}\hat{\mathbf{u}}_j\hat{\mathbf{v}}_\rho = \epsilon_{kij}\epsilon_{k\nu\rho}\hat{\mathbf{u}}_j\hat{\mathbf{v}}_\rho = (\delta_{i\nu}\delta_{j\rho} - \delta_{i\rho}\delta_{j\nu})\,\hat{\mathbf{u}}_j\hat{\mathbf{v}}_\nu = \hat{\mathbf{v}}_i\hat{\mathbf{u}}_\rho - \delta_{i\rho}\hat{\mathbf{u}}_j\hat{\mathbf{v}}_j$$

$$(\mathcal{V}\mathcal{U})_{\mu k} = \epsilon_{\mu\nu\rho}\epsilon_{\rho jk}\hat{\mathbf{v}}_\nu\hat{\mathbf{u}}_j = (\delta_{\mu j}\delta_{\nu k} - \delta_{\mu k}\delta_{\nu j})\,\hat{\mathbf{v}}_\nu\hat{\mathbf{u}}_j = \hat{\mathbf{u}}_\mu\hat{\mathbf{v}}_k - \delta_{\mu k}\hat{\mathbf{v}}_i\hat{\mathbf{u}}_i$$

Then the matrix commutator

$$[\mathcal{U},\mathcal{V}] \equiv \hat{\mathbf{v}}_i\hat{\mathbf{u}}_\rho - \hat{\mathbf{u}}_i\hat{\mathbf{v}}_\rho \equiv \epsilon_{li\rho}\epsilon_{lmk}\hat{\mathbf{v}}_m\hat{\mathbf{u}}_k$$

Then the operator $\mathcal{B}^2$ acts as projector on the perpendicular to the magnetic field line direction : $-\mathbf{I}_\perp = \mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}$

$$\begin{aligned} \mathcal{B}^2 \hat{\perp} &= -\hat{\perp} && (4.141) \\ \mathcal{B}^2 \hat{\boldsymbol{\rho}} &= -\hat{\boldsymbol{\rho}} && (4.142) \end{aligned}$$

The operator $\mathcal{B}^3$ acts as an operator $-\mathcal{B}$:

$$\begin{aligned} \mathcal{B}^3 \hat{\perp} &= -\hat{\boldsymbol{\rho}} && (4.143) \\ \mathcal{B}^3 \hat{\boldsymbol{\rho}} &= \hat{\perp} && (4.144) \end{aligned}$$

This property give us the possibility to introduce the operator of rotation on angle α around the direction $\hat{\mathbf{b}}$ (in other words to pass from algebra to the group) as follows:

$$e^{\alpha\mathcal{B}} = \mathbf{1} - (\cos\alpha - 1)\ \mathcal{B}^2 + (\sin\alpha)\ \mathcal{B} \tag{4.145}$$

To proof the formula (4.145) we need just to decompose the operator $e^{\alpha\mathcal{B}}$ into the Taylor series and then to use the property 4.142, 4.144 in order to sum the series.

$$e^{\alpha\mathcal{B}} = \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!}\ \mathcal{B}^{2n} + \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!}\ \mathcal{B}^{2n+1} \tag{4.146}$$

Now we generalize the proprieties 4.142, 4.144:

$$\begin{aligned} \mathcal{B}^{2n+1} &= (-1)^n\ \mathcal{B} && (4.147) \\ \mathcal{B}^{2n} &= (-1)^{n+1}\ \mathcal{B}^2 && (4.148) \end{aligned}$$

So

$$e^{\alpha\mathcal{B}} = 1 + \underbrace{\sum_{n=1}^{\infty} (-1)^{n+1}\ \frac{\alpha^{2n}}{(2n)!}}_{=1-\cos\alpha}\ \mathcal{B}^2 + \underbrace{\sum_{n=0}^{\infty} (-1)^n\ \frac{\alpha^{2n+1}}{(2n+1)!}}_{=\sin\alpha}\ \mathcal{B} \tag{4.149}$$

Operator $\mathcal{D}$

To deal with separation of scales of motion, we need to define an operator that acts on the observables (functions on phase space) by rotating its arguments in a perpendicular plane to the magnetic direction $\hat{\mathbf{b}}$.

Operator $\mathcal{D}$. Definition on the intrinsic basis vectors Let us consider a scalar differential operator

$$(\hat{\mathbf{p}} \times \hat{\mathbf{b}}) \cdot \partial_{\hat{\mathbf{p}}} \equiv \mathcal{D} \tag{4.150}$$

such an operator appears in the r.h.s. of the general Hamiltonian normal form equation (4.119). Moreover the procedure of solution of this equation comes down to inversion of the operator $\mathcal{D}$. This is why it is important to learn the proprieties of this operator.

Let us start with definition of the operator $\mathcal{D}$ through its action on the basis vectors $(\hat{\mathbf{b}}, \hat{\boldsymbol{\rho}}, \hat{\perp})$.

Proposition 3

$$\begin{aligned} \mathcal{D}\hat{\boldsymbol{\rho}} &= \hat{\perp} & (4.151)\\ \mathcal{D}\hat{\perp} &= -\hat{\boldsymbol{\rho}} & (4.152) \end{aligned}$$

Proof 3 *Here (4.121):*

$$\mathcal{D}\ \hat{\boldsymbol{\rho}} = (\hat{\mathbf{p}} \times \hat{\mathbf{b}}) \cdot \partial_{\hat{\mathbf{p}}} \left[\frac{\hat{\mathbf{b}} \times \hat{\mathbf{p}}}{\sqrt{1-(\hat{\mathbf{b}} \cdot \hat{\mathbf{p}})^2}} \right] \tag{4.153}$$

We need to calculate the following derivative:

$$\begin{aligned} \frac{\partial\ \hat{\boldsymbol{\rho}}_i}{\partial\ \hat{\mathbf{p}}_l} &= \partial_{\hat{\mathbf{p}}_l} \left[\frac{\epsilon_{ijk}\ \hat{\mathbf{b}}_j\ \hat{\mathbf{p}}_k}{\sqrt{1-(\hat{\mathbf{b}}_n\ \hat{\mathbf{p}}_n)^2}} \right] & (4.154)\\ = & \frac{1}{\sqrt{1-(\hat{\mathbf{b}}_n\ \hat{\boldsymbol{\rho}}_n)^2}}\ \epsilon_{lij}\ \hat{\mathbf{b}}_j + \frac{1}{\left(1-(\hat{\mathbf{b}}_n\ \hat{\boldsymbol{\rho}}_n)^2\right)^{3/2}}\ \hat{\mathbf{b}}_{\ l}\ (\hat{\mathbf{b}}_n\ \hat{\mathbf{p}}_n)\ \underbrace{\epsilon_{ijk}\ \hat{\mathbf{b}}_j\ \hat{\mathbf{p}}_k}_{\sin\varphi\ \hat{\boldsymbol{\rho}}_i} \end{aligned}$$

Second, with (4.124)we remark that

$$\hat{\mathbf{p}} \times \hat{\mathbf{b}} = \hat{\perp} \times \hat{\mathbf{b}} \sin\varphi = -\hat{\boldsymbol{\rho}}\ \sqrt{1-(\hat{\mathbf{b}} \cdot \hat{\mathbf{p}})^2} \tag{4.155}$$

Finally

$$\mathcal{D}\ \hat{\boldsymbol{\rho}}_i = -\epsilon_{lij}\ \hat{\boldsymbol{\rho}}_l\ \hat{\mathbf{b}}_j = -\epsilon_{jli}\ \hat{\mathbf{b}}_j\ \hat{\boldsymbol{\rho}}_l = \hat{\perp}_i \tag{4.156}$$

■

Then it turns out that the action of the operator $\mathcal{D}$ on the basis vectors is cyclic and similar to the action of the operator $-\mathcal{B} = -\epsilon_{ijk}\hat{\mathbf{b}}_j$ "cross product with vector $\hat{\mathbf{b}}$":

$$\begin{aligned} \mathcal{D}^2\hat{\perp} &= -\mathcal{D}\hat{\boldsymbol{\rho}} = -\hat{\perp} & (4.157)\\ \mathcal{D}^2\hat{\boldsymbol{\rho}} &= \mathcal{D}\hat{\perp} = -\hat{\boldsymbol{\rho}} & (4.158) \end{aligned}$$

$$
\begin{aligned}
\mathcal{D}^3 \hat{\perp} &= -\mathcal{D}\hat{\perp} = \hat{\boldsymbol{\rho}} && (4.159)\\
\mathcal{D}^3 \hat{\boldsymbol{\rho}} &= -\mathcal{D}\hat{\boldsymbol{\rho}} = -\hat{\perp} && (4.160)
\end{aligned}
$$

Another interesting property of the operator $\mathcal{D}$ is summarized in the following proposition.

Proposition 4

$$
\mathcal{D} = (\hat{\mathbf{p}} \times \hat{\mathbf{b}}) \cdot \partial_{\hat{\mathbf{p}}} = (\hat{\boldsymbol{\rho}} \times \hat{\mathbf{b}}) \cdot \partial_{\hat{\boldsymbol{\rho}}} = (\hat{\perp} \times \hat{\mathbf{b}}) \cdot \partial_{\hat{\perp}} \tag{4.161}
$$

Proof 4 *Because*

$$
\frac{\partial}{\partial \hat{\mathbf{p}}_l} = \frac{\partial \hat{\boldsymbol{\rho}}_i}{\partial \hat{\mathbf{p}}_l}\, \frac{\partial}{\partial \hat{\perp}_i} + \frac{\partial \varphi}{\partial \hat{\mathbf{p}}_l}\, \frac{\partial}{\partial \varphi} \tag{4.162}
$$

By using the formula (4.154)

$$
\frac{\partial \hat{\boldsymbol{\rho}}_k}{\partial\, \hat{\mathbf{p}}_l} = \frac{\epsilon_{ilk}\, \hat{\mathbf{b}}_i}{\sin\varphi} + \Phi\, \frac{\hat{\mathbf{b}}_l\, \hat{\boldsymbol{\rho}}_k}{\sin\varphi} \tag{4.163}
$$

then we have

$$
\begin{aligned}
\mathcal{D} &= (\hat{\mathbf{p}} \times \hat{\mathbf{b}}) \cdot \partial_{\hat{\mathbf{p}}} = -\hat{\boldsymbol{\rho}}_l\, \epsilon_{ljk}\, \hat{\mathbf{b}}_j\, \frac{\partial}{\partial \hat{\boldsymbol{\rho}}_k} && (4.164)\\
&= -\epsilon_{jlk}\, \hat{\mathbf{b}}_j \hat{\boldsymbol{\rho}}_l\, \partial_{\hat{\boldsymbol{\rho}}_k} = (\hat{\boldsymbol{\rho}} \times \hat{\mathbf{b}}) \cdot \partial_{\hat{\boldsymbol{\rho}}} \equiv \hat{\perp} \cdot \partial_{\hat{\boldsymbol{\rho}}}
\end{aligned}
$$

By analogy we can proof that:

$$
\mathcal{D} = (\hat{\perp} \times \hat{\mathbf{b}}) \cdot \partial_{\hat{\perp}} \equiv \hat{\boldsymbol{\rho}} \cdot \partial_{\hat{\perp}} \tag{4.165}
$$

Then we can say that the action of the operator $\mathcal{D}$ on the observables involves only the derivatives over the vectors $(\hat{\perp}, \hat{\boldsymbol{\rho}})$ perpendicular to the magnetic field.

Operator $\mathcal{D}$. Intuitive definition To give an intuition for the origin of the operator $\mathcal{D}$, let us return for a while to the local momentum coordinates (φ, ζ) given by eq.(4.21) involving some fixed basis vectors $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2)$. How is expressed the operator $\mathcal{D}$ in this case ?

The transition between the intrinsic phase space variables and the local phase space variables can be expressed as follows:

$$
\begin{pmatrix} \mathbf{x} \\ \hat{\mathbf{p}} \\ \mathcal{A} \end{pmatrix} \rightarrow \begin{pmatrix} \widetilde{\mathbf{x}} \\ \varphi \\ \zeta \\ \widetilde{\mathcal{A}} \end{pmatrix} \tag{4.166}
$$

with $\mathbf{x} = \widetilde{\mathbf{x}}$ and $\mathcal{A} = \widetilde{\mathcal{A}}$. Now the unit momentum vector is presented as

$$
\hat{\mathbf{p}} = \hat{\mathbf{b}} \cos\phi + \hat{\perp} \sin\phi \tag{4.167}
$$

and the expressions for rotating vectors $(\hat{\perp}, \hat{\boldsymbol{\rho}})$ are taken the same that in (4.16): $\hat{\perp} = -\hat{\mathbf{b}}_1 \sin\zeta - \hat{\mathbf{b}}_2 \cos\zeta$ and $\hat{\boldsymbol{\rho}} = \hat{\mathbf{b}}_1 \cos\zeta - \hat{\mathbf{b}}_2 \sin\zeta$.

This local rotating basis has the same organization: $\hat{\mathbf{b}} \times \hat{\perp} = \hat{\boldsymbol{\rho}}, \quad \hat{\mathbf{b}} \times \hat{\boldsymbol{\rho}} = -\hat{\perp}$ that the intrinsic basis defined in (4.122). To obtain the expression for the operator $\mathcal{D}$ in local coordinates, we should first proceed with the one for the differential operator $\partial/\partial\hat{\mathbf{p}}$, by applying the chain rule:

$$\frac{\partial}{\partial\hat{\mathbf{p}}} = \frac{\partial\phi}{\partial\hat{\mathbf{p}}}\frac{\partial}{\partial\phi} + \frac{\partial\zeta}{\partial\hat{\mathbf{p}}}\frac{\partial}{\partial\zeta} + \frac{\partial\widetilde{\mathcal{A}}}{\partial\hat{\mathbf{p}}}\frac{\partial}{\partial\widetilde{\mathcal{A}}} + \frac{\partial\widetilde{\mathbf{x}}}{\partial\hat{\mathbf{p}}}\frac{\partial}{\partial\widetilde{\mathbf{x}}} \tag{4.168}$$

Accordingly to the basis definition the local momentum coordinates are expressed as:

$$\phi = \arctan\frac{\hat{\mathbf{p}}\cdot\hat{\perp}}{\hat{\mathbf{p}}\cdot\hat{\mathbf{b}}}, \quad \zeta = \arctan\frac{\hat{\mathbf{b}}_1\cdot\hat{\mathbf{p}}}{\hat{\mathbf{b}}_2\cdot\hat{\mathbf{p}}} \tag{4.169}$$

Then its derivatives over the unit momentum variable are:

$$\frac{\partial\varphi}{\partial\hat{\mathbf{p}}} = \hat{\perp}\cos\varphi - \hat{\mathbf{b}}\sin\varphi, \quad \frac{\partial\zeta}{\partial\hat{\mathbf{p}}} = -\frac{\hat{\boldsymbol{\rho}}}{\sin\varphi} \tag{4.170}$$

Due to the fact that the change of variables (4.166) maps $\mathcal{A} = \widetilde{\mathcal{A}}$, $\mathbf{x} = \widetilde{\mathbf{x}}$ and because of the independence of the variables $\mathcal{A}, \mathbf{x}$ from $\hat{\mathbf{p}}$, the last two terms in the expression (4.168) become equal to zero.

Using this information we have:

$$\frac{\partial}{\partial\hat{\mathbf{p}}} = (\hat{\perp}\cos\varphi - \hat{\mathbf{b}}\sin\varphi)\frac{\partial}{\partial\varphi} - \frac{\hat{\boldsymbol{\rho}}}{\sin\varphi}\frac{\partial}{\partial\zeta} \tag{4.171}$$

Then with $\hat{\mathbf{p}} \times \hat{\mathbf{b}} = -\hat{\boldsymbol{\rho}}\sin\varphi$, the operator $\mathcal{D}$ can be expressed as follows:

$$(\hat{\mathbf{p}} \times \hat{\mathbf{b}}) \cdot \partial_{\hat{\mathbf{p}}} = (-\hat{\boldsymbol{\rho}}\sin\varphi) \cdot \left[(\hat{\perp}\cos\varphi - \hat{\mathbf{b}}\sin\varphi)\frac{\partial}{\partial\varphi} - \frac{\hat{\boldsymbol{\rho}}}{\sin\varphi}\frac{\partial}{\partial\zeta}\right] = \frac{\partial}{\partial\zeta} \tag{4.172}$$

Finally, we have find that the scalar differential operator $\mathcal{D}$ in local momentum coordinates is equal to the derivative over the gyroangle ζ.

Note that the expression of the operator $\mathcal{D} = \partial/\partial\zeta$ does not depend on choice of the fixed basis $(\hat{\mathbf{b}}_0\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2)$, but in the same time we emphasize that the definition of the angle ζ will be dependent on this choice. By analogy with operator $e^{\alpha\mathcal{D}}$ one can introduce an operator $e^{\alpha\frac{\partial}{\partial\zeta}}$ that acts similarly.

Operator $\exp(\alpha\,\mathcal{D})$**. Action on the observables** By analogy with the operator of rotation $e^{\alpha\,\mathcal{B}}$ defined in (4.145), we now introduce the operator $e^{\alpha\,\mathcal{D}}$.

As was shown in 4.5.3, the operator $\mathcal{D}$ is equivalent to the operator $-\mathcal{B}$. Due to that we obtain the action of $e^{\alpha\,\mathcal{D}}$ on $(\hat{\perp}, \hat{\boldsymbol{\rho}}, \hat{\mathbf{b}})$:

$$e^{\alpha\,\mathcal{D}}\;\hat{\boldsymbol{\rho}} = \big(\mathbf{1} - (\cos\alpha - 1)\;\mathcal{D}^2 - \sin\alpha\,\mathcal{D}\big)\;\hat{\boldsymbol{\rho}} = \cos\alpha\;\hat{\boldsymbol{\rho}} - \sin\alpha\;\hat{\boldsymbol{\perp}} \equiv e^{-\alpha\,\mathcal{B}}\;\hat{\boldsymbol{\rho}} \quad (4.173)$$

$$e^{\alpha\,\mathcal{D}}\;\hat{\boldsymbol{\perp}} = \big(\mathbf{1} - (\cos\alpha - 1)\;\mathcal{D}^2 - \sin\alpha\,\mathcal{D}\big)\;\hat{\boldsymbol{\perp}} = \cos\alpha\hat{\boldsymbol{\perp}} + \sin\alpha\;\hat{\boldsymbol{\rho}} \equiv e^{-\alpha\,\mathcal{B}}\hat{\boldsymbol{\perp}} \quad (4.174)$$

Similarly to $e^{\alpha\mathcal{B}}$, the operator $e^{\alpha\mathcal{D}}$ does not affect parallel to the magnetic field vector $\hat{\mathbf{b}}$:

$$e^{\alpha\,\mathcal{D}}\;\hat{\mathbf{b}} = \big(\mathbf{1} - (\cos\alpha - 1)\;\mathcal{D}^2 - \sin\alpha\,\mathcal{D}\big)\;\hat{\mathbf{b}} = e^{\alpha\,\mathcal{B}}\hat{\mathbf{b}} \equiv \hat{\mathbf{b}} \quad (4.175)$$

Due to the relation (4.161) between the operators $\mathcal{D}$ and $\mathcal{B}$

$$\mathcal{D} = \hat{\mathbf{p}}\cdot\mathcal{B}\cdot\frac{\partial}{\partial\hat{\mathbf{p}}} = \hat{\boldsymbol{\rho}}\cdot\mathcal{B}\cdot\frac{\partial}{\partial\hat{\boldsymbol{\rho}}} = \hat{\boldsymbol{\perp}}\cdot\mathcal{B}\cdot\frac{\partial}{\partial\hat{\boldsymbol{\perp}}} \quad (4.176)$$

the operator $\mathcal{D}$ can be treated as a generator of dilatations.

Then the action of $\exp(\alpha\;\mathcal{D})$ on an observable $f(\mathbf{x},\hat{\mathbf{p}},\mathcal{A})$ can be expressed as:

$$\exp\left(\hat{\mathbf{p}}\cdot\mathcal{B}\cdot\frac{\partial}{\partial\hat{\mathbf{p}}}\right)\;f(\mathbf{x},\hat{\mathbf{p}},\mathcal{A}) = f(\mathbf{x},e^{-\alpha\,\mathcal{B}}\hat{\mathbf{p}},\mathcal{A}) \quad (4.177)$$

It means that when applied to an observable $f(\mathbf{x},\hat{\mathbf{p}},\mathcal{A})$, the operator $e^{\alpha\mathcal{D}}$ affects only the arguments dependent on $\hat{\mathbf{p}}$ or $\hat{\boldsymbol{\perp}},\hat{\boldsymbol{\rho}}$. Its action comes down to rotation of these vectors through the angle $-\alpha$ around the direction $\hat{\mathbf{b}}$.

The proof of this property can be realized by decomposing the observable in series as follows.

$$f(\hat{\boldsymbol{\perp}}) = f(\vec{0}) + \sum_{n=0}^{\infty}\frac{\hat{\boldsymbol{\perp}}^{\otimes n}}{n!}\circ\partial_{\hat{\boldsymbol{\perp}}}^{\otimes n}f|_{\hat{\boldsymbol{\perp}}=\vec{0}} \quad (4.178)$$

$$\begin{aligned}
e^{\alpha\mathcal{D}}f(\hat{\boldsymbol{\perp}}) &= f(\,\vec{0}\,) + \sum_{n=1}^{\infty}\frac{e^{\alpha\mathcal{D}}\;\left(\hat{\boldsymbol{\perp}}_{i_1}\ldots\hat{\boldsymbol{\perp}}_{i_n}\right)}{n!}\left[\partial_{\hat{\boldsymbol{\perp}}_{i_1}}\ldots\;\partial_{\hat{\boldsymbol{\perp}}_{i_n}}f\right]|_{\hat{\boldsymbol{\perp}}=\vec{0}} \quad (4.179)\\
&= f(\,\vec{0}\,) + \sum_{n=1}^{\infty}\frac{e^{\alpha\mathcal{D}}\hat{\boldsymbol{\perp}}_{i_1}\ldots e^{\alpha\mathcal{D}}\hat{\boldsymbol{\perp}}_{i_n}}{n!}\left[\partial_{\hat{\boldsymbol{\perp}}_{i_1}}\ldots\;\partial_{\hat{\boldsymbol{\perp}}_{i_n}}f\right]|_{\hat{\boldsymbol{\perp}}=\vec{0}} \quad (4.180)\\
&= f(\,\vec{0}\,) + \sum_{n=1}^{\infty}\frac{e^{-\alpha\mathcal{B}}\hat{\boldsymbol{\perp}}_{i_1}\ldots e^{-\alpha\mathcal{B}}\hat{\boldsymbol{\perp}}_{i_n}}{n!}\left[\partial_{\hat{\boldsymbol{\perp}}_{i_1}}\ldots\;\partial_{\hat{\boldsymbol{\perp}}_{i_n}}f\right]|_{\hat{\boldsymbol{\perp}}=\vec{0}} \equiv f\left(e^{-\alpha\mathcal{B}}\hat{\boldsymbol{\perp}}\right) \quad (4.181)
\end{aligned}$$

In order to pass from formula 4.179 to 4.180 we have to proof that:

$$e^{\alpha\mathcal{D}}(\hat{\boldsymbol{\perp}}^{\otimes n}) = (e^{\alpha\mathcal{D}}\hat{\boldsymbol{\perp}})^{\otimes n} \quad (4.182)$$

To do this, we have to iterate Leibnitz rule for scalar differentiation $\mathcal{D}$. Each tensor can be considered as a collection of scalars. We will prove the property 4.182 for two scalars—to expand this proof for tensors it is sufficient to apply it to each component of this tensor. Let f and g be two tensors, so f_i and g_k are they coordinates with respect to the canonical basis in $\mathbb{R}^N$

$$e^{\alpha\mathcal{D}}\,(f \otimes g) = e^{\alpha\mathcal{D}} f \otimes e^{\alpha\mathcal{D}} g \tag{4.183}$$

in coordinates

$$e^{\alpha\mathcal{D}}\,(f_i\; g_k) = e^{\alpha\mathcal{D}} f_i\; g_k \tag{4.184}$$

with series expansion

$$\sum_{n=0}^{\infty} \frac{1}{n!}\,(\alpha\mathcal{D})^n f_i\; g_k = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{1}{m!}\,(\alpha\mathcal{D})^m\; f_i\; \frac{1}{(n-m)!}(\alpha\mathcal{D})^{n-m}\; g_k \tag{4.185}$$

now it is sufficient to prove by induction that

$$(\alpha\mathcal{D})^n(f_i\; g_k) = \sum_{m=0}^{n} C^n_m\;(\alpha\mathcal{D})^m f_i\;(\alpha\mathcal{D})^{n-m} g_k \tag{4.186}$$

for $n \to n+1$

$$C^{n+1}_m\;(\alpha\mathcal{D})^m f_i\;(\alpha\mathcal{D})^{n-m+1} g_k = (n+1)!\;\;(\alpha\mathcal{D})\left(\frac{1}{m!}(\alpha\mathcal{D})^m f_i\;\frac{1}{(n-m)!}(\alpha\mathcal{D})^{n-m} g_k\right) \tag{4.187}$$

this property can be obtained immediately by direct differentiation and then using the property for binomials $C^{n+1}_m = C^n_{m-1} + C^n_m$.

The main idea for formula (4.178) can be found below.

Series decomposition for an analytic function in a fully symmetric tensor space

Let $\mathbb{E}_N$ be an N-dimensional vector space with $\hat{\mathbf{e}}$ its basis vectors. To start let us consider a second order direct product of this space with itself $\mathbb{E}_N \otimes \mathbb{E}_N \equiv (\mathbb{E}_N)^2$. The basis in such a space can be composed by direct product of basis vectors $\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$. Then the direct product of any vector $\mathbf{v} \in \mathbb{E}_N$ with itself lies inside the fully symmetric subspace of $(\mathbb{E}_N)^2$ with basis vectors $\hat{\mathbf{e}}_i \vee \hat{\mathbf{e}}_j \equiv \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j + \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i$

$$\mathbf{v} \otimes \mathbf{v} = \sum \frac{1}{1+\delta_{ij}} v_i \; v_j \; \hat{\mathbf{e}}_i \vee \hat{\mathbf{e}}_j \tag{4.188}$$

Iterating the r-th power of $\mathbf{v}$ which lies within the symmetric subspace of $(\mathbb{E}_N)^r$ with basis

$$\mathbf{e}_{i_1} \vee \cdots \vee \mathbf{e}_{i_r} = \sum_{\text{all elements } \in P_r} \boldsymbol{\sigma} \; \hat{\mathbf{e}}_{i_1} \hat{\mathbf{e}}_{i_2} \ldots \hat{\mathbf{e}}_{i_r} \tag{4.189}$$

Here the sum is taken over all the elements in the group P_r and σ is its group operation of permutation. In the full symmetric basis definition all the permutations (ordered or not) are taken with the sign $+$. The components of $(\mathbf{v})^{\otimes r}$ with respect to the bases are of the form $(v_1)^{j_1} \ldots (v_N)^{j_N}$, with $j_1 + \ldots j_N = r$.

$$\mathbf{v}^{\otimes r} = \sum_{\text{all elements } \in P_r} \left(\sum_{j_1 + j_2 \ldots j_N = r} (v_1)^{j_1} \ldots (v_N)^{j_N} \right) \hat{\mathbf{e}}_{i_1} \vee \hat{\mathbf{e}}_{i_2} \vee \cdots \vee \hat{\mathbf{e}}_{i_r} \tag{4.190}$$

The polynomials $(v_1)^{j_1} \ldots (v_N)^{j_N}$ form a basis for the set of all analytic functions defined on $\mathbb{E}_N$. An analytic function can be represented by its power series expansion. Let $f(\mathbf{v}) \equiv f(v_1, \ldots, v_N)$ be a real valued, analytic scalar function (an observable) on

$$\begin{aligned} f(\mathbf{v}) = f(\,\vec{0}\,) + v_i \left[\partial_{v_i} f\right]|_{\mathbf{v}=\vec{0}} \;\; &+ \;\; v_i \; v_j \;\; \left[\partial_{v_i}\partial_{v_j} f\right]|_{\mathbf{v}=\vec{0}} + \ldots \\ &+ \;\; v_i \; v_j \; v_k \ldots \frac{\left[\partial_{v_i}\partial_{v_j}\partial_{v_k} \ldots f\right]}{n!}|_{\mathbf{v}=\vec{0}} \end{aligned} \tag{4.191}$$

In vector terms this formula can be rewritten as:

$$f(\mathbf{v}) = f(\vec{0}) + \sum_{n=1}^{\infty} \frac{\mathbf{v}^{\otimes n}}{n!} \circ \nabla^{\otimes n} f|_{\mathbf{v}=0} \tag{4.192}$$

where $\nabla = \partial_{\mathbf{v}}$ and the operation $\circ$ represents the operation of n - tensor contraction.

Operator $\mathcal{R}$ of intrinsic gyroaveraging. The introduction of the operator $\exp(\alpha\mathcal{D})$ permits us to define the operation of intrinsic gyroaveraging.

In fact to make the gyroaverage means to sum on all possible rotations around the magnetic field direction $\hat{\mathbf{b}}$. This action can be expressed by the operator:

$$\mathcal{R} \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha \ \exp(\alpha\mathcal{D}) \tag{4.193}$$

We can take this integral formally, considering $\mathcal{D}$ as the integration parameter:

$$\mathcal{R} = \frac{\sinh \pi\mathcal{D}}{\pi\mathcal{D}} \tag{4.194}$$

Note that this expression is not zero

$$\frac{\sinh \pi\mathcal{D}}{\pi\mathcal{D}} = \frac{\pi\mathcal{D} + (1/3!)(\pi\mathcal{D})^3 + \mathcal{O}(\mathcal{D}^5)}{\pi\mathcal{D}} \tag{4.195}$$

From here, it is easy to see that $\mathcal{R}\mathcal{D} = \mathcal{D}\mathcal{R} = 0$, because

$$\sinh \pi\mathcal{D} = 0 \tag{4.196}$$

In fact the last formula can be interpreted geometrically as a subtraction of rotations of argument of observable on the angle ζ and then on the angle $-\zeta$.

$$\begin{aligned} \sinh \pi\mathcal{D}\, f(\mathbf{x}, \hat{\perp}, \varphi, \mathcal{A}) &= \left[e^{\pi\mathcal{D}} - e^{-\pi\mathcal{D}}\right] f(\mathbf{x}, \hat{\perp}, \varphi, \mathcal{A}) \\ &= f(\mathbf{x}, e^{-\pi\mathcal{B}}\hat{\perp}, \varphi, \mathcal{A}) - f(\mathbf{x}, e^{\pi\mathcal{B}}\hat{\perp}, \varphi, \mathcal{A}) \end{aligned} \tag{4.197}$$

moreover accordingly to the formula (4.145) $e^{-\pi\mathcal{D}}\hat{\perp} = e^{\pi\mathcal{D}}\hat{\perp} = -\hat{\perp}$, so we have (4.196). On the other hand, the properties (4.195), (4.196) permit us to prove that the operator $\mathcal{R}$ is a projector. It is sufficient to show that $\mathcal{R}^2 = \mathcal{R}$, but

$$\mathcal{R}(1-\mathcal{R}) = \sinh \pi\mathcal{D} \left(\frac{\pi\mathcal{D} - \sinh \pi\mathcal{D}}{\pi^2\mathcal{D}^2}\right) = 0 \tag{4.198}$$

Now we can also introduce a complementary to $\mathcal{R}$ projector $\mathcal{N} = \mathbf{1} - \mathcal{R}$.

Finally, the application of the operator $\mathcal{R}$ to any observable $f(\mathbf{x}, \hat{\mathbf{p}}, \mathcal{A})$ gives its gyroaveraged part $\langle f(\mathbf{x}, \varphi, \hat{\perp}, \mathcal{A})\rangle$ and application of the operator $\mathcal{N}$ to the same observable, gives its fluctuating part$\widetilde{f}(\mathbf{x}, \varphi, \hat{\perp}, \mathcal{A})$. By considering this operators as a complementary projectors on the set of the averaged and fluctuating part of the observables correspondingly, for any observable we can make a decomposition:

$$\begin{aligned} f(\mathbf{x}, \hat{\perp}(\mathbf{x}), \varphi(\mathbf{x}), \mathcal{A}) &= \mathcal{R} f(\mathbf{x}, \hat{\perp}(\mathbf{x}), \varphi(\mathbf{x}), \mathcal{A}) + \mathcal{N} f(\mathbf{x}, \hat{\perp}(\mathbf{x}), \varphi(\mathbf{x}), \mathcal{A}) \\ &\equiv \langle f(\mathbf{x}, \varphi(\mathbf{x}); \mathcal{A})\rangle + \widetilde{f}(\mathbf{x}, \hat{\perp}(\mathbf{x}), \varphi(\mathbf{x}), \mathcal{A}) \end{aligned} \tag{4.199}$$

Pseudo-inverse operator $\mathcal{G}$ As we can see in previous paragraph, the operator $\mathcal{D}$ has a non-zero kernel $\mathcal{DR} = 0$, composed by all the observables that do not depend on $\hat{\perp}$ and $\hat{\boldsymbol{\rho}}$. So it can not be inverted on the set of all the observables defined on the phase space $(\mathbf{x}, \hat{\perp}(\mathbf{x}), \varphi(\mathbf{x}), \mathcal{A})$. However its left inverse $\mathcal{G}$ can be defined as follows: $\mathcal{GD} = \mathcal{N} \equiv \mathbf{1} - \mathcal{R}$.

Now using the spectral expression for the operator $\mathcal{N}$ we have formally:

$$\mathcal{GD} = \mathcal{N}, \Rightarrow \mathcal{G} = \frac{\pi\mathcal{D} - \sinh \pi\mathcal{D}}{\pi\mathcal{D}^2} \tag{4.200}$$

This formal expression for the operator $\mathcal{G}$ can be also rewritten into the integral form.

Theorem 4

$$\mathcal{G} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha\ (\alpha - \pi \mathrm{sign}\alpha)\ \exp(\alpha\mathcal{D}) \tag{4.201}$$

We can then define its action on the basis vectors

$$\mathcal{G}\hat{\perp} = \hat{\boldsymbol{\rho}} \tag{4.202}$$

$$\mathcal{G}\hat{\boldsymbol{\rho}} = -\hat{\perp} \tag{4.203}$$

Note that the action of the intrinsic operator $\mathcal{G}$ on rotating basis vectors is similar to the action of the operator $\mathcal{G}$ (4.66)in the non-intrinsic case. In fact with $\hat{\perp} = -\hat{\mathbf{b}}_1 \sin\zeta - \hat{\mathbf{b}}_2 \cos\zeta$ and $\hat{\boldsymbol{\rho}} = \hat{\mathbf{b}}_1 \cos\zeta - \hat{\mathbf{b}}_2 \sin\zeta$.

$$\int^{\zeta} d\zeta\ \hat{\perp} = \hat{\boldsymbol{\rho}} \ \text{ and } \ \int^{\zeta} d\zeta\ \hat{\boldsymbol{\rho}} = -\hat{\perp} \tag{4.204}$$

4.6 Intrinsic Hamiltonian normal form equation

Let us now return to consideration of the general Hamiltonian normal form equation (4.119). Now we will make use of the intrinsic tools introduced above for its resolution.

First we rewrite the r.h.s. of this equation by introducing the scalar differential operator[17] $\mathcal{D} = (\hat{\mathbf{p}} \times \hat{\mathbf{b}}) \cdot \partial_{\hat{\mathbf{p}}}$:

$$\frac{p}{eB} \underbrace{\hat{\mathbf{p}} \cdot \nabla p}_{\text{slow variables derivative}} = - \underbrace{\mathcal{D}p}_{\text{fast variable derivative}} \tag{4.205}$$

That after the introduction of the intrinsic basis in coordinates $(\mathbf{x}', \varphi, \hat{\perp}, \mathcal{A}')$ became:

$$\frac{p}{eB}\left(\hat{\mathbf{b}}(\mathbf{x}')\sin\varphi + \hat{\perp}\cos\varphi\right) \cdot \left(\partial_{\mathbf{x}'} + \partial_{\mathbf{x}}\varphi\,\partial_{\phi} + \partial_{\mathbf{x}}\hat{\perp}\cdot\partial_{\hat{\perp}}\right) p = -(\hat{\perp} \times \hat{\mathbf{b}})\partial_{\hat{\perp}} p \tag{4.206}$$

[17]In what follows we define the differential operator ∇ as: $\nabla \equiv \partial_{\mathbf{x}} = \partial_{\mathbf{x}'} + \partial_{\mathbf{x}}\varphi\,\partial_{\phi} + \partial_{\mathbf{x}}\hat{\perp}\cdot\partial_{\hat{\perp}}$

that can be also interpreted by using the variables $(\mathbf{x}, \varphi(\mathbf{x},)\hat{\perp}(\mathbf{x}))$ as:

$$\frac{p}{eB}\left(\hat{\mathbf{b}}(\mathbf{x})\sin\varphi(\mathbf{x}) + \hat{\perp}(\mathbf{x})\cos\varphi(\mathbf{x})\right)\cdot\partial_{\mathbf{x}}p = -(\hat{\mathbf{p}}\times\hat{\mathbf{b}}(\mathbf{x}))\cdot\partial_{\hat{\mathbf{p}}}p \tag{4.207}$$

The operator $\mathcal{D}$ involves the derivatives only on the perpendicular to magnetic field directions $(\hat{\perp}, \hat{\boldsymbol{\rho}})$. As we can see in 4.5.3 $\mathcal{D}$ is a fixed-basis-independent representation of the differentiation over the gyroangle. In intrinsic basis it becomes:

$$\mathcal{D} = (\hat{\boldsymbol{\rho}}\times\hat{\mathbf{b}})\cdot\partial_{\hat{\boldsymbol{\rho}}} = (\hat{\perp}\times\hat{\mathbf{b}})\cdot\partial_{\hat{\perp}} \dashleftarrow\dashrightarrow \frac{\partial}{\partial\zeta} \tag{4.208}$$

Moreover we remark that the l.h.s. of the equation (4.205) contains small parameter $\varepsilon = p/eB$. The separation of dynamical scales appears naturally.

4.6.1 Solution

Similarly to the general equation for the constant of motion in the non-intrinsic case (4.59), an iterative procedure for the resolution of (4.205) can be implemented by using the intrinsic operators $\mathcal{R}$ defined in (4.193) and $\mathcal{G}$ defined in (4.201).

Iterative procedure Application of the operator $\mathcal{R}$ to the r.h.s. and the l.h.s. of (4.205) gives us the equation for the averaged part of rescaled Hamiltonian p :

$$\mathcal{R}\left(\frac{p}{eB}\,\hat{\mathbf{p}}\cdot\nabla p\right) = 0 \tag{4.209}$$

Note that this equation can be interpreted as a solvability condition of the general equation (4.205).

Application of the operator $\mathcal{G}$ to the r.h.s. and the l.h.s. gives us the equation for the fluctuating of p :

$$\mathcal{N}p = \mathcal{G}\left(\frac{p}{eB}\,\hat{\mathbf{p}}\cdot\nabla p\right) \tag{4.210}$$

The first step leads to the expression for the gyroaveraged part of p at the order n. The second step consists to obtain the fluctuating part of p at the order $n+1$. As in the local case we suppose that the rescaled Hamiltonian p is independent of the gyroangle ζ at the zeroth order of ϵ.

At any following order we suppose that

$$p = \langle p\rangle(\mathbf{x}', \varphi) + \widetilde{p}(\mathbf{x}', \varphi, \hat{\perp}) \longleftrightarrow p = \langle p\rangle(\mathbf{x}, \varphi(\mathbf{x})) + \widetilde{p}(\mathbf{x}, \varphi(\mathbf{x}), \hat{\perp}(\mathbf{x})) \tag{4.211}$$

Then the corresponding iterative procedure can be organized as follows:

$$\begin{aligned}
\varepsilon^0 \quad &: \ \mathcal{D}p_0 &&= 0 && (4.212)\\
\varepsilon^1 \quad &: \ -\mathcal{D}p_1 &&= \frac{p_0}{eB}\hat{\mathbf{p}}\cdot\nabla p_0 && (4.213)\\
\varepsilon^2 \quad &: \ -\mathcal{D}p_2 &&= \frac{\widetilde{p}_1}{eB}\hat{\mathbf{p}}\cdot\nabla p_0 + \frac{\langle p_1\rangle}{eB}\hat{\mathbf{p}}\cdot\nabla\hat{\mathbf{p}}_0 && (4.214)\\
& &&\quad + \frac{p_0}{eB}\hat{\mathbf{p}}\cdot\nabla\widetilde{p}_1 + \frac{p_0}{eB}\hat{\mathbf{p}}\cdot\nabla\langle p_1\rangle && (4.215)
\end{aligned}$$

In what follows all the spatial derivatives will be taken over the variable $\mathbf{x}$, then the rotating frame vectors and the pitch angle will be supposed dependent on $\mathbf{x}$.

First order solution

In previous section the zeroth order constant of motion $\mathcal{A}_0 \equiv \langle \mathcal{A}_0 \rangle$ was obtained by applying the separation of variables method to first order partial differential equation (4.59). In order to make the connection with the familiar expression for magnetic momentum $\mu_0 = (p^2 \sin\varphi^2)/2B \equiv 2m\mathcal{A}_0$, we start our series decomposition for the Hamiltonian p into the new variables $(\mathbf{x}, \varphi(\mathbf{x}), \hat{\perp}(\mathbf{x}), \mathcal{A})$ with[18]

$$p_0 = p_0(\mathbf{x}, \phi(\mathbf{x}), \hat{\perp}(\mathbf{x})) = \frac{\sqrt{\mathcal{A}B(\mathbf{x})}}{\sin\varphi(\mathbf{x})} e^{-\varrho(\mathbf{x})/2} \tag{4.216}$$

where $\varrho = \varrho(\mathbf{x})$ is some function of the space coordinate. It will be needed in order to obtain the second order terms. Its nature will be discussed below.

Note that the implementation of such an ansatz into the iterative procedure (4.215) for rescaled Hamiltonian p leads to its series expansion in powers of the variable $\mathcal{A}^{1/2}$. In what follows we deal with construction of its two first orders, i.e. we find the terms in $\mathcal{A}^{1/2}$ and $\mathcal{A}$.

By introducing p_0 into the general equation (4.205) for Hamiltonian normal form, at the first order, we have

$$\frac{p_0}{eB}\, \hat{\mathbf{p}} \cdot \nabla p_0 = -\mathcal{D} p_1 \tag{4.217}$$

By expanding ∇p_0

$$\nabla p_0 = \frac{\sqrt{\mathcal{A}B}}{\sin\varphi} e^{-\varrho/2} \left(\frac{\nabla B}{2B} - \frac{\nabla \varrho}{2} + \Phi \nabla \hat{\mathbf{b}} \cdot \hat{\perp} \right) \tag{4.218}$$

and then by substituting it into (4.217) we have:

$$\frac{p_0}{eB}\, \hat{\mathbf{p}} \cdot \nabla p_0 = \frac{\mathcal{A} e^{-\varrho}}{e \sin^2\varphi} (\hat{\mathbf{b}} \cos\varphi + \hat{\perp} \sin\varphi) \cdot \left(\frac{\nabla B}{2B} - \frac{\nabla \varrho}{2} + \Phi \nabla \hat{\mathbf{b}} \cdot \hat{\perp} \right) \tag{4.219}$$

This equation give us the possibility to obtain the fluctuating part, that we call $\widetilde{p}_1$ for the first order in $\mathcal{A}$ Hamiltonian. The averaged part of the first order Hamiltonian $\langle p_1 \rangle$ can be obtained when considering the second order equation. This calculation is considered in the following section.

Here we deal with the solution of the first order equation. Due to the fact that $\mathcal{RD} = 0$, the gyroaverage of the r.h.s. of the equation (4.205) is always equal to

[18] Here we have inverted the functional dependence of zeroth order constant of motion and the zeroth order Hamiltonian:

$$\mathcal{A}_0 = \frac{(p \sin\varphi)^2}{B} \longleftrightarrow p_0 = \frac{\sqrt{\mathcal{A}B}}{\sin\varphi}$$

zero. This implies the necessity to verify that the gyroaverage of the l.h.s. of the same equation is also equal to zero. The gyroaverage of the both parts of the (4.219) leads to the condition for the function ϱ:

$$\hat{\mathbf{b}} \cdot \nabla \varrho = 0 \tag{4.220}$$

In fact, with $\mathcal{R}\,(\hat{\perp}\,\hat{\perp}) = 1/2\,(\hat{\perp}\,\hat{\perp} + \hat{\boldsymbol{\rho}}\,\hat{\boldsymbol{\rho}}\,)$ and $\nabla\hat{\mathbf{b}} \cdot \hat{\mathbf{b}} = 0$, we obtain:

$$\mathcal{R}\left(\frac{p_0}{eB}\,\hat{\mathbf{p}} \cdot \nabla p_0\right) = \tag{4.221}$$
$$\frac{\mathcal{A}e^{-\varrho}}{e\sin\varphi}\left(\Phi\frac{\hat{\mathbf{b}} \cdot \nabla B}{2B} - \Phi\frac{\hat{\mathbf{b}} \cdot \nabla\varrho}{2} + \frac{1}{2}\Phi\,(\hat{\mathbf{b}}\hat{\mathbf{b}} + \hat{\perp}\hat{\perp} + \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}}) : \nabla\hat{\mathbf{b}}\right)$$

After using the electromagnetic constraint $\nabla \cdot \mathbf{B} = \nabla \cdot (B\hat{\mathbf{b}}) = 0$, that can be rewritten as $\nabla \cdot \hat{\mathbf{b}} = -\dfrac{\hat{\mathbf{b}} \cdot \nabla B}{B}$, we obtain the condition (4.220).

The next step is to apply the operator $\mathcal{G}$ to the both parts of the equation (4.219), in order to obtain the fluctuation of the Hamiltonian $\widetilde{p}_1$ at the first order.

$$\widetilde{p}_1 = \mathcal{G}\left(\frac{p_0}{eB}\,\hat{\mathbf{p}} \cdot \nabla p_0\right) = \tag{4.222}$$
$$\frac{\mathcal{A}e^{-\varrho}}{e\sin\varphi}\left(-\Phi^2\,\hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}} : \nabla\hat{\mathbf{b}} - \frac{1}{2}\hat{\boldsymbol{\rho}} \cdot (\frac{\nabla B}{B} - \nabla\varrho) - \frac{1}{4}\Phi\,(\hat{\boldsymbol{\rho}}\hat{\perp} + \hat{\perp}\hat{\boldsymbol{\rho}}) : \nabla\hat{\mathbf{b}}\right)$$

Here we have used that $\mathcal{G}\hat{\boldsymbol{\rho}} = -\hat{\perp}$, $\mathcal{G}\hat{\perp} = \hat{\boldsymbol{\rho}}$, $\mathcal{G}\hat{\perp}\hat{\perp} = 1/4\,(\hat{\boldsymbol{\rho}}\hat{\perp} + \hat{\perp}\hat{\boldsymbol{\rho}})$.

At this stage the fluctuating part of the first order solution $\widetilde{p}_1$ have been obtained. Now we proceed with the second order differential equation in order to find the gyroaveraged part to the first order Hamiltonian $\langle p_1 \rangle$.

Obtaining $\langle p_1 \rangle$. Solution of the second order averaged equation

The procedure of the intrinsic gyroaveraging applied to the eq. (4.215) leads to the partial differential equation [19] [20]:

$$\cos\varphi\,\hat{\mathbf{b}} \cdot \partial_{\mathbf{x}'}\langle p_1 \rangle - \frac{1}{2}\,\sin\varphi\,\left(\nabla \cdot \hat{\mathbf{b}}\right)\,\partial_\varphi\langle p_1 \rangle =$$
$$\frac{\mathcal{A}\,e^{-\varrho}}{e}\,\left[\frac{\Phi^2}{2}\left[\hat{\mathbf{b}} \cdot \left(\nabla \times (\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}})\right) - \left(\hat{\mathbf{b}} \times \nabla\varrho\right) \cdot (\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}})\right] \right. \tag{4.223}$$
$$\left. + \frac{1}{4}\left[(\hat{\mathbf{b}} \times \nabla) \cdot \hat{\mathbf{b}}\right]\,\left[\nabla \cdot \hat{\mathbf{b}}\right]\right]$$

[19] Here we continue to distinguish two spatial positions $\mathbf{x}$ and $\mathbf{x}'$, before and after the projection on the intrinsic basis. We remind that $\nabla \equiv \partial_{\mathbf{x}} = \partial_{\mathbf{x}'} + \partial_{\mathbf{x}}\varphi\partial_\varphi + \partial_{\mathbf{x}}\hat{\perp} \cdot \partial_{\hat{\perp}}$

[20] Such an equation is consistent with the first order differential equation for the constant of motion $\mathcal{A}$ (4.70)

The details about its obtaining can be found in the Appendix G.1.

Now we are looking for the solution in the following form[21]:

$$\langle p_1 \rangle = \frac{\mathcal{A}\, e^{-\varrho}}{e \sin\varphi}\, \Phi\, \varpi(\mathbf{x}) \tag{4.224}$$

In this case

$$\cos\varphi\, \hat{\mathbf{b}} \cdot \partial_{\mathbf{x}} \langle p_1 \rangle = \frac{\mathcal{A}\, e^{-\varrho}}{e}\, \Phi^2\, \hat{\mathbf{b}} \cdot \partial_{\mathbf{x}} \varpi(\mathbf{x}) \tag{4.225}$$

$$\sin\varphi\, \partial_\varphi \langle p_1 \rangle = -\frac{\mathcal{A}\, e^{-\varrho}}{e}\, \varpi(\mathbf{x}) \left(2\Phi^2 + 1\right) \tag{4.226}$$

This ansatz permits us to separate the terms of the equation according to the power of Φ. Each group of terms give us an independent equation. Two groups of terms appears.

The first one is the group of the terms multiplied by Φ^0, its cancelation leads to the expression for $\varpi(\mathbf{x})$

$$\varpi(\mathbf{x}) = \frac{1}{2}\, \left[(\hat{\mathbf{b}} \times \nabla) \cdot \hat{\mathbf{b}}\right] \tag{4.227}$$

The following group of a terms, multiplied by Φ^2 will give us the second condition for the function ϱ

$$\frac{1}{2}\, \left(\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}})\right) \cdot \nabla\varrho = \hat{\mathbf{b}} \cdot \partial_{\mathbf{x}} \varpi(\mathbf{x}) + (\nabla \cdot \hat{\mathbf{b}})\, \varpi(\mathbf{x}) - \frac{1}{2}\, \hat{\mathbf{b}} \cdot \left(\nabla \times (\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}})\right) \tag{4.228}$$

$$\left(\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}})\right) \cdot \nabla\varrho = \nabla \cdot \left(\left[(\hat{\mathbf{b}} \times \nabla) \cdot \hat{\mathbf{b}}\right]\, \hat{\mathbf{b}}\right) - \hat{\mathbf{b}} \cdot \left(\nabla \times (\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}})\right) \tag{4.229}$$

Finally:

$$\langle p_1 \rangle = \frac{\mathcal{A}\, e^{-\varrho}}{2\, e\, \sin\varphi}\, \Phi\, \left[(\hat{\mathbf{b}} \times \nabla) \cdot \hat{\mathbf{b}}\right] \tag{4.230}$$

4.6.2 Final result for second order solution

The first and the second order (in powers of square root of the constant of motion $\sqrt{\mathcal{A}}$) decomposition of the Hamiltonian normal form are given by:

$$\begin{aligned} p\, (\mathbf{x}, \hat{\perp}(\mathbf{x}), \varphi(\mathbf{x}), \mathcal{A}) &= \frac{\sqrt{\mathcal{A}}\, B}{\sin\varphi} e^{-\varrho/2} \\ &+ \frac{\mathcal{A} e^{-\varrho}}{e \sin\varphi} \left(-\Phi^2\, \hat{\boldsymbol{\rho}}\, \hat{\mathbf{b}} : \nabla\hat{\mathbf{b}} - \frac{1}{2} \hat{\boldsymbol{\rho}} \cdot \left(\frac{\nabla B}{B} - \nabla\varrho\right) \right) \\ &+ \frac{\mathcal{A}\, e^{-\varrho}}{2\, e\, \sin\varphi}\, \Phi\, \left[(\hat{\mathbf{b}} \times \nabla) \cdot \hat{\mathbf{b}} - \frac{1}{2} (\hat{\boldsymbol{\rho}} \hat{\perp} + \hat{\perp} \hat{\boldsymbol{\rho}}) : \nabla\hat{\mathbf{b}}\right] + \mathcal{O}\left(\mathcal{A}^{3/2}\right) \end{aligned} \tag{4.231}$$

[21] Note that this is one of possible solutions. Probably another solution can be more suitable. This opportunity need to be exploited.

where the function $\varrho = \varrho(\mathbf{x})$ must satisfy two following conditions:

$$\hat{\mathbf{b}} \cdot \nabla \varrho(\mathbf{x}) \;=\; 0 \tag{4.232}$$

$$k\, \hat{\mathbf{b}}_2 \cdot \nabla \varrho(\mathbf{x}) \;=\; \nabla \cdot \left(\left((\hat{\mathbf{b}} \times \nabla) \cdot \hat{\mathbf{b}} \right) \hat{\mathbf{b}} \right) - \hat{\mathbf{b}} \cdot \left(\nabla \times (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \right) \equiv \sigma \tag{4.233}$$

where $\hat{\mathbf{b}}_2 \equiv \hat{\mathbf{b}} \times \frac{(\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}})}{k}$ and $k \equiv |\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}|$ denotes the curvature of magnetic field.

In fact this is two conditions on the the directional derivatives of the function ϱ in two perpendicular directions.

From the first order equation, we know that in parallel to magnetic field direction, its derivative is equal to zero. From the second order equation, we obtain that its derivative in the directions perpendicular to the magnetic field $\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}})$ must be equal to some quantity σ

Such a quantity can be expressed using only the second order derivatives of the magnetic field direction $\hat{\mathbf{b}}$. These two conditions may be interpreted as solvability conditions. We can suppose here, that we have an ordering in magnetic field line derivatives, so that, the second condition must be taken into account when solving the general equation at the next order.

4.6.3 Discussion

First of all we remark that the conditions on the function ϱ obtained just above are similar to the condition imposed by Littlejohn [3] on the gyrogauge function ξ:

$$\hat{\mathbf{b}} \cdot \nabla \xi = \frac{1}{2} \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} + \hat{\mathbf{b}} \cdot \mathbf{R} \tag{4.234}$$

Such a condition allows to remove geometrical terms from the guiding-center Hamiltonian $H = \mu\, B + 1/2\, p_{||}^2$.

The crucial difference here is that no fixed basis vectors $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$ was used when arriving at the conditions (4.232),(4.233). Therefore no derivatives of the fixed basis vectors are involved in the expression for the function σ and then the gyrogauge vector $\mathbf{R}$ does not appear explicitly.

Here we discuss several issues for the function $\xi = \xi(\mathbf{x})$.

The first one is to deal with solution of the system of directional differential equations (4.232),(4.233) in order to find the corresponding solution for ξ. Further discussion of solvability of such a system of differential equations will be needed. For example it will be necessary to verify some Newcomb - like condition[22] on the

[22] The Newcomb condition states that the differential equation $\hat{\mathbf{b}} \cdot \nabla \xi = \sigma$ possesses a single valued solution if the following condition is accomplish: $\oint \frac{dl}{B} \sigma = 0$ where the integral is taken around the closed magnetic field line

function σ. Note that it was shown by Hagan and Frieman that in the case of the Littlejohn equation on the gyrogauge function ξ the Newcomb condition is violated. However no inconsistency within the general method occurs because of the angle-like nature of the gyrophase function ζ.

Another opportunity is to impose a standard gyrokinetic ordering on the magnetic field derivatives $L_B\,\nabla\hat{\mathbf{b}} \sim \varepsilon$, where L_B represents a characteristic length scale for magnetic field variation. Then the function σ containing second order magnetic field derivatives, can be omitted at the first order and will need to be considered at the second order.

The other way to treat the function ξ is to put it equal to zero. In this case the first condition will be automatically satisfied, and no inconsistency in solution of the zeroth order differential equation will appear, because $p_0 = \sqrt{\mathcal{A}B}/\sin\varphi$ satisfies the first order differential equation (4.217).

Then the second condition should be treated as a geometrical restriction for magnetic field:

$$\nabla \cdot \left(\left((\hat{\mathbf{b}} \times \nabla) \cdot \hat{\mathbf{b}} \right) \hat{\mathbf{b}} \right) = \hat{\mathbf{b}} \cdot \left(\nabla \times (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \right) \tag{4.235}$$

Such a condition is satisfied in slab geometry by the vanishing of both sides of this equation. In a the general magnetic geometry case such a condition adds a supplementary constraint on the magnetic field and the solubility of the iterative procedure (4.215).

This way to proceed with conditions (4.232) is similar to those proposed by L. Sugiyama in [59] when discussing the solvability of the Littlejohn condition (4.234) on the gyrogauge function. As was shown by Hagan and Frieman, this equation does not possess any single valued solution because of violation of Newcomb's condition.

Due to this fact, in order to find another manner to obtain a single - valued solution of the eq.(4.234), Sugiyama claim that each of the terms $\tau = \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}$ $\tau_g = \hat{\mathbf{b}} \cdot \mathbf{R}$ in the r.h.s. of the eq. (4.234) is equal to zero. This implies the serious restriction on the magnetic field that it be torsion free.

On the other hand, it was remarked by Brizard in [65] that there is no inconsistency in the fact that the solution of the eq. (4.234) cannot be single valued. Because the function ξ is angle-like in nature and therefore multi-valued. Finally as Littlejohn said in his work on phase anholonomy in the classical adiabatic motion [58], *"there will inevitably occur terms depending on the perpendicular unit vectors which cannot be transformed away, and that it is best to live with these terms and to understand their gauge dependence"*. Perturbative Lie-transform theory introduced in [3], developed in [5] and explicit in [45] possesses a perturbation expansion, based on gyrogauge invariant Lie generators, that leads to gyrogauge invariant equations for parallel dynamics and to the gyrophase dynamical equation that explicitly depends naturally on the gyrogauge vector $\mathbf{R}$. The geometrical origin of $\mathbf{R}$ insures the validity of this approach. The difference of our method with respect to the Lie-transform approach is such that no explicit dependence on the fixed unit vectors $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_1)$ during the dynamical reduction procedure appears. The constraints imposed on the function ξ are defined only by the magnetic field itself. On the other hand, further discussion of the solvability conditions (4.232),(4.233) as well as an alternative manner to deal with second order partial differential equation (4.223) will be necessary in the following in order to make sure the solvability of the intrinsic iterative procedure at any order.

4.7 Summary

New abstract methods for the guiding-center dynamical reduction have been introduced in this work. The rigorous derivation does not rely on the definition of the basis vectors in the perpendicular plane and thus is free from the gyrogauge and "Sugiyama" problems. The derivation presented in this work may result in the formulation of a gyrokinetic theory that is accurate and includes consistently all terms associated with the non-uniformity of the magnetic field.

Chapter 5

Conclusions and discussion

In this thesis a theoretical investigation into improvement of fusion plasma confinement by plasma control, with possible barrier formation, was undertaken from different points of view.

In Chapter 2 Hamiltonian control tools were applied for considering transport reduction for the $\mathbf{E} \times \mathbf{B}$ drift model suited for test particles. Then, in Chapter 3 an investigation of intrinsic plasma rotation mechanisms was pursued through the derivation of the momentum conservation law for the gyrokinetic Maxwell-Vlasov model. Here the dynamical reduction for the Maxwell-Vlasov equations was realized by using Lie-transform perturbation methods and a suitable constrained variational principle. There are some important remarks to make about these two studies.

First of all, in both cases electrostatic turbulence (coupling particle motion with electric field) was considered. Such an approach is well suited to magnetically confined plasmas and is widely used by physicists. However it would be interesting in the future to explore the problem of barrier formation and to reveal intrinsic rotation mechanisms in the case of electromagnetic turbulence. This could provide an opportunity to go into depth in the understanding of effects related to the self-consistency of field-particle interaction.

Concerning the implementation of the Hamiltonian control tools for the $\mathbf{E} \times \mathbf{B}$ model, an important step was the implementation of methods of abstract Hamiltonian theory for the concrete physical problem. The analytical expression for electric potential used here is well suited to the theoretical investigation presented in [66]. In such a situation, the corresponding control term possesses also an analytical expression. In order to obtain a transport barrier that completely stops particle diffusion at a chosen position, the control term must be implemented at each point of the poloidal section of a machine. In a real situation, the electric field can be measured at a finite number of points and the implementation of the control term is limited by engineering features like for example restriction on the number and position of the actuators that control the electric field. The first step in exploration of the robustness of such a control term by truncation of Fourier series was explored in this dissertation. We saw that for a rather chaotic system,by inducing only the two first Fourier harmonics the turbulent transport throughout the barrier could be halved. On the other hand, one of the powerful points of this method is its low additional cost of energy.

Moreover, the experimental realization of this control in a Traveling Wave Tube (PIIM laboratory, Marseille, [21]), opens the possibility to practically achieve the control of a wide range of systems. In such a device, the interaction between electrons and electrostatic waves was considered. An interesting final issue for application of this method could be its implementation in a fusion device taking into account all experimental constraints. Presently there is a work in progress by the Non-Linear Dynamics team (Marseille) concerning its implementation for the linear device Vineta, Greifswald; the corresponding results will be published in [67].

Concerning the derivation of gyrokinetic momentum conservation law through a constrained variational principle for the full and electrostatic gyrokinetic Maxwell-Vlasov system, the strong point of this method is providing an exact statement that depends on the nonlinear gyrokinetic physics. General mechanisms of intrinsic plasma rotation were identified for the electrostatic turbulence case. Currently an article in collaboration with A. J. Brizard is in preparation. Its goal is to derive the momentum conservation law in the case of the gyrokinetic electromagnetic Maxwell-Vlasov system and to identify new intrinsic rotation mechanisms that lie behind it.

At the same time, these investigations were accompanied by construction of an alternative dynamical reduction method for the Maxwell-Vlasov system by applying Hamiltonian perturbation tools. As was previously remarked, for the Maxwell-Vlasov system the electric field plays the role of the mechanism that couples fields and particles. Then, as in the case of the Lie-transform perturbation method, the first stage in the strategy consists of dynamical reduction for a particle moving in an external non-uniform magnetic field in a six-dimensional phase space. As shown in Chapter 4, those problem reveals fundamental questions related to the geometry of the magnetic field configuration. For example, concerning the definition of gyroangle, necessary for separating the scales of motion, the gyrogauge dependence of dynamics is induced.

In order to encompass these problems, an intrinsic formalism for construction of constants of motion, in the case of an uncoupled system is built in Chapter 4 of this dissertation. Such a procedure does not involve the definition of the gyroangle and thereby avoids the problems related to the gyrogauge dependence. Here some questions related to the iterative construction strategy for the resolution of the final system of partial differential equations will need to be discussed.

The next step of such a reduction procedure consists of reintroduction of a perturbative field-particle coupling into the system.

Then at each order, by introducing a small modification into the system, the constant of motion should be reconstituted. One of the possible issues is to use the general Hamiltonian control method for this purpose. Finally, the perturbative field-particle coupling series for the constant of motion of the full Maxwell-Vlasov system should be constructed. This part of the work is still presently being undertaken.

Appendix A

Eulerian variations for Maxwell-Vlasov action

This Appendix deals with a detailed decomposition of the general expression for the Eulerian variation of the Maxwell-Vlasov Lagrangian density (3.6) in its Noether's part and its dynamical part (3.14).

$$\delta\mathcal{A} = \int d^4x \, \delta\mathcal{L} \equiv \underbrace{\int d^4x \, \delta\mathcal{L}_M}_{\equiv \delta\mathcal{A}_M} + \underbrace{\int d^4x \, \delta\mathcal{L}_{Vl}}_{\equiv \delta\mathcal{A}_{Vl}} \tag{A.1}$$

A.1 Eulerian variation for Maxwell part of action

In this Section we consider the perturbed Maxwell-Vlasov equations and the contribution to the Noether terms of Maxwell's part of action. The Eulerian variation of Maxwell's part of Lagrangian density is given by:

$$\delta\mathcal{L}_M \equiv \frac{1}{4\pi}\left(\epsilon^2 \, \delta\mathbf{E}_1 \cdot \mathbf{E}_1 - \epsilon \, \delta\mathbf{B}_1 \cdot \mathbf{B}\right) \tag{A.2}$$

We use that

$$\mathbf{E}_1 = -\nabla\Phi_1 - c^{-1}\partial_t\mathbf{A}_1 \tag{A.3}$$

and $\delta\mathbf{E}_1 = -\nabla\delta\Phi_1 - c^{-1}\partial_t\delta\mathbf{A}_1$ in order to rearrange the electrostatic part of the variation:

$$\delta\mathbf{E}_1 \cdot \mathbf{E}_1 = \nabla\delta\Phi_1 \cdot \nabla\Phi_1 + \frac{1}{c}\nabla\delta\Phi_1 \cdot \partial_t\mathbf{A}_1 + \frac{1}{c}\partial_t\delta\mathbf{A}_1 \cdot \nabla\Phi_1 + \frac{1}{c^2}\partial_t\delta\mathbf{A}_1 \cdot \partial_t\mathbf{A}_1 \tag{A.4}$$

Then we apply the Leibnitz rule on each term in this expression in order to separate it in the full derivative and the terms multiplied by the variations $(\delta\Phi_1, \delta\mathbf{A}_1)$. For example, for the first term we have:

$$\delta\nabla\Phi_1 \cdot \nabla\Phi_1 = \nabla \cdot (\delta\Phi_1\nabla\Phi_1) - \delta\Phi_1\nabla^2\Phi_1 \tag{A.5}$$

Similarly proceeding with the rest three terms, and using the relationA.3, we can rewrite the electrostatic part of the variation as:

$$\begin{aligned} \delta\mathbf{E}_1\cdot\mathbf{E}_1 &= -\nabla\cdot(\delta\Phi_1\,\mathbf{E}_1) - \frac{1}{c}\frac{\partial}{\partial t}(\delta\mathbf{A}_1\cdot\mathbf{E}_1) && \text{(A.6)}\\ &+ \delta\Phi_1(\nabla\cdot\mathbf{E}_1) + \delta\mathbf{A}_1\frac{1}{c}\left(\frac{\partial\mathbf{E}_1}{\partial t}\right) && \text{(A.7)} \end{aligned}$$

The next step consists in rearranging similarly the magnetic part of the variation. We need to use the following tensor relation:

$$(\nabla\times\mathbf{C})\cdot\mathbf{D} = \nabla\cdot(\mathbf{C}\times\mathbf{D}) + \mathbf{C}\cdot(\nabla\times\mathbf{D}) \quad \text{(A.8)}$$

for any tensors $\mathbf{C}$ and $\mathbf{D}$. Then we obtain:

$$\delta\mathbf{B}_1\cdot\mathbf{B} = (\nabla\times\delta\mathbf{A}_1)\cdot\mathbf{B} = \nabla\cdot(\delta\mathbf{A}_1\times\mathbf{B}) + \delta\mathbf{A}_1\cdot(\nabla\times\mathbf{B}) \quad \text{(A.9)}$$

By combining the rearranged expressions for the variations od electric and magnetic field (A.3) and (A.9), we can group Noether's terms and dynamic terms (multiplied by the variations $(\delta\Phi_1, \delta\mathbf{A}_1)$:

$$\begin{aligned} \delta\mathcal{L}_M &= \frac{1}{4\pi}\left[-\nabla\cdot\left(\epsilon^2\delta\Phi_1\,\mathbf{E}_1 + \epsilon\,\delta\mathbf{A}_1\times\mathbf{B}\right) - \frac{1}{c}\frac{\partial}{\partial t}\left(\epsilon^2\delta\mathbf{A}_1\cdot\mathbf{E}_1\right)\right]\\ &+ \frac{1}{4\pi}\left[\epsilon^2\delta\Phi_1(\nabla\cdot\mathbf{E}_1) + \delta\mathbf{A}_1\cdot\left(\epsilon^2\frac{1}{c}\,\partial_t\,\mathbf{E}_1 - \epsilon\,(\nabla\times\mathbf{B})\right)\right] && \text{(A.10)} \end{aligned}$$

The two first terms represents contribution to the Noether's terms from Maxwell's part of Lagrangian density. The two last terms will be considered during derivation of the Maxwell-Vlasov perturbed equations. The latter calculation is discussed in the general part of this text, see Section (3.3.2).

A.2 Eulerian variation for Vlasov part of action

This section deals with detailed decomposition of the term $\delta\mathcal{A}_{Vl} \equiv -\int d^8\mathcal{Z}\;\mathcal{H}\;\{S,\mathcal{F}\}_{ext}$ in its dynamical part and its Noether's part.[1] By using the Leibnitz rule for the extended canonical Poisson bracket, we can write:

$$\begin{aligned} \int d^8\mathcal{Z}\;\mathcal{H}\;\{S,\mathcal{F}\}_{ext} &= \int d^8\mathcal{Z}\;\{S\mathcal{H},\mathcal{F}\}_{ext} + \int d^8\mathcal{Z}\;S\;\{\mathcal{F},\mathcal{H}\}_{ext}\\ &\equiv -\int d^4x\;\delta\mathcal{L}_N^{Vl} - \int d^4x\;\delta\mathcal{L}_{dyn}^{Vl} && \text{(A.11)} \end{aligned}$$

The first term here is an exact bracket, so we will group it to others Noether's components. The second term will give us Vlasov equation (see Appendix A.2.2 for details). In two following sections we give details about each of these terms.

[1]Note that the following decomposition can also be applied to the case of reduced gyrocenter dynamics. One should replace the extended Poisson bracket by the reduced gyrocenter bracket.

A.2.1 Noether's term for Vlasov part

Our work in this subsection is related to the rearrangement of the first term into the expression A.11. First, we rewrite the Poisson bracket as follows:

$$
\begin{aligned}
\{S\mathcal{H}, \mathcal{F}\}_{ext} = -\{\mathcal{F}, S\mathcal{H}\}_{ext} = -\frac{\partial \mathcal{F}}{\partial \mathcal{Z}^a} J^{ab} \frac{\partial (S\mathcal{H})}{\partial \mathcal{Z}^b} = & - \frac{\partial}{\partial \mathcal{Z}^a}\left(\mathcal{F}\,\frac{\partial}{\partial \mathcal{Z}^b}(S\mathcal{H})\right) J^{ab} \\
& + \mathcal{F}\,\underbrace{\frac{\partial^2 (S\mathcal{H})}{\partial \mathcal{Z}^a \partial \mathcal{Z}^b} J^{ab}}_{=0}
\end{aligned} \tag{A.12}
$$

the latter term here is equal to zero because of symmetry of second derivative and antisymmetry of the extended Poisson matrix J^{ab}, further we multiply and we divide our expression by the Jacobian $\mathcal{J}$, and we apply one more time the Leibnitz rule:

$$
\frac{1}{\mathcal{J}}\frac{\partial}{\partial \mathcal{Z}^a}\left(\mathcal{F}\,\frac{\partial}{\partial \mathcal{Z}^b}(S\mathcal{H})\right)\mathcal{J}\;J^{ab} = -\underbrace{\frac{1}{\mathcal{J}}\frac{\partial}{\partial \mathcal{Z}^a}\left(\mathcal{J}\;J^{ab}\right)}_{=0} + \frac{1}{\mathcal{J}}\frac{\partial}{\partial \mathcal{Z}^a}\left(\mathcal{J}\;\mathcal{F}\{\mathcal{Z}^a, S\mathcal{H}\}_{ext}\right) \tag{A.13}
$$

where we use the Liouville identity for Poisson bracket and that $J^{ab}\dfrac{\partial (S\mathcal{H})}{\partial \mathcal{Z}^b} \equiv \{\mathcal{Z}^a, S\mathcal{H}\}_{ext}$.

Finally we obtain that:

$$
\{S\mathcal{H}, \mathcal{F}\}_{ext} = -\frac{1}{\mathcal{J}}\frac{\partial}{\partial \mathcal{Z}^a}\left(\mathcal{J}\;\mathcal{F}\{\mathcal{Z}^a, S\mathcal{H}\}\right) \tag{A.14}
$$

The integral over all extended phase space of an exact Poisson bracket is equal to zero. It suffice to prove it in the canonical coordinates, by integrating by parts. To translate this proof in the case of the reduced phase space we have to use the fact the there is a diffeomorphism between the canonical coordinates and the guiding-center (gyrocenter) coordinates.

In order to obtain the contribution of the integral $\int d^8\mathcal{Z}\;\{S\mathcal{H}, \mathcal{F}\}_{ext}$ to the Noether part of the Lagrangian density variation, we should first integrate over momentum part of the phase space and then evaluate all non-vanishing terms

$$
\begin{aligned}
& \int d^8\mathcal{Z}\;\{S\mathcal{H}, \mathcal{F}\}_{ext} \\
= & \; -\int d^{\,4}x \int d^{\,3}p\; c^{-1}\, dw\;\; \frac{1}{\mathcal{J}}\frac{\partial}{\partial \mathcal{Z}^a}\left(\mathcal{J}\;\mathcal{F}\{\mathcal{Z}^a, S\mathcal{H}\}\right) = 0
\end{aligned} \tag{A.15}
$$

here we use that $d^8\mathcal{Z} \equiv d^4x\; d^4p \equiv\; d^4x d^3p\; c^{-1}\; dw$ Further we remark that:

$$
\{\mathcal{Z}^a, S\mathcal{H}\}_{ext} = S\;\{\mathcal{Z}^a, \mathcal{H}\}_{ext} + \mathcal{H}\;\{\mathcal{Z}^a, S\}_{ext} \tag{A.16}
$$

and we apply the physical constraint $\mathcal{H} \equiv 0$ in order to vanish the second term, so only the first term will contribute:

$$\begin{aligned}
0 &= \int d^3p\, dw\, \frac{\partial}{\partial \mathcal{Z}^a}\left(\mathcal{J}\ \mathcal{F}\ \ S\ \{\mathcal{Z}^a, \mathcal{H}\}_{ext}\right) \\
&= \underbrace{\int d^3p\, dw \left(\frac{\partial}{\partial w}\left(\mathcal{J}\ \mathcal{F}\ S\{w, \mathcal{H}\}\right) + \frac{\partial}{\partial p_i}\left(\mathcal{J}\ \mathcal{F}\ S\{p_i, \mathcal{H}\}\right)\right)}_{\equiv 0} \\
&+ \int d^3p\, dw\ \left(\frac{\partial}{\partial x_i}\left(\mathcal{J}\ \mathcal{F}\ S\{x_i, \mathcal{H}\}\right) + \frac{\partial}{\partial t}\left(\mathcal{J}\ \mathcal{F}\ S\{ct, \mathcal{H}\}\right)\right) \\
&= \frac{\partial}{\partial x_i}\left(\int d^3p\, dw\ \left(\mathcal{J}\ \mathcal{F}\ S\{x_i, \mathcal{H}\}\right)\right) \\
&+ \frac{\partial}{\partial t}\left(\int d^3p\, dw\ \left(\mathcal{J}\ \mathcal{F}\ S\{ct, \mathcal{H}\}\right)\right)
\end{aligned} \tag{A.17}$$

With $\{ct, \mathcal{H}\}_{ext} = c$ and $\{\mathbf{x}, \mathcal{H}\}_{ext} = \dot{\mathbf{x}}$ [2] we have the following expression for Noether's part of Vlasov Lagrangian density :

$$\delta \mathcal{L}_N^{Vl} = -\frac{\partial}{\partial t}\int d^{\,4}p\, \mathcal{F}\, S - \nabla \cdot \int d^{\,4}p\, \mathcal{F}\, \dot{\mathbf{x}}\, S \tag{A.18}$$

A.2.2 Vlasov equation on a 6 dimensional phase space

Here we present details about obtaining the Vlasov equation on a 6 dimensional phase space from the relation of commutation on extended phase space between the extended Vlasov distribution function and the extended Hamiltonian: $\int dw\ \{\mathcal{F}, \mathcal{H}\}_{ext} = 0$. First we explicitly rewrite this expression into the canonical variables [3]:

$$\{\mathcal{F}, \mathcal{H}\}_{ext} = \nabla \mathcal{F} \cdot \frac{\partial \mathcal{H}}{\partial \mathbf{p}} - \frac{\partial \mathcal{F}}{\partial \mathbf{p}} \cdot \nabla \mathcal{H} - \left(\frac{\partial \mathcal{F}}{\partial t}\frac{\partial \mathcal{H}}{\partial w} - \frac{\partial \mathcal{F}}{\partial w}\frac{\partial \mathcal{H}}{\partial t}\right) \tag{A.19}$$

[2]This expression in the case of gyrocenter reduced dynamics is replaced by $\{\mathbf{X}, \mathcal{H}_{gy}\}_\epsilon = \dot{\mathbf{X}}_{gy}$ where $\{\mathbf{X}, \mathcal{H}_{gy}\}_\epsilon$ is an extended guiding center bracket, $\mathbf{X}$ is the guiding-center position and $\mathbf{X}_{gy}$ is the gyrocenter position

[3]This proof will be also convenient in the case of the reduced gyrocenter dynamics due to the diffeomorphism between the canonical variables and the gyrocenter variables [44]

We rearrange each term by using the Leibnitz rule $\nabla \cdot (\mathbf{AB}) = (\nabla \cdot \mathbf{A})\mathbf{B} + \mathbf{A} \cdot \nabla \mathbf{B}$:

$$\nabla \mathcal{F} \cdot \frac{\partial \mathcal{H}}{\partial \mathbf{p}} = \nabla \cdot \left(\mathcal{F} \, \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \right) - \mathcal{F} \, \nabla \cdot \left(\frac{\partial \mathcal{H}}{\partial \mathbf{p}} \right) \tag{A.20}$$

$$\frac{\partial \mathcal{F}}{\partial \mathbf{p}} \cdot \nabla \mathcal{H} = \frac{\partial}{\partial \mathbf{p}} \cdot (\mathcal{F} \, \nabla \mathcal{H}) - \mathcal{F} \, \frac{\partial}{\partial \mathbf{p}} \cdot (\nabla \mathcal{H}) \tag{A.21}$$

$$\frac{\partial \mathcal{F}}{\partial t} \frac{\partial \mathcal{H}}{\partial w} = \frac{\partial}{\partial t} \left(\mathcal{F} \, \frac{\partial \mathcal{H}}{\partial w} \right) - \mathcal{F} \, \left(\frac{\partial^2 \mathcal{H}}{\partial t \partial w} \right) \tag{A.22}$$

$$\frac{\partial \mathcal{F}}{\partial w} \frac{\partial \mathcal{H}}{\partial t} = \frac{\partial}{\partial w} \left(\mathcal{F} \, \frac{\partial \mathcal{H}}{\partial t} \right) - \mathcal{F} \, \left(\frac{\partial^2 \mathcal{H}}{\partial w \partial t} \right) \tag{A.23}$$

Then we obtain the following expression for the extended Poisson bracket:

$$\{\mathcal{F}, \mathcal{H}\}_{ext} = \nabla \cdot \left(\mathcal{F} \, \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \right) - \frac{\partial}{\partial \mathbf{p}} \cdot (\mathcal{F} \, \nabla \mathcal{H}) - \frac{\partial}{\partial t} \left(\mathcal{F} \, \frac{\partial \mathcal{H}}{\partial w} \right) + \frac{\partial}{\partial w} \left(\mathcal{F} \, \frac{\partial \mathcal{H}}{\partial t} \right) \tag{A.24}$$

Now we should integrate this expression over variable w by substituting the expression for the extended Vlasov distribution function $\mathcal{F} = \delta(w - H) \, F$ and the extended Hamiltonian $\mathcal{H} = H - w$:

$$\int dw \, \{\mathcal{F}, \mathcal{H}\}_{ext} = \int dw \, \delta(w - H) \left[\frac{\partial F}{\partial t} + \nabla \cdot \left(F \, \frac{\partial H}{\partial \mathbf{p}} \right) - \frac{\partial}{\partial \mathbf{p}} \cdot (F \, \nabla H) \right] \tag{A.25}$$

Note that here we exchange the derivative and the integral over independent variables $(\mathbf{x}, \mathbf{p})$ and we use that: $\partial_w \mathcal{H} = -1$. The key moment of this proof is vanishing of the integral:

$$\int dw F \, \delta(w - H) \, (H - w) = 0 \tag{A.26}$$

in fact $\int dw \, \delta(w - H) = 1$ if and only if $H = w$, and then automatically the integral is equal to zero.

By rearranging terms into the expression A.25 according to the Leibnitz rule:

$$\nabla \cdot \left(F \, \frac{\partial H}{\partial \mathbf{p}} \right) = \nabla F \cdot \frac{\partial H}{\partial \mathbf{p}} + F \, \nabla \cdot \frac{\partial H}{\partial \mathbf{p}} \tag{A.27}$$

$$\frac{\partial}{\partial \mathbf{p}} \cdot (F \, \nabla H) = \frac{\partial F}{\partial \mathbf{p}} \cdot \nabla H + F \, \frac{\partial}{\partial \mathbf{p}} \cdot \nabla H \tag{A.28}$$

and according to the Hamilton principle, we obtain:

$$0 = \frac{\partial F}{\partial t} + \nabla F \cdot \frac{\partial H}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}} \cdot \nabla H \equiv \frac{\partial F}{\partial t} + \{F, H\} \tag{A.29}$$

Gyrokinetic Vlasov equation on a 6 dimensional phase space

In the case of electrostatic gyrokinetic Maxwell-Vlasov model considered in the section 3.5 , the Poisson bracket in the expression (A.29) is the guiding-center Poisson bracket defined on the reduced guiding-center phase space $z^a = (\mathbf{X}, p_{\,||}, \theta, \mu)$:

$$\begin{aligned} \{F, G\}_{gc} &= \epsilon_{gc}^{-1} \frac{\Omega}{B} \left(\frac{\partial F}{\partial \theta} \frac{\partial G}{\partial \mu} - \frac{\partial F}{\partial \mu} \frac{\partial G}{\partial \theta} \right) && \text{(A.30)} \\ &+ \frac{\mathbf{B}^*}{B^*_{||}} \cdot \left(\nabla^* F \frac{\partial G}{\partial p_{\,||}} - \frac{\partial F}{\partial p_{\,||}} \nabla^* G \right) - \epsilon_{gc} \frac{c\, \hat{\mathbf{b}}}{e\, B^*_{||}} \left(\nabla^* F \times \nabla^* G \right) \end{aligned}$$

where $\nabla^* = \nabla + \mathbf{R}\, \partial/\partial\theta$, $\mathbf{R}^* = \nabla \hat{\perp} \cdot \hat{\boldsymbol{\rho}} + 1/2\, (\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}})\, \hat{\mathbf{b}}$ and

$$\mathbf{B}^* = \mathbf{B} + \epsilon\, B\, \frac{p_{\,||}}{m\Omega} \nabla \times \hat{\mathbf{b}} + \ldots \tag{A.31}$$

from which $B_{\,||} = \mathbf{B}^* \cdot \hat{\mathbf{b}}$.

To obtain the gyrokinetic Vlasov equation we use the phase-space divergence form of the guiding-center Poisson bracket:

$$\{F, H_{gy}\}_{gc} = \frac{1}{B^*_{||}} \frac{\partial}{\partial z^a} \left(B^*_{||} F\, \{z^a, H_{gy}\}_{gc} \right) \tag{A.32}$$

where H_{gy} is the electrostatic gyrocenter gyrophase-independent Hamiltonian

$$H_{gy} = \mu B_0 + \frac{p^2_{||}}{2m} + \epsilon\, e \langle \Phi_{1gc} \rangle - \frac{\epsilon^2}{2}\, e\, \langle \{ \Psi_{1gc}, \Phi_{1gc} \}_{gc} \rangle \tag{A.33}$$

with $\partial_\theta \Psi_{1gc} = \widetilde{\Phi}_{1gc}$. Using the gyrocenter equations of motion [4]

$$\begin{aligned} \dot{\mathbf{X}} &= \frac{\mathbf{B}^*}{B^*_{||}} \frac{\partial H_{gy}}{\partial p_{\,||}} + \frac{c\, \hat{\mathbf{b}}}{e B^*_{||}} \times \nabla H_{gy} && \text{(A.34)} \\ \dot{p}_{\,||} &= -\frac{\mathbf{B}^*}{B^*_{||}} \cdot \nabla H_{gy} && \text{(A.35)} \\ \dot{\theta} &= \frac{\Omega}{B} \frac{\partial H_{gy}}{\partial \mu} + \frac{\mathbf{B}^*}{B^*_{||}} \cdot \left(\mathbf{R}^* \frac{\partial H_{gy}}{\partial p_{\,||}} \right) && \text{(A.36)} \\ \dot{\mu} &= -\frac{\Omega}{B^*_{||}} \frac{\partial H_{gy}}{\partial \theta} = 0 && \text{(A.37)} \end{aligned}$$

[4]Note that by supposing that all the fields here $B^*_{||}$, $\hat{\mathbf{b}}_0$ and $\mathbf{R}^*$ are evaluated into the gyrocenter position, the corresponding gyrocenter dynamic is completely independent of the gyroangle coordinate. We have for example $\dot{\mathbf{X}} = \langle \dot{\mathbf{X}} \rangle$

with $\partial_\theta F \equiv 0$ we obtain that

$$\{F, H_{gy}\}_{gc} = \frac{1}{B^*_{||}} \nabla \cdot \left(B^*_{||} \, \dot{\mathbf{X}} \, F \right) + \frac{1}{B^*_{||}} \frac{\partial}{\partial p_{\,||}} \left(B^*_{||} \, \dot{p}_{\,||} \, F \right) \tag{A.38}$$

Then taking into the account the Liouville identity

$$\frac{1}{B^*_{||}} \frac{\partial}{\partial z^a} \left(B^*_{||} \, \dot{z}^a \right) = 0 \tag{A.39}$$

we can rewrite the gyrokinetic Vlasov equation as

$$\frac{\partial F}{\partial t} = -\dot{\mathbf{X}} \cdot \nabla F - \dot{p}_{\,||} \frac{\partial F}{\partial p_{\,||}} \tag{A.40}$$

Note that in the case of time-independent background magnetic field $\mathbf{B}_0$ phase-space diverge form of the Vlasov equation is

$$\frac{\partial (B^*_{||} F)}{\partial t} + \nabla \cdot \left(B^*_{||} \, \dot{\mathbf{X}} \, F \right) + \frac{\partial}{\partial p_{\,||}} \left(B^*_{||} \, \dot{p}_{\,||} \, F \right) = 0 \tag{A.41}$$

Appendix B

Proof of Momentum conservation

In this Appendix we give an explicit proof of momentum conservation. More precisely we show how to simplify the eq.(3.55) by using the equations of motion for the perturbed Maxwell-Vlasov system. We start with the first term in the r.h.s. of the expression (3.55). We substitute the Vlasov equation in its phase-space divergence form (3.56) and then we apply the Leibnitz rule.

$$\begin{aligned} \int \frac{\partial F}{\partial t} \left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right) d^3p &= -\int \frac{\partial}{\partial \mathbf{p}} \cdot \left(F\,\dot{\mathbf{p}}\,\left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right)\right) d^3p \\ - \quad \nabla \cdot \int \left(F\,\dot{\mathbf{x}}\,\left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right)\right) d^3p \\ + \quad \int F\,\dot{\mathbf{x}} \cdot \nabla \left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right) d^3p &+ \int F\,\dot{\mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{p}} \left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right) d^3p \end{aligned} \tag{B.1}$$

The first term in the r.h.s. of this equation is equal to zero as an integral over momentum part of the phase space (d^3p) of momentum divergence $\partial_{\mathbf{p}}$. The two latter terms can be rewritten as

$$\int F\,\dot{\mathbf{x}} \cdot \nabla \left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right) d^3p + \int F\,\dot{\mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{p}} \left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right) d^3p \tag{B.2}$$

$$= -\int F\,\{\left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right), H\} d^3p \tag{B.3}$$

We now continue the simplification of the equation (3.55) by considering Maxwell terms $\partial_t \mathbf{E}_1 \times \mathbf{B}_1$ and $\mathbf{E}_1 \times \partial_t \mathbf{B}_1$. The first term transforms according to the perturbed Ampere equation (3.23) and the tensor identity above combined with the electromagnetic constraint $\nabla \cdot \mathbf{B} = 0$

$$(\nabla \times \mathbf{C}) \times \mathbf{D} \quad = \quad \nabla \cdot (\mathbf{D}\,\mathbf{C}) - (\nabla \cdot \mathbf{D})\,\mathbf{C} - (\nabla \mathbf{C}) \cdot \mathbf{D} \tag{B.4}$$

$$\frac{\epsilon^2}{4\pi c}\frac{\partial \mathbf{E}_1}{\partial t}\times \mathbf{B}_1 = \frac{\epsilon^2}{4\pi}\left(\nabla\times\mathbf{B}\right)\times\mathbf{B}_1 - \epsilon\, e\int F\,\left(\frac{\mathbf{v}}{c}\times\mathbf{B}_1\right)d^3p$$
$$= -\frac{\epsilon}{4\pi}\nabla\mathbf{B}_0\cdot\mathbf{B}_1 - \frac{\epsilon^2}{8\pi}\nabla\left(\mathbf{B}_1\cdot\mathbf{B}_1\right) + \frac{\epsilon}{4\pi}\nabla\cdot\left(\mathbf{B}_1\cdot\mathbf{B}\right)$$
$$- \epsilon\, e\int F\,\left(\frac{\mathbf{v}}{c}\times\mathbf{B}_1\right)d^3p \tag{B.5}$$

The second term transforms according to the perturbed Poisson equation (3.22) and the second electromagnetic constraint $\partial_t\mathbf{B}_1 = -c\left(\nabla\times\mathbf{E}_1\right)$.

$$\frac{\epsilon^2}{4\pi\, c}\,\mathbf{E}_1\times\frac{\partial\mathbf{B}_1}{\partial t} = -\frac{\epsilon^2}{4\pi}\mathbf{E}_1\times\left(\nabla\times\mathbf{E}_1\right)$$
$$= -\frac{\epsilon^2}{8\pi}\nabla\left(\mathbf{E}_1\cdot\mathbf{E}_1\right) + \frac{\epsilon^2}{4\pi}\nabla\cdot\left(\mathbf{E}_1\,\mathbf{E}_1\right) - \epsilon\, e\int F\,\mathbf{E}_1\,d^3p \tag{B.6}$$

here we have also applied the tensor identity (B.4). Finally we can rewrite the Maxwell part of the eq.(3.55) as

$$\frac{\epsilon^2}{4\pi\, c}\left(\frac{\partial\mathbf{E}_1}{\partial t}\times\mathbf{B}_1 + \mathbf{E}_1\times\frac{\partial\mathbf{B}_1}{\partial t}\right)$$
$$= -\frac{\epsilon}{4\pi}\nabla\mathbf{B}_0\cdot\mathbf{B}_1 - \frac{\epsilon^2}{8\pi}\nabla\left(|\mathbf{E}_1|^2 + |\mathbf{B}_1|^2\right) + \frac{\epsilon}{4\pi}\nabla\cdot\left(\mathbf{B}_1\mathbf{B} + \epsilon\mathbf{E}_1\mathbf{E}_1\right)$$
$$- \epsilon\, e\int F\,\left(\left(\frac{\mathbf{v}}{c}\times\mathbf{B}_1\right) + \mathbf{E}_1\right)d^3p \tag{B.7}$$

By combining equations (B.7,B.5 and B.6) we obtain the equation 3.57.

In order to obtain the eq.(3.60) we use

$$\nabla\mathbf{B}_0\cdot\frac{\partial\mathcal{L}_M}{\partial\mathbf{B}_0} = \frac{1}{4\pi}\left(\nabla\mathbf{B}_0\cdot\mathbf{B}\right) = \frac{1}{8\pi}\nabla\left(\mathbf{B}_0\cdot\mathbf{B}_0\right) + \frac{\epsilon}{8\pi}\nabla\mathbf{B}_0\cdot\mathbf{B}_1 \tag{B.8}$$

where $\mathcal{L}_M$ denotes the Maxwell part of Lagrangian density. Then we remark that

$$\nabla\cdot\mathbf{\Pi}_M - \nabla\mathbf{B}_0\cdot\frac{\partial\mathcal{L}_M}{\partial\mathbf{B}_0} =$$
$$- \frac{\epsilon}{4\pi}\nabla\mathbf{B}_0\cdot\mathbf{B}_1 - \frac{\epsilon^2}{8\pi}\nabla\left(|\mathbf{E}_1|^2 + |\mathbf{B}_1|^2\right) + \frac{\epsilon}{4\pi}\nabla\cdot\left(\mathbf{B}_1\mathbf{B} + \epsilon\mathbf{E}_1\mathbf{E}_1\right) \tag{B.9}$$

where $\mathbf{\Pi}_M$ denotes the Maxwell part of the canonical momentum stress tensor (3.54).

By substituting the fundamental relation $\mathbf{v}\equiv\dot{\mathbf{x}} = \{H,\mathbf{x}\}$ into the Vlasov part of the canonical momentum stress tensor (3.54), we can associate with $\nabla\cdot\mathbf{\Pi}_{Vl}$ with $\nabla\cdot\int\,\left(F\,\dot{\mathbf{x}}\,\left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right)\right)d^3p$ from eq. (B.2), then we obtain

$$\frac{\partial\mathbf{P}}{\partial t} = -\nabla\cdot\mathbf{\Pi} + \nabla\mathbf{B}_0\cdot\frac{\partial\mathcal{L}_M}{\partial\mathbf{B}_0} + \nabla\mathbf{B}_0\cdot\frac{\partial\mathcal{L}_{Vl}}{\partial\mathbf{B}_0} \tag{B.10}$$
$$+ \int F\left[-\frac{d}{dt}\left(\mathbf{p}_* - \epsilon\frac{e}{c}\mathbf{A}_1\right) + \epsilon\, e\left(\mathbf{E}_1 + \frac{\mathbf{v}}{c}\times\mathbf{B}_1\right) - \nabla\mathbf{B}_0\cdot\frac{\partial\mathcal{L}_{Vl}}{\partial\mathbf{B}_0}\right]d^3p$$

here we add and we subtract $\dfrac{\partial \mathcal{L}_{Vl}}{\partial \mathbf{B}_0} = -\int d^3p\ F\ \dfrac{\partial H}{\partial \mathbf{B}_0}$ in order to complete the expression above up to the eq.(3.60).

Appendix C

Particle canonical equation of motion

In this Appendix we show how to derive the fundamental dynamical equation for particle moving into external electromagnetic fields. The Hamiltonian in canonical variables $(\mathbf{p}, \mathbf{x})$

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right) + e\,\Phi \tag{C.1}$$

then the Hamiltonian equations (with using the canonical Poisson bracket) are

$$\dot{\mathbf{x}} \;=\; \frac{\partial H}{\partial \mathbf{p}} = \frac{1}{m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right) \equiv \mathbf{v} \tag{C.2}$$

$$\dot{\mathbf{p}} \;=\; -\frac{\partial H}{\partial \mathbf{x}} = \frac{e}{c}\nabla\mathbf{A}\cdot\mathbf{v} - \; e\nabla\Phi \tag{C.3}$$

Then we substitute the eq.(C.2) in the l.h.s. of the eq.(C.3) and we use that $\dfrac{d}{dt} \equiv \dfrac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$

$$m\,\frac{d\mathbf{v}}{dt} = \underbrace{\frac{e}{c}\nabla\mathbf{A}\cdot\mathbf{v} - \frac{e}{c}\mathbf{v}\cdot\nabla\mathbf{A}}_{=\frac{e}{c}\mathbf{v}\times\mathbf{B}} + e\underbrace{\left(-\nabla\Phi - \frac{1}{c}\frac{\partial}{\partial}\mathbf{A}_1\right)}_{=\mathbf{E}} \tag{C.4}$$

here we have used that

$$(\nabla\mathbf{D})\cdot\mathbf{C} - (\mathbf{C}\cdot\nabla)\,\mathbf{D} = \mathbf{C}\times(\nabla\times\mathbf{D}) \tag{C.5}$$

and $\mathbf{B} = \nabla\times\mathbf{A}$, $\mathbf{E} = -\nabla\Phi - c^{-1}\partial_t\mathbf{A}$. Then we obtain the equation of motion driven by the Lorenz force

$$m\,\frac{d\mathbf{v}}{dt} = e\left(\frac{\mathbf{v}}{c}\times\mathbf{B} + \mathbf{E}\right) \tag{C.6}$$

Appendix D

Gyrocenter magnetization

In this Appendix, we derive the first-order gyrocenter contribution to the partial derivative:

$$\frac{\partial H_{gy}}{\partial \mathbf{B}_0} = \frac{\partial H_{gc}}{\partial \mathbf{B}_0} + \epsilon\, e\, \frac{\partial \langle \phi_{1gc} \rangle}{\partial \mathbf{B}_0} + \ldots \tag{D.1}$$

of the gyrocenter Hamiltonian (3.69).

D.1 Functional dependence on B_0

Before starting the calculation of $\dfrac{\partial H_{gy}}{\partial \mathbf{B}_0}$, we need to emphasize some important details. First of all, here we take into account the fact that the magnetic momentum μ is an independent of $\mathbf{B}_0$ phase space variable and the Larmor frequency Ω is expressed as $\Omega = \frac{eB_0}{mc}$.

So in further calculations:

$$\boldsymbol{\rho}_0 \equiv \sqrt{\frac{2\mu\; B_0}{m\Omega^2}}\; \hat{\boldsymbol{\rho}} \tag{D.2}$$

then

$$\boldsymbol{\rho}_0 = \sqrt{\frac{2\mu B_0}{m} \frac{m^2 c^2}{e^2 B_0^2}}\, \hat{\boldsymbol{\rho}} = \frac{c}{e}\sqrt{\frac{2\mu m}{B_0}}\, \hat{\boldsymbol{\rho}} \tag{D.3}$$

$$B_0 \equiv (\mathbf{B}_0 \cdot \mathbf{B}_0)^{1/2} \;\Rightarrow\; \frac{\partial B_0}{\partial \mathbf{B}_0} = \frac{\mathbf{B}_0}{B_0} \equiv \hat{\mathbf{b}}_0 \tag{D.4}$$

$$\frac{\partial \boldsymbol{\rho}}{\partial \mathbf{B}_0} = -\frac{1}{2}\frac{c}{e}\sqrt{2\mu m}\frac{\mathbf{B}_0}{B_0^{5/2}} = -\frac{c}{e}\sqrt{\frac{\mu m}{2B_0}}\frac{\hat{\mathbf{b}}_0}{B_0} \tag{D.5}$$

We remark also that

$$\frac{\partial \hat{\mathbf{b}}_0}{\partial \mathbf{B}_0} = \frac{\mathbf{1}}{B_0} - \frac{\hat{\mathbf{b}}_0 \hat{\mathbf{b}}_0}{B_0} \equiv \frac{\mathbf{1}_\perp}{B_0} \tag{D.6}$$

D.2 H_{gc}

$$H_{gc} = \mu\ B_0 + \frac{p_{||}^2}{2m} \tag{D.7}$$

here $B_0 = (\mathbf{B}_0 \cdot \mathbf{B}_0)^{1/2}$ is the norm of the background magnetic field. so

$$\frac{\partial H_{gc}}{\partial \mathbf{B}_0} = \mu \frac{\mathbf{B}_0}{B_0} \equiv \mu \hat{\mathbf{b}}_0 \tag{D.8}$$

D.3 $\epsilon \left\langle \phi_{1gc} \right\rangle$

In order to realize this calculation, we make a series expansion on the guiding center Larmor radius $\boldsymbol{\rho}_0$ for scalar electric potential ϕ_{1gc}:

$$\phi_{1gc} = \phi_1(\mathbf{X} + \boldsymbol{\rho}_0) = \Phi_1(\mathbf{X}) + \boldsymbol{\rho}_0 \cdot \nabla\phi_1(\mathbf{X}) + \frac{1}{2}\ \boldsymbol{\rho}_0\boldsymbol{\rho}_0 : \nabla\nabla\phi_1(\mathbf{X}) + \ldots \tag{D.9}$$

$$\langle\phi_{1gc}\rangle = \phi_1(\mathbf{X}) + \frac{1}{2}\frac{c^{\,2}}{e^{\,2}}\frac{\mu m}{B_0}(\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}} + \hat{\perp}\hat{\perp}) : \nabla\nabla\phi_1(\mathbf{X}) + \ldots \tag{D.10}$$

$$\begin{aligned} \frac{\partial\langle\phi_{1gc}\rangle}{\partial \mathbf{B}_0} &= -\frac{c^{\,2} m\mu}{2\ e^{\,2}}\frac{\mathbf{B}_0}{B_0^3}(\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}} + \hat{\perp}\hat{\perp}) : \nabla\nabla\phi_1 = \\ &= -\frac{\mu}{2m\Omega^2}\ \hat{\mathbf{b}}_0\ (\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}} + \hat{\perp}\hat{\perp}) : \nabla\nabla\phi_1 \\ &= -\frac{\mu}{2m\Omega^2}\ \hat{\mathbf{b}}_0\ \mathbf{1}_\perp : \nabla\nabla\phi_1 \end{aligned} \tag{D.11}$$

Using the fact that $\mathbf{E}_1 = -\nabla\phi_1$ and

$$\mathbf{1}_\perp : \nabla\mathbf{E}_1 = (\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}} + \hat{\perp}\hat{\perp}) : \nabla\mathbf{E}_1 = (\hat{\boldsymbol{\rho}} \cdot \nabla)(\mathbf{E}_1 \cdot \hat{\boldsymbol{\rho}}) + (\hat{\perp} \cdot \nabla)(\mathbf{E}_1 \cdot \hat{\perp}) = \nabla \cdot \mathbf{E}_\perp \tag{D.12}$$

we have:

$$\frac{\partial}{\partial \mathbf{B}_0}\langle\phi_{1gc}\rangle = \frac{\mu}{2m\Omega^2}\hat{\mathbf{b}}_0\,(\nabla \cdot \mathbf{E}_{1\perp}) \tag{D.13}$$

Appendix E

Gyrokinetic momentum conservation application

Gyrokinetic momentum conservation law in axisymmetric geometry In this Appendix we give a detailed projection of the gyrokinetic momentum conservation law (3.96) in axisymmetric geometry. Following [68] we start with introducing some generalities about the curvilinear coordinates.

E.1 Curvilinear coordinates

It is well known that the convenient choice of coordinates plays an important role in classical physics. Let us discuss here the procedure of introduction of the curvilinear coordinates.

In general case any three quantities, which in follows will be denoted as (y^1, y^2, y^3), can be used as coordinates if they are well-behaved (diffeomorphism) functions of the Cartesian coordinates and vice versa. The functions that give the direct transformation (from curvilinear to Cartesian) are:

$$
\begin{aligned}
x &= x(y^1, y^2, y^3) && \text{(E.1)}\\
y &= y(y^1, y^2, y^3) && \text{(E.2)}\\
z &= y(y^1, y^2, y^3) && \text{(E.3)}
\end{aligned}
$$

The inverse transformation (from curvilinear to Cartesian) can be obtained by solving the above system of equations for the arguments (y^1, y^2, y^3):

$$
\begin{aligned}
y^1 &= y^1(x, y, z) && \text{(E.4)}\\
y^2 &= y^2(x, y, z) && \text{(E.5)}\\
y^3 &= y^3(x, y, z) && \text{(E.6)}
\end{aligned}
$$

A given point $\mathbf{x} \in \mathbb{R}^3$ may be described by specifying either the set (x, y, z) or (y^1, y^2, y^3). Each of the equations $y^i = y^i(x, y, z)$, that define the inverse transformation (E.6), describes a surface in the new coordinates and the intersection of three such surfaces locates the point in the three-dimensional space. The surfaces $y^i = \text{const}$ are called the *c*oordinate surfaces; the space curves formed by their intersection in pairs are called the *c*oordinate lines. The *c*oordinate axes are determined by the tangents to the coordinate lines at the intersection of three surfaces. They are not in general fixed directions in space, as is true for simple Cartesian coordinates. The quantities (y^1, y^2, y^3) are the curvilinear coordinates of a point $\mathbf{x}$.

Then the Jacobian of the direct transformation:

$$\mathcal{J} = \det \frac{\partial (x, y, z)}{\partial (y^1, y^2, y^3)} = \det \begin{pmatrix} \dfrac{\partial x}{\partial y^1}, \dfrac{\partial x}{\partial y^2}, \dfrac{\partial x}{\partial y^3} \\ \\ \dfrac{\partial y}{\partial y^1}, \dfrac{\partial y}{\partial y^2}, \dfrac{\partial y}{\partial y^3} \\ \\ \dfrac{\partial z}{\partial y^1}, \dfrac{\partial z}{\partial y^2}, \dfrac{\partial z}{\partial y^3} \end{pmatrix} \tag{E.7}$$

cannot be infinite. Note that the expression for the Jacobian $\mathcal{J}$ can be rewritten as (here we use the decomposition of the Jacobian matrix by the first column)

$$\mathcal{J} \equiv \frac{\partial \mathbf{x}}{\partial y^1} \cdot \left(\frac{\partial \mathbf{x}}{\partial y^2} \times \frac{\partial \mathbf{x}}{\partial y^3} \right) \tag{E.8}$$

that represents an elementary volume.

The Jacobian of the inverse transformation:

$$\mathcal{J}^{-1} = \det \frac{\partial (y^1, y^2, y^3)}{\partial (x, y, z)} = \det \begin{pmatrix} \dfrac{\partial y^1}{\partial x}, \dfrac{\partial y^1}{\partial y}, \dfrac{\partial y^1}{\partial z} \\ \\ \dfrac{\partial y^2}{\partial x}, \dfrac{\partial y^2}{\partial y}, \dfrac{\partial y^2}{\partial z} \\ \\ \dfrac{\partial y^3}{\partial x}, \dfrac{\partial z}{\partial y}, \dfrac{\partial y^3}{\partial z} \end{pmatrix} \tag{E.9}$$

cannot be correspondingly equal to zero. Similarly to direct transform case, this relation can be rewritten as (here we use the decomposition of the Jacobian matrix by the first line)

$$\mathcal{J}^{-1} \equiv \nabla y^1 \cdot \left(\nabla y^2 \times \nabla y^3 \right) \tag{E.10}$$

where $\nabla y^i \equiv (\partial_x y^i, \partial_y y^i, \partial_z y^i)$

E.1.1 Covariant and contravariant representation

Basis vectors are usually associated with a coordinate system by two methods:

- they can be built along the coordinate axes (collinear to axes), tangent vectors $\partial\mathbf{x}/\partial y^a$
- they can be built to be perpendicular (normal) to the coordinate surfaces given by gradient of three coordinates ∇y^i

In the first case we deal with *t*he covariant basis vector representation and in the second case with *t*he contravariant basis vector representation: They are related by the orthogonality relation

$$\nabla y^a \cdot \frac{\partial \mathbf{x}}{\partial y^b} = \delta^a_b \tag{E.11}$$

The corresponding relation for the Jacobian is

$$\mathcal{J} = \frac{\partial \mathbf{x}}{\partial y^1} \cdot \left(\frac{\partial \mathbf{x}}{\partial y^2} \times \frac{\partial \mathbf{x}}{\partial y^3} \right) \equiv \left(\nabla y^1 \cdot \left(\nabla y^2 \times \nabla y^3 \right) \right)^{-1} \tag{E.12}$$

Cylindrical coordinates

Direct coordinate transformation:

$$\mathbf{x}(R, \theta, Z) = R \sin\phi \; \hat{\mathbf{x}} + R\cos\phi \; \hat{\mathbf{y}} + Z \; \hat{\mathbf{z}} \tag{E.13}$$

with Cartesian unit vectors: $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$. Note that here we do not make a difference between covariant and contravariant Cartesian vectors, for example: $\dfrac{\partial \mathbf{x}}{\partial x} = \nabla x \equiv \hat{\mathbf{x}}$, because the corresponding metric tensor is equal to the identity tensor in the both cases.

The covariant (tangent) vectors in the new coordinates (r, ϕ, Z)

$$\frac{\partial \mathbf{x}}{\partial R} = \begin{pmatrix} \sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \frac{\partial \mathbf{x}}{\partial \phi} = R \begin{pmatrix} \cos\phi \\ -\sin\phi \\ 0 \end{pmatrix}, \frac{\partial \mathbf{x}}{\partial Z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{E.14}$$

The contravariant vectors in new coordinates

$$\nabla R = \begin{pmatrix} \sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \nabla\phi = \frac{1}{R} \begin{pmatrix} -\cos\phi \\ \sin\phi \\ 0 \end{pmatrix}, \nabla Z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{E.15}$$

The covariant vectors in old (Cartesian) coordinates

$$\frac{\partial \mathbf{x}}{\partial R} = \frac{1}{\sqrt{x^2 + y^2}} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}, \frac{\partial \mathbf{x}}{\partial \phi} = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}, \frac{\partial \mathbf{x}}{\partial Z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{E.16}$$

and the contravariant vectors

$$\nabla R = \frac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}, \nabla \phi = \frac{1}{x^2+y^2} \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}, \nabla Z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{E.17}$$

Here $R = \sqrt{x^2+y^2}$ and $\tan\phi = \frac{x}{y} \equiv \frac{\partial_x R}{\partial_y R}$

E.1.2 Metric tensor

The metric tensor in the covariant and the contravariant representations is given by

$$g_{ab} = \frac{\partial \mathbf{x}}{\partial y^a} \cdot \frac{\partial \mathbf{x}}{\partial y^b} \tag{E.18}$$

$$g^{ab} = \nabla y^a \cdot \nabla y^b \tag{E.19}$$

In what follows we will consider axisymmetric coordinates system (ϕ, θ, ψ) where $\phi = y^1$ denotes the toroidal coordinate and θ and ψ corresponds to two remaining orthogonal directions, which we will represent for instance as y^a, where $a \in \{2, 3\}$.

When the basis vectors are orthogonal, the metric tensor is diagonal

$$g_{ab} = \begin{pmatrix} g_{\phi\phi} & 0 & 0 \\ 0 & g_{\theta\theta} & 0 \\ 0 & 0 & g_{\psi\psi} \end{pmatrix} \tag{E.20}$$

$$g^{ab} = \begin{pmatrix} g^{\phi\phi} & 0 & 0 \\ 0 & g^{\theta\theta} & 0 \\ 0 & 0 & g^{\psi\psi} \end{pmatrix} \tag{E.21}$$

where the coefficients are given by

$$g_{\phi\phi} = \frac{\partial \mathbf{x}}{\partial \phi} \cdot \frac{\partial \mathbf{x}}{\partial \phi} = \left| \frac{\partial \mathbf{x}}{\partial \phi} \right|^2 = R^2 \tag{E.22}$$

$$g_{\theta\theta} = \frac{\partial \mathbf{x}}{\partial \theta} \cdot \frac{\partial \mathbf{x}}{\partial \theta} = \left| \frac{\partial \mathbf{x}}{\partial \theta} \right|^2 \tag{E.23}$$

$$g_{\psi\psi} = \frac{\partial \mathbf{x}}{\partial \psi} \cdot \frac{\partial \mathbf{x}}{\partial \psi} = \left| \frac{\partial \mathbf{x}}{\partial \psi} \right|^2 \tag{E.24}$$

$$g^{\phi\phi} = \nabla\phi \cdot \nabla\phi = |\nabla\phi|^2 = 1/\left| \frac{\partial \mathbf{x}}{\partial \phi} \right|^2 = \frac{1}{R^2} \tag{E.25}$$

$$g^{\theta\theta} = \nabla\theta \cdot \nabla\theta = |\nabla\theta|^2 = 1/\left| \frac{\partial \mathbf{x}}{\partial \theta} \right|^2 \tag{E.26}$$

$$g^{\psi\psi} = \nabla\psi \cdot \nabla\psi = |\nabla\psi|^2 = 1/\left| \frac{\partial \mathbf{x}}{\partial \psi} \right|^2 \tag{E.27}$$

The Jacobian

$$\det(g_{ab}) = g_{\phi\phi}\, g_{\theta\theta}\, g_{\psi\psi} = \mathcal{J}^2 \tag{E.28}$$
$$\det(g^{ab}) = g^{\phi\phi}\, g^{\theta\theta}\, g^{\psi\psi} = \mathcal{J}^{-2} \tag{E.29}$$

Coefficients transformation

The correspondence between the coefficients of the tensor C in the covariant and the contravariant representations

$$\mathbf{C} = C_a \nabla y^a + C_\phi \nabla\phi = C^a \frac{\partial \mathbf{x}}{\partial y^a} + C^\phi \frac{\partial \mathbf{x}}{\partial \phi} \tag{E.30}$$

is given by the metric tensor $C_a = g_{ab} C^b$, $C^a = g^{ab} C_b$

E.1.3 Dyadic identity tensor and gradient

The covariant basis vectors

$$\frac{\partial \mathbf{x}}{\partial \phi}, \frac{\partial \mathbf{x}}{\partial \theta}, \frac{\partial \mathbf{x}}{\partial \psi} \tag{E.31}$$

with their norm

$$\left|\frac{\partial \mathbf{x}}{\partial \phi}\right| = \sqrt{g_{\phi\phi}} = R\ , \quad \left|\frac{\partial \mathbf{x}}{\partial \theta}\right| = \sqrt{g_{\theta\theta}}\ , \quad \left|\frac{\partial \mathbf{x}}{\partial \psi}\right| = \sqrt{g_{\psi\psi}}\ , \tag{E.32}$$

So the dyadic tensor has a form

$$\mathbf{I} = \frac{1}{R^2}\frac{\partial \mathbf{x}}{\partial \phi}\frac{\partial \mathbf{x}}{\partial \phi} + \frac{1}{g_{\theta\theta}}\frac{\partial \mathbf{x}}{\partial \theta}\frac{\partial \mathbf{x}}{\partial \theta} + \frac{1}{g_{\psi\psi}}\frac{\partial \mathbf{x}}{\partial \psi}\frac{\partial \mathbf{x}}{\partial \psi} \tag{E.33}$$

Note also that

$$\nabla = \frac{\partial}{\partial \mathbf{x}} \tag{E.34}$$

so

$$\frac{\partial \mathbf{x}}{\partial \phi} \cdot \nabla = \frac{\partial}{\partial \phi} \tag{E.35}$$

Finally into the covariant basis

$$\nabla = \mathbf{I} \cdot \nabla = \frac{1}{R^2}\frac{\partial \mathbf{x}}{\partial \phi}\left(\frac{\partial}{\partial \phi}\right) + \frac{1}{g_{\theta\theta}}\frac{\partial \mathbf{x}}{\partial \theta}\left(\frac{\partial}{\partial \theta}\right) + \frac{1}{g_{\psi\psi}}\frac{\partial \mathbf{x}}{\partial \psi}\left(\frac{\partial}{\partial \psi}\right) \tag{E.36}$$

For the contravariant basis decomposition:

$$|\nabla\phi| = \frac{1}{\sqrt{g_{\phi\phi}}} = \frac{1}{R},\ |\nabla\theta| = \frac{1}{\sqrt{g_{\theta\theta}}},\ |\nabla\psi| = \frac{1}{\sqrt{g_{\psi\psi}}}, \tag{E.37}$$

So the dyadic tensor

$$\mathbf{I} = \frac{1}{g^{\phi\phi}}\nabla\phi\nabla\phi+\frac{1}{g^{\theta\theta}}\nabla\theta\nabla\theta+\frac{1}{g^{\psi\psi}}\nabla\psi\nabla\psi = R^2\nabla\phi\nabla\phi+g_{\theta\theta}\nabla\theta\nabla\theta+g_{\psi\psi}\nabla\psi\nabla\psi \quad \text{(E.38)}$$

$$\nabla\phi\cdot\nabla = \frac{1}{R^2}\frac{\partial}{\partial\phi},\ \nabla\theta\cdot\nabla = \frac{1}{g_{\theta\theta}}\frac{\partial}{\partial\theta},\ \nabla\psi\cdot\nabla = \frac{1}{g_{\psi\psi}}\frac{\partial}{\partial\psi}, \quad \text{(E.39)}$$

And finally

$$\nabla = \nabla\phi\frac{\partial}{\partial\phi}+\nabla\theta\frac{\partial}{\partial\theta}+\nabla\psi\frac{\partial}{\partial\psi} \quad \text{(E.40)}$$

E.2 Momentum conservation law projection

In this Section we consider toroidal gyrokinetic momentum conservation equation (3.112).

The gyrokinetic momentum stress tensor in the electrostatic perturbation case (E.38) has a dyadic form. Here we evaluate

$$\frac{\partial\mathbf{x}}{\partial\phi}\cdot\nabla\cdot\bar{\mathbf{\Pi}}_{gy} = \quad \text{(E.41)}$$

$$\frac{\epsilon^2}{8\pi}\frac{\partial\mathbf{x}}{\partial\phi}\cdot\nabla\cdot\left(\,|\mathbf{E}_1|^2\,\mathbf{I}\right) - \frac{\epsilon^2}{4\pi}\frac{\partial\mathbf{x}}{\partial\phi}\cdot\nabla\cdot\ (\mathbf{E}_1\mathbf{E}_1)$$
$$+\ \int m\,\frac{\partial\mathbf{x}}{\partial\phi}\cdot\nabla\cdot\left(\,F\,\dot{\mathbf{X}}_{gy}\,\dot{\mathbf{X}}_{gy}\right)\,d^3p \quad \text{(E.42)}$$

In what follows we use the next formula for projection of the divergence of the dyadic tensor $\mathbf{CC}$ on the toroidal direction $\partial\mathbf{x}/\partial\phi$

$$\frac{\partial\mathbf{x}}{\partial\phi}\cdot\nabla\cdot(\mathbf{CC}) = C_\phi\left(\frac{1}{\mathcal{J}}\frac{\partial}{\partial y^a}(\mathcal{J}C^a)+\frac{\partial C^\phi}{\partial\phi}\right) - C^a\left(\frac{\partial C_\phi}{\partial y^a}-\frac{\partial C_a}{\partial\phi}\right)+\frac{1}{2}\frac{\partial}{\partial\phi}\left(C_aC^a+C_\phi C^\phi\right) \quad \text{(E.43)}$$

E.2.1 $\dfrac{\partial\mathbf{x}}{\partial\phi}\cdot\nabla\cdot|\mathbf{E}_1|^2\mathbf{I}$

First we need to identify the covariant and the contravariant coordinates for the tensor $|\mathbf{E}_1|^2\mathbf{I}$.

Following the equation (E.36) for covariant representation of the dyadic identity tensor, we identify its coordinates in the covariant basis :

$$|\mathbf{E}_1|^2\ \mathbf{I} = \left(|\mathbf{E}_1|\frac{1}{R}\frac{\partial \mathbf{x}}{\partial \phi}|\mathbf{E}_1|\frac{1}{R}\frac{\partial \mathbf{x}}{\partial \phi} + |\mathbf{E}_1|\frac{1}{\sqrt{g_{\theta\theta}}}\frac{\partial \mathbf{x}}{\partial \theta}|\mathbf{E}_1|\frac{1}{\sqrt{g_{\theta\theta}}}\frac{\partial \mathbf{x}}{\partial \theta} + |\mathbf{E}_1|\frac{1}{\sqrt{g_{\psi\psi}}}\frac{\partial \mathbf{x}}{\partial \psi}|\mathbf{E}_1|\frac{1}{\sqrt{g_{\psi\psi}}}\frac{\partial \mathbf{x}}{\partial \psi}\right)$$

$$\Rightarrow\ \mathbf{C} = |\mathbf{E}_1|\frac{1}{R}\frac{\partial \mathbf{x}}{\partial \phi} + |\mathbf{E}_1|\frac{1}{\sqrt{g_{\theta\theta}}}\frac{\partial \mathbf{x}}{\partial \theta} + |\mathbf{E}_1|\frac{1}{\sqrt{g_{\psi\psi}}}\frac{\partial \mathbf{x}}{\partial \psi} \tag{E.44}$$

$$\Rightarrow\ C^{\phi} = |\mathbf{E}_1|\frac{1}{R},\quad C^{a} = |\mathbf{E}_1|\frac{1}{\sqrt{g_{aa}}} \tag{E.45}$$

At the same time using the eq.(E.38)for contravariant representation of the dyadic identity tensor, we identify its coordinates in the contravariant basis :

$$|\mathbf{E}_1|^2\ \mathbf{I} = \left(|\mathbf{E}_1|\ R\ \nabla\phi|\mathbf{E}_1|\ R\ \nabla\phi + |\mathbf{E}_1|\sqrt{g_{\theta\theta}}\frac{\partial \mathbf{x}}{\partial \theta}|\mathbf{E}_1|\sqrt{g_{\theta\theta}}\frac{\partial \mathbf{x}}{\partial \theta} + |\mathbf{E}_1|\sqrt{g_{\psi\psi}}\frac{\partial \mathbf{x}}{\partial \psi}|\mathbf{E}_1|\sqrt{g_{\psi\psi}}\frac{\partial \mathbf{x}}{\partial \psi}\right)$$

$$\Rightarrow\ \mathbf{C} = |\mathbf{E}_1|R\nabla\phi + |\mathbf{E}_1|\sqrt{g_{\theta\theta}}\nabla\theta + |\mathbf{E}_1|\sqrt{g_{\psi\psi}}\nabla\psi \tag{E.46}$$

$$\Rightarrow\ C_{\phi} = |\mathbf{E}_1|R,\quad C_{a} = |\mathbf{E}_1|\sqrt{g_{aa}} \tag{E.47}$$

Now we can apply the formula (E.43) in order to obtain $\frac{\partial \mathbf{x}}{\partial \phi}\cdot\nabla\cdot|\mathbf{E}_1|^2\mathbf{I}$.

$$C_{\phi}\left(\frac{1}{\mathcal{J}}\frac{\partial}{\partial y^a}(\mathcal{J}C^a) + \frac{\partial C^{\phi}}{\partial \phi}\right) =$$

$$(|E|R)\left(\frac{1}{\mathcal{J}}\frac{\partial}{\partial \theta}\left(\mathcal{J}|E|\frac{1}{\sqrt{g_{\theta\theta}}}\right)\right) + (|E|R)\left(\frac{1}{\mathcal{J}}\frac{\partial}{\partial \psi}\left(\mathcal{J}|E|\frac{1}{\sqrt{g_{\psi\psi}}}\right)\right) + |E|R\frac{1}{R}\frac{\partial}{\partial \phi}|E|$$

$$= |E|\frac{1}{\sqrt{g_{\theta\theta}g_{\psi\psi}}}\left(\frac{\partial}{\partial \theta}\left(R\sqrt{g_{\psi\psi}}|E|\right) + \frac{\partial}{\partial \psi}\left(R\sqrt{g_{\theta\theta}}|E|\right)\right) + \frac{1}{2}\frac{\partial}{\partial \phi}|E|^2 \tag{E.48}$$

$$-C^{a}\left(\frac{\partial C_{\phi}}{\partial y^a} - \frac{\partial C_a}{\partial \phi}\right) =$$

$$-|E|\frac{1}{\sqrt{g_{\theta\theta}}}\left(\frac{\partial(|E|R)}{\partial \theta} - \frac{\partial(|E|\sqrt{g_{\theta\theta}})}{\partial \phi}\right) - |E|\frac{1}{\sqrt{g_{\psi\psi}}}\left(\frac{\partial(|E|R)}{\partial \psi} - \frac{\partial(|E|\sqrt{g_{\psi\psi}})}{\partial \phi}\right)$$

$$= \quad -|E|\frac{1}{\sqrt{g_{\theta\theta}}}\frac{\partial(|E|R)}{\partial \theta} - |E|\frac{1}{\sqrt{g_{\psi\psi}}}\frac{\partial(|E|R)}{\partial \psi} + \frac{\partial|E|^2}{\partial \phi} \tag{E.49}$$

$$\frac{1}{2}\frac{\partial}{\partial \phi}\left(C_a C^a + C_{\phi}C^{\phi}\right) = \frac{1}{2}\frac{\partial}{\partial \phi}|E|^2 \tag{E.50}$$

So finally we obtain:

$$\frac{\partial \mathbf{x}}{\partial \phi}\cdot\nabla\cdot|\mathbf{E}_1|^2\mathbf{I} \quad = \tag{E.51}$$

$$= \quad \frac{1}{2}\frac{|E_1|^2 R}{\sqrt{g_{\theta\theta}g_{\psi\psi}}}\left(\frac{\partial}{\partial \theta}\sqrt{g_{\psi\psi}} + \frac{\partial}{\partial \psi}\sqrt{g_{\theta\theta}}\right) + \frac{\partial}{\partial \phi}|E_1|^2$$

E.2.2 $\frac{\partial \mathbf{x}}{\partial \phi} \cdot \nabla \cdot \mathbf{E}_1\mathbf{E}_1$

Here we deal with the second term in the expression (E.42). By applying the formula (E.43) we obtain:

$$
\begin{aligned}
&C_\phi \left(\frac{1}{\mathcal{J}} \frac{\partial}{\partial y^a} (\mathcal{J} C^a) + \frac{\partial C^\phi}{\partial \phi} \right) \\
&= E_\phi \left(\frac{1}{\mathcal{J}} \frac{\partial}{\partial \theta} \left(\mathcal{J} E^\theta \right) \right) + E_\phi \left(\frac{1}{\mathcal{J}} \frac{\partial}{\partial \psi} \left(\mathcal{J} E^\psi \right) \right) + E_\phi \frac{\partial E^\phi}{\partial \phi} = \\
&E_\phi \frac{\partial E^\theta}{\partial \theta} + E_\phi \frac{\partial E^\psi}{\partial \psi} + \frac{1}{\mathcal{J}} \left(E^\theta \frac{\partial \mathcal{J}}{\partial \theta} + E^\psi \frac{\partial \mathcal{J}}{\partial \psi} \right) + E_\phi \frac{\partial E^\phi}{\partial \phi}
\end{aligned}
\tag{E.52}
$$

$$
\begin{aligned}
&-C^a \left(\frac{\partial C_\phi}{\partial y^a} - \frac{\partial C_a}{\partial \phi} \right) \\
&= -E^\theta \frac{\partial E_\phi}{\partial \theta} + E^\theta \frac{\partial E_\theta}{\partial \phi} - E^\psi \frac{\partial E_\phi}{\partial \psi} + E^\psi \frac{\partial E_\psi}{\partial \phi}
\end{aligned}
\tag{E.53}
$$

$$
\frac{1}{2} \frac{\partial}{\partial \phi} \left(C_a C^a + C_\phi C^\phi \right) = \frac{1}{2} \frac{\partial}{\partial \phi} |E_1|^2
\tag{E.54}
$$

Note that

$$
E_\phi \frac{\partial E^\phi}{\partial \phi} = E^\phi \frac{\partial E_\phi}{\partial \phi} = \frac{\partial \left(E_\phi E^\phi \right)}{\partial \phi}
\tag{E.55}
$$

because

$$
E_\phi = R^2 E^\phi \,, E^\phi = \frac{1}{R^2} E_\phi \Rightarrow \; E_\phi E^\phi = E^\phi E_\phi
\tag{E.56}
$$

and

$$
\frac{\partial \mathcal{J}}{\partial \phi} = 0
\tag{E.57}
$$

Then finally the blue terms gives the derivative of the norm of E

$$
E_\phi \frac{\partial E^\phi}{\partial \phi} + E_\theta \frac{\partial E^\theta}{\partial \phi} + E_\psi \frac{\partial E^\psi}{\partial \phi} = \frac{1}{2} \frac{\partial |E|^2}{\partial \phi}
\tag{E.58}
$$

Lastly, for the second term we have:

$$
\begin{aligned}
&\frac{\partial \mathbf{x}}{\partial \phi} \cdot \nabla \cdot \mathbf{E}_1\mathbf{E}_1 = \\
&\left(E_\phi \frac{\partial E^\theta}{\partial \theta} - E^\theta \frac{\partial E_\phi}{\partial \theta} \right) + \left(E_\phi \frac{\partial E^\psi}{\partial \psi} - E^\psi \frac{\partial E_\phi}{\partial \psi} \right) + \frac{1}{\mathcal{J}} E_\phi \left(E^\theta \frac{\partial \mathcal{J}}{\partial \theta} + E^\psi \frac{\partial \mathcal{J}}{\partial \psi} \right) + \frac{\partial}{\partial \phi} |E_1|^2
\end{aligned}
\tag{E.59}
$$

E.2.3 Vlasov term

Here we will need to use the following tensor relation:

$$\frac{\partial \mathbf{x}}{\partial \phi} \cdot \nabla \cdot (f\ \mathbf{CC}) = f\ \frac{\partial \mathbf{x}}{\partial \phi} \cdot \nabla \cdot (\mathbf{CC}) + C_\phi\, (\mathbf{C} \cdot \nabla f) \tag{E.60}$$

where f is some scalar function.

then we apply this formula on order to project the Vlasov part of eq. (E.42).

Using the eq.(E.43) we obtain:

$$m \int d^3p\ F\ \dot{\mathbf{X}}\dot{\mathbf{X}} = m \int d^3p\ F \left[\left(\dot{\mathbf{X}}_\phi \frac{\partial \dot{\mathbf{X}}^\theta}{\partial \theta} - \dot{\mathbf{X}}^\theta \frac{\partial \dot{\mathbf{X}}_\phi}{\partial \theta} \right) \right. \tag{E.61}$$

$$\left. + \left(\dot{\mathbf{X}}_\phi \frac{\partial \dot{\mathbf{X}}^\psi}{\partial \psi} - \dot{\mathbf{X}}^\psi \frac{\partial \dot{\mathbf{X}}_\phi}{\partial \psi} \right) + \frac{1}{\mathcal{J}} \dot{\mathbf{X}}_\phi \left(\dot{\mathbf{X}}^\theta \frac{\partial \mathcal{J}}{\partial \theta} + \dot{\mathbf{X}}^\psi \frac{\partial \mathcal{J}}{\partial \psi} \right) + \frac{\partial}{\partial \phi} |\dot{\mathbf{X}}|^2 \right] \tag{E.62}$$

$$+ m \int d^3p\ \dot{\mathbf{X}}_\phi \left(\dot{\mathbf{X}} \cdot \nabla F \right)$$

here $\mathbf{X} \equiv \mathbf{X}_{gy}$

E.2.4 Final result: general axisymmetric geometry

By combining the equations (E.51),(E.59),(E.63) we have:

$$\frac{\partial \mathbf{x}}{\partial \phi} \cdot \nabla \cdot \bar{\mathbf{\Pi}}_{gy} = \tag{E.63}$$

$$\frac{\epsilon^2}{8\pi} \frac{|E_1|^2 \sqrt{g_{\phi\phi}}}{\sqrt{g_{\theta\theta} g_{\psi\psi}}} \left(\frac{\partial}{\partial \theta} \sqrt{g_{\psi\psi}} + \frac{\partial}{\partial \psi} \sqrt{g_{\theta\theta}} \right) - \frac{\epsilon^2}{4\pi} \frac{\partial}{\partial \phi} |\mathbf{E}_1|^2 + \tag{E.64}$$

$$-\frac{\epsilon^2}{4\pi} \left[\left(E_\phi \frac{\partial E^\theta}{\partial \theta} - E^\theta \frac{\partial E_\phi}{\partial \theta} \right) + \left(E_\phi \frac{\partial E^\psi}{\partial \psi} - E^\psi \frac{\partial E_\phi}{\partial \psi} \right) + \frac{1}{\mathcal{J}} E_\phi \left(E^\theta \frac{\partial \mathcal{J}}{\partial \theta} + E^\psi \frac{\partial \mathcal{J}}{\partial \psi} \right) \right]$$

$$+ m \int d^3p\ F \left[\left(\dot{\mathbf{X}}_\phi \frac{\partial \dot{\mathbf{X}}^\theta}{\partial \theta} - \dot{\mathbf{X}}^\theta \frac{\partial \dot{\mathbf{X}}_\phi}{\partial \theta} \right) \right. \tag{E.65}$$

$$\left. + \left(\dot{\mathbf{X}}_\phi \frac{\partial \dot{\mathbf{X}}^\psi}{\partial \psi} - \dot{\mathbf{X}}^\psi \frac{\partial \dot{\mathbf{X}}_\phi}{\partial \psi} \right) + \frac{1}{\mathcal{J}} \dot{\mathbf{X}}_\phi \left(\dot{\mathbf{X}}^\theta \frac{\partial \mathcal{J}}{\partial \theta} + \dot{\mathbf{X}}^\psi \frac{\partial \mathcal{J}}{\partial \psi} \right) + \frac{\partial}{\partial \phi} |\dot{\mathbf{X}}|^2 \right] \tag{E.66}$$

$$+ m \int d^3p\ \dot{\mathbf{X}}_\phi \left(\dot{\mathbf{X}} \cdot \nabla F \right)$$

E.2.5 Final result:cylindrical geometry

In the case of cylindrical geometry $g^{RR} = g_{RR} = g^{ZZ} = g_{ZZ} = 1$ and $g_{\phi\phi} = R^2$, then $\mathcal{J} \equiv R$

In this case the result of the projection on the toroidal direction for momentum conservation law has the following form:

$$
\begin{aligned}
&\frac{\partial \mathbf{x}}{\partial \phi} \cdot \nabla \cdot \bar{\mathbf{\Pi}}_{gy} = \\
&-\frac{\epsilon^2}{4\pi}\left[\frac{\partial}{\partial \phi}|\mathbf{E}_1|^2 + \left(E_\phi \frac{\partial E^Z}{\partial Z} - E^Z \frac{\partial E_\phi}{\partial Z}\right) + \left(E_\phi \frac{\partial E^R}{\partial R} - E^R \frac{\partial E_\phi}{\partial R}\right) + \frac{1}{R} E_\phi E^R\right] \\
&+m \int d^3p\, F\, \dot{\mathbf{X}}_\phi \left[\frac{\partial \dot{\mathbf{X}}^Z}{\partial Z} + \frac{1}{R}\dot{\mathbf{X}}^R + \frac{\partial \dot{\mathbf{X}}^Z}{\partial Z} + \dot{\mathbf{X}} \cdot \nabla F\right] - m \int d^3p\, F\, \left[\dot{\mathbf{X}}^Z \frac{\partial \mathbf{X}^\phi}{\partial Z} + \frac{\partial}{\partial \phi}|\dot{\mathbf{X}}|^2\right]
\end{aligned}
\tag{E.67}
$$

Appendix F

Local Poisson bracket

F.1 Calculation of the brackets $\{z_i, z_j\}_{old}$

Here is presented the calculation which leads to the expression (4.39)

$$
\begin{aligned}
\{\zeta, \mathbf{x}\} &= \frac{\partial \zeta}{\partial \mathbf{p}} \cdot \frac{\partial \mathbf{x}}{\partial \mathbf{r}} = -\frac{\hat{\mathbf{p}}_2}{p \sin\varphi} \\
\{\varphi, \mathbf{x}\} &= \frac{\partial \varphi}{\partial \mathbf{p}} \cdot \frac{\partial \mathbf{x}}{\partial \mathbf{r}} = \frac{\hat{\mathbf{p}}_1}{p} \\
\{p, \zeta\} &= \frac{\partial p}{\partial \mathbf{p}} \cdot \frac{\partial \zeta}{\partial \mathbf{r}} - \frac{\partial p}{\partial \mathbf{r}} \cdot \frac{\partial \zeta}{\partial \mathbf{p}} - e\mathbf{B} \frac{\partial p}{\partial \mathbf{p}} \times \frac{\partial \zeta}{\partial \mathbf{p}} = -\frac{\hat{\mathbf{p}}_0 \cdot M \cdot \hat{\mathbf{p}}_1}{\sin\varphi} + e\mathbf{B} \cdot \left(\hat{\mathbf{p}}_0 \times \frac{\hat{\mathbf{p}}_2}{p \sin\varphi} \right) = \\
& -\frac{\hat{\mathbf{p}}_1 \cdot M \cdot \hat{\mathbf{p}}_0}{\sin\varphi} - e\mathbf{B} \cdot \left(\frac{\hat{\mathbf{p}}_1}{p \sin\varphi} \right) = -\frac{\hat{\mathbf{p}}_1 \cdot M \cdot \hat{\mathbf{p}}_0}{\sin\varphi} - eB \frac{\hat{\mathbf{b}}_0 \cdot \hat{\mathbf{p}}_1}{p \sin\varphi} = -\frac{\hat{\mathbf{p}}_1 \cdot M \cdot \hat{\mathbf{p}}_0}{\sin\varphi} + \frac{eB}{p} \\
\{p, \varphi\} &= \frac{\partial p}{\partial \mathbf{p}} \cdot \frac{\partial \varphi}{\partial \mathbf{r}} - \frac{\partial p}{\partial \mathbf{r}} \cdot \frac{\partial \varphi}{\partial \mathbf{p}} - e\mathbf{B} \frac{\partial p}{\partial \mathbf{p}} \times \frac{\partial \varphi}{\partial \mathbf{p}} = -\hat{\mathbf{p}}_0 \cdot M \cdot \hat{\mathbf{p}}_2 - e\mathbf{B} \left(\frac{\hat{\mathbf{p}}_0 \times \hat{\mathbf{p}}_1}{p} \right) = \\
& -\hat{\mathbf{p}}_0 \cdot M \cdot \hat{\mathbf{p}}_2 - eB \frac{\hat{\mathbf{b}}_0 \cdot \hat{\mathbf{p}}_2}{p} = -\hat{\mathbf{p}}_0 \cdot M \cdot \hat{\mathbf{p}}_2 \\
\{\zeta, \varphi\} &= \frac{\partial \zeta}{\partial \mathbf{p}} \cdot \frac{\partial \varphi}{\partial \mathbf{r}} - \frac{\partial \zeta}{\partial \mathbf{r}} \cdot \frac{\partial \varphi}{\partial \mathbf{p}} - e\mathbf{B} \cdot \left(\frac{\partial \zeta}{\partial \mathbf{p}} \times \frac{\partial \varphi}{\partial \mathbf{p}} \right) = \\
& \frac{\hat{\mathbf{p}}_1 \cdot M \cdot \hat{\mathbf{p}}_1}{p \sin\varphi} + \frac{\hat{\mathbf{p}}_2 \cdot M \cdot \hat{\mathbf{p}}_2}{p \sin\varphi} + eB \left(\frac{\hat{\mathbf{p}}_2}{p \sin\varphi} \times \frac{\hat{\mathbf{p}}_1}{p} \right) = \\
& \frac{1}{p \sin\varphi} \left(\hat{\mathbf{p}}_1 \cdot M \cdot \hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2 \cdot M \cdot \hat{\mathbf{p}}_2 - \frac{eB \cos\varphi}{p} \right)
\end{aligned}
$$

Appendix G

Series decomposition for Hamiltonian normal form.

G.1 Second order

Here we proceed with detailed calculation which leads to the second order partial differential equation for the Hamiltonian normal form. This equation provides the averaged part of the first order correction $\langle p_1 \rangle$ and the fluctuating part of the second order correction $\widetilde{p}_2$

$$
\begin{aligned}
\mathcal{D}\widetilde{p}_2 &= -\frac{p_0}{eB}\left(\hat{\mathbf{b}}\cos\varphi + \hat{\perp}\sin\varphi\right)\cdot\nabla\left(\langle p_1\rangle + \widetilde{p}_1\right) \\
&- \frac{(\langle p_1\rangle + \widetilde{p}_1)}{eB}\left(\hat{\mathbf{b}}\cos\varphi + \hat{\perp}\sin\varphi\right)\cdot\nabla p_0 \qquad \text{(G.1)}
\end{aligned}
$$

We start with deriving $\nabla\widetilde{p}_1$

$$
\begin{aligned}
\nabla\widetilde{p}_1 &= \frac{\mathcal{A}e^{-\varrho}}{e\ \sin\varphi}\Big(-\nabla\Phi^2\hat{\boldsymbol{\rho}}\ \hat{\mathbf{b}} : \nabla\hat{\mathbf{b}} - \Phi^2(\nabla\hat{\mathbf{b}})\cdot(\nabla\hat{\mathbf{b}})\cdot\hat{\boldsymbol{\rho}} - \Phi^2\hat{\mathbf{b}}\cdot\nabla\nabla\hat{\mathbf{b}}\cdot\hat{\boldsymbol{\rho}} - \Phi^2\ \hat{\mathbf{b}}_i\ (\partial_i\ \hat{\mathbf{b}}_k)\ (\partial_j\ \hat{\boldsymbol{\rho}}_k) \\
&- \frac{1}{2}\nabla\hat{\boldsymbol{\rho}}\cdot\left(\frac{\nabla B}{B} - \nabla\varrho\right) - \frac{1}{2}\hat{\boldsymbol{\rho}}_i\ \partial_j\ \frac{\partial_i B}{B} + \frac{1}{2}\hat{\boldsymbol{\rho}}\cdot\nabla\nabla\varrho - \frac{1}{4}\nabla\Phi(\hat{\boldsymbol{\rho}}\hat{\perp} + \hat{\perp}\hat{\boldsymbol{\rho}}) : \nabla\hat{\mathbf{b}} \qquad \text{(G.2)} \\
&- \frac{1}{4}\Phi(\nabla\hat{\boldsymbol{\rho}}\cdot\nabla\hat{\mathbf{b}}\cdot\hat{\perp} + \nabla\hat{\perp}\cdot\nabla\hat{\mathbf{b}}\cdot\hat{\boldsymbol{\rho}}) \\
&- \frac{1}{4}\Phi(\hat{\perp}\cdot\nabla\nabla\hat{\mathbf{b}}\cdot\hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\rho}}\cdot\nabla\nabla\hat{\mathbf{b}}\cdot\hat{\perp}) - \frac{1}{4}\Phi(\hat{\perp}_i\ \partial_i\ \hat{\mathbf{b}}_k\ \partial_j\ \hat{\boldsymbol{\rho}}_k + \hat{\boldsymbol{\rho}}_i\ \partial_i\ \hat{\mathbf{b}}_k\ \partial_j\ \hat{\perp}_k) \\
&+ (-\nabla\varrho + \Phi\nabla\hat{\mathbf{b}}\cdot\hat{\perp})(-\Phi^2\hat{\boldsymbol{\rho}}\ \hat{\mathbf{b}} : \nabla\hat{\mathbf{b}} - \frac{1}{2}\hat{\boldsymbol{\rho}}\cdot(\frac{\nabla B}{B} - \nabla\varrho) - \frac{1}{4}\Phi(\hat{\boldsymbol{\rho}}\hat{\perp} + \hat{\perp}\hat{\boldsymbol{\rho}}) : \nabla\hat{\mathbf{b}}\Big) \qquad \text{(G.3)}
\end{aligned}
$$

Obtaining $\langle p_1 \rangle$.Second order averaged equation

In this subsection we show how to get the gyroaveraged part of the first order correction of our Hamiltonian, which we express as $\langle p_1 \rangle$. First we apply the operator

$\mathcal{R}$ to both parts of the equation (G.1). Here $\langle p_1\rangle = \langle p_1\rangle(\mathbf{x}, \varphi(\mathbf{x}))$, but $\nabla\langle p_1\rangle = \nabla\langle p_1\rangle(\mathbf{x}, \varphi(\mathbf{x}), \hat{\perp}(\mathbf{x}))$. Due to the fact that $\hat{\mathbf{b}} = \hat{\mathbf{b}}(\mathbf{x})$, we can identify operators ∇ and $\partial_{\mathbf{x}}$.

$$\begin{aligned}
& \frac{p_0}{eB}\cos\varphi\,\mathcal{R}\left(\hat{\mathbf{b}}\cdot\ \nabla\,\langle p_1\rangle\right)_1 + \frac{p_0}{eB}\sin\varphi\,\mathcal{R}\left(\hat{\perp}\cdot\nabla\,\langle p_1\rangle\right)_2 + \frac{p_0}{eB}\cos\varphi\,\mathcal{R}\left(\hat{\mathbf{b}}\cdot\ \nabla\,\widetilde{p}_1\right)_3 \\
+ \;& \frac{p_0}{eB}\sin\varphi\,\mathcal{R}\left(\hat{\perp}\cdot\nabla\,\widetilde{p}_1\right)_4 + \frac{\langle p_1\rangle}{eB}\cos\varphi\,\mathcal{R}\left(\hat{\mathbf{b}}\cdot\nabla\,p_0\right)_5 + \frac{\langle p_1\rangle}{eB}\sin\varphi\,\mathcal{R}\left(\hat{\perp}\cdot\nabla\,p_0\right)_6 \\
+ \;& \cos\varphi\,\mathcal{R}\left(\frac{\widetilde{p}_1}{eB}\,\hat{\mathbf{b}}\cdot\nabla\,p_0\right)_7 + \sin\varphi\,\mathcal{R}\left(\frac{\widetilde{p}_1}{eB}\,\hat{\perp}\cdot\nabla\,p_0\right)_8 = 0 \qquad \text{(G.4)}
\end{aligned}$$

Theorem 5 *The second order gyroaveraged equation is given by*

$$\begin{aligned}
& \cos\varphi\,\hat{\mathbf{b}}\cdot\partial_{\mathbf{x}}\langle p_1\rangle - \frac{1}{2}\,\sin\varphi\,\left(\nabla\cdot\hat{\mathbf{b}}\right)\,\partial_\varphi\langle p_1\rangle \qquad \text{(G.5)} \\
= \;& \mathcal{A}\,e^{-\varrho}\,\left[\frac{\Phi^2}{2}\left[\hat{\mathbf{b}}\cdot\left(\nabla\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\right) - \left(\hat{\mathbf{b}}\times\nabla\varrho\right)\cdot(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\right] + \frac{1}{4}\left[(\hat{\mathbf{b}}\times\nabla)\cdot\hat{\mathbf{b}}\right]\,\left[\nabla\cdot\hat{\mathbf{b}}\right]\right]
\end{aligned}$$

Proof 5

In what follows we treat each term of this equation. First of all we remind the expression for the spatial derivative $\nabla\equiv\partial_{\mathbf{x}}$ in new coordinates $(\mathbf{x}', \phi, \hat{\perp})$

$$\begin{aligned}
\nabla \;&\equiv\; \partial_{\mathbf{x}'} + \partial_{\mathbf{x}}\varphi\,\partial_\varphi + \partial_{\mathbf{x}}\hat{\perp}\cdot\partial_{\hat{\perp}} \qquad \text{(G.6)} \\
&=\; \partial_{\mathbf{x}'} - \hat{\perp}\cdot\partial_{\mathbf{x}}\hat{\mathbf{b}}\,\partial_\varphi - \left[(\nabla\,\hat{\mathbf{b}}\cdot\hat{\perp})\,\hat{\mathbf{b}} + \Phi(\nabla\,\hat{\mathbf{b}}\cdot\hat{\boldsymbol{\rho}})\,\hat{\boldsymbol{\rho}}\right]\,\partial_{\hat{\perp}}
\end{aligned}$$

term 1

$$\frac{p_0}{eB}\cos\varphi\,\mathcal{R}\left(\hat{\mathbf{b}}\cdot\ \nabla\,\langle p_1\rangle\right) = \frac{p_0}{eB}\cos\varphi\,\left(\hat{\mathbf{b}}\cdot\ \partial_{\mathbf{x}}\langle p_1\rangle\right) \qquad \text{(G.7)}$$

term 2 Here we need to use:

$$\mathcal{R}(\hat{\perp}\,\hat{\perp}) = \frac{1}{2}\left(\hat{\perp}\,\hat{\perp} + \hat{\boldsymbol{\rho}}\,\hat{\boldsymbol{\rho}}\right) \qquad \text{(G.8)}$$

$$(\hat{\perp}\,\hat{\perp} + \hat{\boldsymbol{\rho}}\,\hat{\boldsymbol{\rho}} + \hat{\mathbf{b}}\,\hat{\mathbf{b}}) : \nabla\hat{\mathbf{b}} = \mathbf{1} : \nabla\hat{\mathbf{b}} = \nabla\cdot\hat{\mathbf{b}} \qquad \text{(G.9)}$$

Note that due to the Leibnitz rule, $\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot\hat{\mathbf{b}} = 0$, so we can add this term anywhere we need.

$$\begin{aligned}
\frac{p_0}{eB}\sin\varphi\,\mathcal{R}\left(\hat{\perp}\cdot\nabla\,\langle p_1\rangle\right) \;&=\; \frac{p_0}{eB}\sin\varphi\,\mathcal{R}\left(\hat{\perp}\cdot\partial_{\mathbf{x}}\,\langle p_1\rangle - \hat{\perp}\cdot\nabla\hat{\mathbf{b}}\cdot\hat{\perp}\,\partial_\varphi\,\langle p_1\rangle\right) \\
&=\; -\frac{1}{2}\frac{p_0}{eB}\sin\varphi\,\left(\nabla\cdot\hat{\mathbf{b}}\right) \qquad \text{(G.10)}
\end{aligned}$$

term 3 The evaluation of this term contains several steps. The first one consists to prove the following theorem.

Theorem 6 *$\mathcal{R}(\hat{\mathbf{b}}\cdot\nabla\widetilde{p_1})$=0*

Proof 6 *We know that the average of the monomials containing odd number of vectors $\hat{\boldsymbol{\rho}},\hat{\perp}$ is equal to zero. So we will only evaluate the terms containing monomials with even number of these vectors.*

First of all we present a short list of the key proprieties necessary for this proof.

$$\mathcal{R}\left(\hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}}\,\hat{\perp}\,\hat{\mathbf{b}}\right)\,\vdots\,\nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}}=\mathcal{R}\,\left(\hat{\perp}\,\hat{\mathbf{b}}\,\hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}}\right)\,\vdots\,\nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}}=0 \tag{G.11}$$

In fact

$$\begin{aligned}\mathcal{R}\left(\hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}}\,\hat{\perp}\,\hat{\mathbf{b}}\right)\,\vdots\,\nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}}&=\frac{1}{2}\left(\hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}}\,\hat{\perp}\,\hat{\mathbf{b}}-\hat{\perp}\,\hat{\mathbf{b}}\,\hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}}\right)\,\vdots\,\nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}}\\ =\;\frac{1}{2}&\left(\left[\hat{\perp}\,\hat{\mathbf{b}}\;:\nabla\hat{\mathbf{b}}\right]\,\left[\hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}}\;:\nabla\hat{\mathbf{b}}\right]-\left[\hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}}\;:\nabla\hat{\mathbf{b}}\right]\,\left[\hat{\perp}\,\hat{\mathbf{b}}\;:\nabla\hat{\mathbf{b}}\right]\right)=0\end{aligned} \tag{G.12}$$

$$\mathcal{R}\,(\hat{\boldsymbol{\rho}}\,\hat{\perp})=-\mathcal{R}\,(\hat{\perp}\,\hat{\boldsymbol{\rho}}) \tag{G.13}$$

The derivative of the norm of an unit vector is equal to zero, Leibnitz rule give us some useful information in this case.

$$\partial_k\,(\hat{\mathbf{b}}_i\,\hat{\mathbf{b}}_i)=2\,(\partial_k\,\hat{\mathbf{b}}_i)\hat{\mathbf{b}}_i=0\quad\Rightarrow(\nabla\hat{\mathbf{b}})\cdot\hat{\mathbf{b}}=0 \tag{G.14}$$

Here we note $\Phi\equiv\cot\,\varphi$, so by applying the chain rule, we have $\nabla\Phi=-(1+\Phi^2)\nabla\varphi=(1+\Phi^2)\;\nabla\hat{\mathbf{b}}\;\cdot\hat{\perp}$. In what follows we group the terms according to the key property which we use for it canceling. The first key property concerns the 3 following terms:

$$\mathcal{R}\left(\hat{\mathbf{b}}\cdot\;(\nabla\;\Phi^2)\;\hat{\boldsymbol{\rho}}\;\hat{\mathbf{b}}\;:\nabla\hat{\mathbf{b}}\right)\;=\;2\,\Phi\,(1+\Phi^2)\,\mathcal{R}\,\left(\hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}}\,\hat{\perp}\,\hat{\mathbf{b}}\right)\,\vdots\,\nabla\hat{\mathbf{b}}\,\nabla\hat{\mathbf{b}}=0 \tag{G.15}$$

similarly

$$\begin{aligned}\mathcal{R}\left(\Phi^2\,\hat{\mathbf{b}}_j\,\hat{\mathbf{b}}_i\,(\partial_i\,\hat{\mathbf{b}}_k)\,(\partial_j\,\hat{\boldsymbol{\rho}}_k)\right)\;&=\;\Phi^3\,\mathcal{R}\left(\hat{\mathbf{b}}_i\,(\partial_i\,\hat{\mathbf{b}}_k)\,\hat{\mathbf{b}}_j\,(\;\partial_j\,\hat{\mathbf{b}}_n\;)\,\hat{\boldsymbol{\rho}}_n\,\hat{\perp}_k\right)\\ &=\;\Phi^3\,\mathcal{R}\,\left(\hat{\perp}\,\hat{\mathbf{b}}\,\hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}}\;:\nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}}\right)=0\end{aligned} \tag{G.16}$$

and

$$\begin{aligned}&-\frac{\Phi}{4}\,\mathcal{R}\,\left(\hat{\mathbf{b}}\cdot(\nabla\hat{\boldsymbol{\rho}})\cdot(\nabla\hat{\mathbf{b}})\cdot\hat{\perp}+\hat{\mathbf{b}}\cdot(\nabla\hat{\perp})\cdot(\nabla\hat{\mathbf{b}})\cdot\hat{\boldsymbol{\rho}}\right)\\ =\;&-\frac{\Phi}{8}\,\mathcal{R}\,\left((\hat{\perp}\,\hat{\mathbf{b}}\,\hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}}\;+\hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}}\,\hat{\perp}\,\hat{\mathbf{b}})\vdots\nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}}\right)=0\end{aligned} \tag{G.17}$$

The second key property is useful for the 3 following terms. First it permit us to cancel two of this terms together.

$$
\begin{aligned}
-\frac{1}{2}\mathcal{R}\left(\hat{\mathbf{b}}\cdot\nabla\hat{\boldsymbol{\rho}}\cdot(\frac{\nabla B}{B}-\nabla\varrho)\right) &= -\frac{\Phi}{2}\hat{\mathbf{b}}_j(\partial_j\,\hat{\mathbf{b}}_n)\,\mathcal{R}\left(\,\hat{\boldsymbol{\rho}}_n\,\hat{\perp}_k\,\right)\left(\frac{\partial_k B}{B}-\partial_k\,\varrho\right)\\
&= -\frac{\Phi}{4}\left(\hat{\boldsymbol{\rho}}_n\,\hat{\perp}_k-\hat{\perp}_n\,\hat{\boldsymbol{\rho}}_k\right)\left(\frac{\partial_k B}{B}-\partial_k\varrho\right)\,\hat{\mathbf{b}}_j\,\partial_j\,\hat{\mathbf{b}}_n =\\
&= -\frac{\Phi}{4}\left((\hat{\mathbf{b}}\,\cdot\nabla\,\hat{\mathbf{b}})\times\hat{\mathbf{b}}\right)\cdot\left(\frac{\nabla B}{B}-\nabla\varrho\right) \qquad \text{(G.18)}
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\Phi}{2}\,\mathcal{R}\,\left((\hat{\perp}\,\hat{\mathbf{b}}\;:\nabla\hat{\mathbf{b}})\,\hat{\boldsymbol{\rho}}\cdot\left((\frac{\nabla B}{B}-\nabla\varrho\right)\right) &= -\frac{\Phi}{2}\,\hat{\mathbf{b}}_j\,(\partial_j\,\hat{\mathbf{b}}_n)\,\mathcal{R}\,\left(\hat{\perp}_n\,\hat{\boldsymbol{\rho}}_k\right)\,\left(\frac{\partial_k B}{B}-\partial_k\varrho\right)\\
&= \frac{\Phi}{4}\left((\hat{\mathbf{b}}\,\cdot\nabla\,\hat{\mathbf{b}})\times\hat{\mathbf{b}}\right)\cdot\left(\frac{\nabla B}{B}-\nabla\varrho\right) \qquad \text{(G.19)}
\end{aligned}
$$

In fact, the sum of (G.18) and (G.19) is equal to zero. Finally for the third term we have:

$$
\begin{aligned}
&-\frac{\Phi}{4}\,\mathcal{R}\,\left(\hat{\mathbf{b}}_i\,\hat{\perp}_j(\partial_i\,\partial_j\,\hat{\mathbf{b}}_k)\,\hat{\boldsymbol{\rho}}_k+\hat{\mathbf{b}}_i\,\hat{\boldsymbol{\rho}}_j(\partial_i\,\partial_j\,\hat{\mathbf{b}}_k)\,\hat{\perp}_k\right)\\
=&\;-\frac{\Phi}{8}\,\mathcal{R}\left(\hat{\boldsymbol{\rho}}_k\,\hat{\perp}_j+\hat{\perp}_k\,\hat{\boldsymbol{\rho}}_j\right)\hat{\mathbf{b}}_i\,\partial_i\,\hat{\mathbf{b}}_k \qquad \text{(G.20)}\\
=&\;-\frac{\Phi}{8}\left(\hat{\boldsymbol{\rho}}_j\,\hat{\perp}_k-\hat{\perp}_j\,\hat{\boldsymbol{\rho}}_k+\hat{\perp}_k\,\hat{\boldsymbol{\rho}}_j-\hat{\boldsymbol{\rho}}_k\,\hat{\perp}_j\right)\hat{\mathbf{b}}_i\,\partial_i\,\hat{\mathbf{b}}_k=0
\end{aligned}
$$

The latter property is applied to the following term:

$$
\begin{aligned}
&-\frac{\Phi}{4}\left((\hat{\perp}_i\,\partial_i\,\hat{\mathbf{b}}_k)\,(\hat{\mathbf{b}}_j\,\partial_j\,\hat{\boldsymbol{\rho}}_k)+(\hat{\boldsymbol{\rho}}_i\,\partial_i\,\hat{\mathbf{b}}_k)\,(\hat{\mathbf{b}}_j\,\partial_j\,\hat{\perp}_k)\right) \qquad \text{(G.21)}\\
=&\;\frac{\Phi}{4}\left(\hat{\mathbf{b}}_j\,(\partial_j\,\hat{\mathbf{b}}_n)\,\hat{\boldsymbol{\rho}}_n\,(\hat{\perp}_i\,\partial_i\,\hat{\mathbf{b}}_k)\,\hat{\mathbf{b}}_k+\hat{\mathbf{b}}_j\,(\partial_j\,\hat{\mathbf{b}}_n)\,\hat{\perp}_n\,\hat{\boldsymbol{\rho}}_i\,\underbrace{(\partial_i\,\hat{\mathbf{b}}_k)\,\hat{\mathbf{b}}_k}_{=0}\right)=0
\end{aligned}
$$

■

The average of all the other terms obtained as a result of contraction $\hat{\mathbf{b}}\cdot\nabla\,\widetilde{p}_1$ is equal to zero because of containing the odd number of vectors $\hat{\perp}, \hat{\boldsymbol{\rho}}$.

Finally the term 3 does not give the contribution to the (G.4).

term 4

Theorem 7

$$\mathcal{R}(\hat{\perp}\cdot\nabla\widetilde{p_1}) = \frac{\mathcal{A}\,e^{-\varrho}}{e\,\sin\varphi}\left[-\frac{\Phi^2}{2}\,(\epsilon_{ljk}\,\hat{\mathbf{b}}_l\,\partial_j\,\hat{\mathbf{b}}_i)\,(\partial_i\,\hat{\mathbf{b}}_k) - \frac{\Phi^2}{2}\,\Big(\epsilon_{ljk}\,\hat{\mathbf{b}}_l\,(\hat{\mathbf{b}}_i\,\partial_i)\,\partial_j\Big)\,\hat{\mathbf{b}}_k\right.$$
$$+\ \frac{1}{4}\,\Big[(\hat{\mathbf{b}}\times\nabla)\cdot\hat{\mathbf{b}}\Big]\ \left[\hat{\mathbf{b}}\cdot\left(\frac{\nabla B}{B}-\nabla\varrho\right)\right]$$
$$\left.+\ \frac{\Phi^2}{2}\,\Big((\hat{\mathbf{b}}\times\nabla)\,\varrho\Big)\cdot\Big(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\Big)\right] \tag{G.22}$$

Proof 7

We start this proof with listing the proprieties which will be useful here. the first one:

$$\mathcal{R}(\hat{\boldsymbol{\rho}}\hat{\perp}) = \frac{1}{2}(\hat{\boldsymbol{\rho}}\hat{\perp}-\hat{\perp}\hat{\boldsymbol{\rho}}) \tag{G.23}$$

Further we remark that:

$$\hat{\boldsymbol{\rho}}\times\hat{\perp} = (\hat{\mathbf{b}}\times\hat{\perp})\times\hat{\perp} = -\hat{\perp}\times(\hat{\mathbf{b}}\times\hat{\perp}) = -\hat{\mathbf{b}} \tag{G.24}$$

Each time when it is possible we reorganize the vectors as follows:

$$\hat{\boldsymbol{\rho}}_k\,(\hat{\perp}_i\,\partial_i) - \hat{\perp}_k\,(\hat{\boldsymbol{\rho}}_i\,\partial_i) = -(\hat{\boldsymbol{\rho}}\times\hat{\perp})\times\nabla = \hat{\mathbf{b}}\times\nabla \tag{G.25}$$

Using the proprieties below $(G.24, G.25)$ permits to evaluate the gyroaverage of the following terms:

$$-\mathcal{R}\left(\Phi^2\hat{\perp}_j\,(\partial_j\,\hat{\mathbf{b}}_i)\,(\partial_i\,\hat{\mathbf{b}}_k)\,\hat{\boldsymbol{\rho}}_k\right) = -\Phi^2\mathcal{R}(\hat{\boldsymbol{\rho}}_k\hat{\perp}_j)\,(\partial_j\,\hat{\mathbf{b}}_i)\,(\partial_i\,\hat{\mathbf{b}}_k) =$$
$$-\ \frac{\Phi^2}{2}(\hat{\boldsymbol{\rho}}_k\hat{\perp}_j-\hat{\perp}_k\hat{\boldsymbol{\rho}}_j)\,(\partial_j\,\hat{\mathbf{b}}_i)\,(\partial_i\,\hat{\mathbf{b}}_k)\ = -\frac{\Phi^2}{2}(\epsilon_{ljk}\,\hat{\mathbf{b}}_l\,\partial_j\,\hat{\mathbf{b}}_i)\,(\partial_i\,\hat{\mathbf{b}}_k)$$

$$-\mathcal{R}\left(\Phi^2\hat{\mathbf{b}}_i\hat{\perp}_j\,(\partial_j\partial_i\,\hat{\mathbf{b}}_k)\,\hat{\boldsymbol{\rho}}_k\right) = -\Phi^2\mathcal{R}(\hat{\boldsymbol{\rho}}_k\,\hat{\perp}_j)(\partial_j\partial_i\,\hat{\mathbf{b}}_k)\,\hat{\mathbf{b}}_i = -\frac{\Phi^2}{2}(\hat{\boldsymbol{\rho}}_k\hat{\perp}_j-\hat{\perp}_k\hat{\boldsymbol{\rho}}_j)(\partial_j\partial_i\hat{\mathbf{b}}_k)\,\hat{\mathbf{b}}_i$$
$$= -\frac{\Phi^2}{2}\epsilon_{ljk}\,\hat{\mathbf{b}}_l\,(\hat{\mathbf{b}}_i\,\partial_i)\,\partial_j\,\hat{\mathbf{b}}_k \tag{G.26}$$

$$-\frac{1}{2}\mathcal{R}\,\Big(\hat{\perp}\cdot\nabla\hat{\boldsymbol{\rho}}\Big)\cdot\left(\frac{\nabla B}{B}-\nabla\varrho\right) = \frac{1}{2}\,\mathcal{R}\,\Big(\hat{\perp}_j\,(\partial_j\,\hat{\mathbf{b}}_l)\,\hat{\boldsymbol{\rho}}_l\,\hat{\mathbf{b}}_k\Big)\,\left(\frac{\partial_k B}{B}-\partial_k\varrho\right) =$$
$$\frac{1}{4}\,(\hat{\boldsymbol{\rho}}_l\,\hat{\perp}_j-\hat{\perp}_l\,\hat{\boldsymbol{\rho}}_j)\,(\partial_j\,\hat{\mathbf{b}}_l)\,\hat{\mathbf{b}}_k\left(\frac{\partial_k B}{B}-\partial_k\varrho\right)$$
$$= \frac{1}{4}(\epsilon_{ij\,l}\,\hat{\mathbf{b}}_i\,\partial_j\,\hat{\mathbf{b}}_l)\,\hat{\mathbf{b}}_k\,\left(\frac{\partial_k B}{B}-\partial_k\varrho\right)$$
$$\equiv \frac{1}{4}(\hat{\mathbf{b}}\times\nabla)\cdot\hat{\mathbf{b}}\ \ \hat{\mathbf{b}}\cdot\left(\frac{\nabla B}{B}-\nabla\varrho\right) \tag{G.27}$$

$$\begin{aligned}\Phi^2\,\mathcal{R}\left(\hat{\perp}\cdot\nabla\varrho\ \hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}}\ :\nabla\hat{\mathbf{b}}\right) &= \Phi^2\,\mathcal{R}\left(\left(\hat{\perp}_j\,\partial_j\varrho\right)\ \hat{\mathbf{b}}_i\,\partial_i\,\hat{\mathbf{b}}_k\,\hat{\boldsymbol{\rho}}_k\right) = \Phi^2\,\mathcal{R}\left(\hat{\boldsymbol{\rho}}_k\hat{\perp}_j\right)\,\partial_j\varrho\ \hat{\mathbf{b}}_i\,\partial_i\,\hat{\mathbf{b}}_k \\ &= \frac{\Phi^2}{2}\left(\hat{\boldsymbol{\rho}}_k\hat{\perp}_j-\hat{\perp}_k\hat{\boldsymbol{\rho}}_j\right)\,\partial_j\varrho\ \hat{\mathbf{b}}_i\,\partial_i\,\hat{\mathbf{b}}_k \\ &= \frac{\Phi^2}{2}\left((\hat{\mathbf{b}}\times\nabla)\,\varrho\right)\cdot\left(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\right) \end{aligned} \tag{G.28}$$

$$\begin{aligned}\frac{1}{2}\,\mathcal{R}\left(\hat{\perp}\ \cdot\nabla\varrho\,\hat{\boldsymbol{\rho}}\cdot\frac{\nabla B}{B}\right) &= \frac{1}{4}\left(\hat{\boldsymbol{\rho}}_k\,\hat{\perp}_j\ -\hat{\perp}_k\,\hat{\boldsymbol{\rho}}_j\right)\partial_j\varrho\,\frac{\partial_k B}{B} \\ &= \frac{1}{4}\left((\hat{\mathbf{b}}\times\nabla)\varrho\right)\cdot\frac{\nabla B}{B}\end{aligned} \tag{G.29}$$

$$\partial_j\,\hat{\boldsymbol{\rho}}_k = -(\partial_j\,\hat{\mathbf{b}}_l)\,\hat{\boldsymbol{\rho}}_l\,\hat{\mathbf{b}}_k + \Phi\,(\partial_j\,\hat{\mathbf{b}}_l)\,\hat{\boldsymbol{\rho}}_l\,\hat{\perp}_k \tag{G.30}$$

Using the Leibnitz rule, (G.14) we have

$$\begin{aligned}\mathcal{R}\left(-\Phi^2\,\hat{\mathbf{b}}_i\,(\partial_i\,\hat{\mathbf{b}}_k)\,\hat{\perp}_j\,(\partial_j\,\hat{\boldsymbol{\rho}}_k)\right) &= -\Phi^2\,\mathcal{R}\left(\hat{\perp}_j\,\partial_j\,\hat{\boldsymbol{\rho}}_k\right)\,\hat{\mathbf{b}}_i\,\partial_i\,\hat{\mathbf{b}}_k \\ &= \Phi^2\mathcal{R}\left(\hat{\perp}_j\,(\partial_j\hat{\mathbf{b}}_l)\,\hat{\boldsymbol{\rho}}_l\,\hat{\mathbf{b}}_k\right)\,\hat{\mathbf{b}}_i\,\partial_i\,\hat{\mathbf{b}}_k \\ &= \Phi^2\,\mathcal{R}\left(\hat{\boldsymbol{\rho}}_l\,\hat{\perp}_j\right)(\partial_j\,\hat{\mathbf{b}}_l)\,\hat{\mathbf{b}}_i\,\underbrace{(\partial_i\,\hat{\mathbf{b}}_k)\,\hat{\mathbf{b}}_k}_{=0} = 0\end{aligned} \tag{G.31}$$

Because of the symmetry of the second derivative $\partial_k\partial_i B = \partial_i\partial_k B$, $\partial_k B\partial_i B = \partial_i B\partial_k B$ and $\partial_k\partial_i\varrho = \partial_i\partial_k\varrho$

$$-\frac{1}{2}\mathcal{R}\,\left(\hat{\perp}_k\hat{\boldsymbol{\rho}}_i\right)\,\left(\partial_k\,\frac{\partial_i B}{B}-\partial_i\,\partial_k\varrho\right) \tag{G.32}$$

$$= -\frac{1}{4}\left(\hat{\perp}_k\,\hat{\boldsymbol{\rho}}_i-\hat{\boldsymbol{\rho}}_k\,\hat{\perp}_i\right)\,\left(\partial_k\frac{\partial_i B}{B}-\partial_k\partial_i\varrho\right) = 0 \tag{G.33}$$

The gyroaverage of the four latter terms containing the odd number of vectors $\hat{\boldsymbol{\rho}},\hat{\perp}$ is equal to zero. To obtain this result, we need to use the following property, which can be easily obtained by computing the gyroaverage:

$$\begin{aligned}\mathcal{R}\left(\hat{\boldsymbol{\rho}}\,\hat{\perp}\,\hat{\perp}\,\hat{\perp}+\hat{\perp}\,\hat{\boldsymbol{\rho}}\,\hat{\perp}\,\hat{\perp}\right) &= -\mathcal{R}\left(\hat{\boldsymbol{\rho}}\,\hat{\perp}\,\hat{\boldsymbol{\rho}}\,\hat{\boldsymbol{\rho}}-\hat{\boldsymbol{\rho}}\,\hat{\perp}\,\hat{\perp}\hat{\perp}\right) \\ &= \mathcal{R}\left(\hat{\boldsymbol{\rho}}\,\hat{\boldsymbol{\rho}}\,\hat{\boldsymbol{\rho}}\,\hat{\perp}-\hat{\perp}\,\hat{\perp}\,\hat{\boldsymbol{\rho}}\,\hat{\perp}\right)\end{aligned} \tag{G.34}$$

$$\begin{aligned}\mathcal{R}\left(\hat{\boldsymbol{\rho}}\,\hat{\perp}\,\hat{\perp}\,\hat{\perp}+\hat{\perp}\,\hat{\boldsymbol{\rho}}\,\hat{\perp}\,\hat{\perp}\right) &= \frac{1}{4}\left(\hat{\boldsymbol{\rho}}\,\hat{\boldsymbol{\rho}}\,\hat{\boldsymbol{\rho}}\,\hat{\perp}-\hat{\boldsymbol{\rho}}\,\hat{\perp}\,\hat{\boldsymbol{\rho}}\,\hat{\boldsymbol{\rho}}\right)+\frac{1}{4}\left(\hat{\boldsymbol{\rho}}\,\hat{\boldsymbol{\rho}}\,\hat{\perp}\,\hat{\boldsymbol{\rho}}-\hat{\perp}\,\hat{\boldsymbol{\rho}}\,\hat{\boldsymbol{\rho}}\,\hat{\boldsymbol{\rho}}\right) \\ &+\ \frac{1}{4}\left(\hat{\perp}\,\hat{\boldsymbol{\rho}}\,\hat{\perp}\,\hat{\perp}\ -\hat{\perp}\,\hat{\perp}\,\hat{\perp}\,\hat{\boldsymbol{\rho}}\right)+\frac{1}{4}\left(\hat{\boldsymbol{\rho}}\,\hat{\perp}\,\hat{\perp}\,\hat{\perp}\ -\hat{\perp}\,\hat{\perp}\,\hat{\boldsymbol{\rho}}\,\hat{\perp}\right)\end{aligned} \tag{G.35}$$

Then we remark that for tensor $\nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}}$ the two following contractions give the same result:

$$a\ a\ c\ a \vdots \nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}} = c\ a\ a\ a \vdots \nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}} \tag{G.36}$$

because one can always exchange the place of two scalars:

$$\begin{aligned} a\ a\ c\ a \vdots \nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}} &= \left[a_j\ \partial_j\hat{\mathbf{b}}_k\ c_k\right]\ \left[a_i\ \partial_i\hat{\mathbf{b}}_n\ a_n\right] = \\ \left[a_i\ \partial_i\hat{\mathbf{b}}_n\ a_n\right]\ \left[b_j\ \partial_j\hat{\mathbf{b}}_k\ c_k\right] &= c\ a\ a\ a \vdots \nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}} \end{aligned} \tag{G.37}$$

replacing a and b by $\hat{\boldsymbol{\rho}}$ and $\hat{\perp}$ correspondingly, we will see that all the terms in the expression (G.35) will be canceled.

$$\begin{aligned} &-\frac{1}{4}(1+\Phi^2)\ \mathcal{R}\left(\ \hat{\perp}\ \hat{\perp}\ :\nabla\hat{\mathbf{b}}\ (\hat{\boldsymbol{\rho}}\ \hat{\perp} + \hat{\perp}\ \hat{\boldsymbol{\rho}})\ :\nabla\hat{\mathbf{b}}\right) = \\ &-\frac{1}{4}(1+\Phi^2)\ \mathcal{R}\left(\hat{\perp}\ \hat{\perp}\ \hat{\boldsymbol{\rho}}\ \hat{\perp} + \hat{\perp}\ \hat{\perp}\ \hat{\perp}\ \hat{\boldsymbol{\rho}}\right) \vdots \nabla\hat{\mathbf{b}}\ \nabla\hat{\mathbf{b}} = 0 \end{aligned} \tag{G.38}$$

Using the following expressions for derivatives of basis vectors which we have obtained in the previous chapter:

$$\partial_j\ \hat{\perp}_k = -(\partial_j\ \hat{\mathbf{b}}_n)\ \hat{\perp}_n\ \hat{\mathbf{b}}_k - \Phi\ (\partial_j\ \hat{\mathbf{b}}_n)\ \hat{\boldsymbol{\rho}}_n\ \hat{\boldsymbol{\rho}}_k \tag{G.39}$$

$$\partial_j\ \hat{\boldsymbol{\rho}}_k = -(\partial_j\ \hat{\mathbf{b}}_n)\ \hat{\boldsymbol{\rho}}_n\ \hat{\mathbf{b}}_k + \Phi\ (\partial_j\ \hat{\mathbf{b}}_n)\ \hat{\boldsymbol{\rho}}_n\ \hat{\perp}_k \tag{G.40}$$

and keeping only the terms with even number of vectors $\hat{\perp}, \hat{\boldsymbol{\rho}}$, we obtain:

$$\begin{aligned} -\ & \frac{\Phi}{4}\mathcal{R}\ \left(\hat{\perp}_j\ (\partial_j\ \hat{\perp}_k)\ \hat{\boldsymbol{\rho}}_l + \hat{\perp}_j\ (\partial_j\ \hat{\boldsymbol{\rho}}_k)\ \hat{\perp}_l\right)\ (\partial_k\ \hat{\mathbf{b}}_l) \\ =\ & -\frac{\Phi^2}{4}\mathcal{R}\ \left(\hat{\perp}_j\ (\partial_j\ \hat{\mathbf{b}}_n)\ \hat{\boldsymbol{\rho}}_n\ \hat{\boldsymbol{\rho}}_k\ \hat{\boldsymbol{\rho}}_l - \hat{\perp}_j\ (\partial_j\ \hat{\mathbf{b}}_n)\ \hat{\boldsymbol{\rho}}_n\ \hat{\perp}_k\ \hat{\perp}_l\right)\ (\partial_k\ \hat{\mathbf{b}}_l) \\ =\ & -\frac{\Phi^2}{4}\mathcal{R}\ \left(\left[\hat{\perp}_j\ (\partial_j\ \hat{\mathbf{b}}_n)\ \hat{\boldsymbol{\rho}}_n\ \right]\ \left[\hat{\boldsymbol{\rho}}_k\ (\partial_k\ \hat{\mathbf{b}}_l)\hat{\boldsymbol{\rho}}_l\right]\ - \left[\hat{\perp}_j\ (\partial_j\ \hat{\mathbf{b}}_n)\ \hat{\boldsymbol{\rho}}_n\right]\ \left[\hat{\perp}_k\ (\partial_k\ \hat{\mathbf{b}}_l)\hat{\perp}_l\ \right]\right) \\ =\ & -\frac{\Phi^2}{4}\mathcal{R}\ \left(\left[\hat{\boldsymbol{\rho}}\ \hat{\perp}\ :\nabla\hat{\mathbf{b}}\right]\ \left[\hat{\boldsymbol{\rho}}\ \hat{\boldsymbol{\rho}}\ :\nabla\hat{\mathbf{b}}\right] - \left[\hat{\boldsymbol{\rho}}\ \hat{\perp}\ :\nabla\hat{\mathbf{b}}\right]\ \left[\hat{\perp}\ \hat{\perp}\ :\nabla\hat{\mathbf{b}}\right]\right) \\ \equiv\ & -\frac{\Phi^2}{4}\mathcal{R}\ \left(\left[\hat{\boldsymbol{\rho}}\ \hat{\boldsymbol{\rho}}\ \hat{\boldsymbol{\rho}}\ \hat{\perp} - \hat{\perp}\ \hat{\perp}\ \hat{\boldsymbol{\rho}}\ \hat{\perp}\right] \vdots \nabla\hat{\mathbf{b}}\ \nabla\hat{\mathbf{b}}\right) = 0 \end{aligned} \tag{G.41}$$

The similar procedure is applied for the next non-zero average term:

$$\begin{aligned} &-\frac{\Phi}{4}\ \mathcal{R}\left(\hat{\perp}_i\ (\partial_i\ \hat{\mathbf{b}}_k)\ \hat{\perp}_j\ (\partial_j\ \hat{\boldsymbol{\rho}}_k) + \hat{\boldsymbol{\rho}}_i\ (\partial_i\ \hat{\mathbf{b}}_k)\ \hat{\perp}_j\ (\partial_j\ \hat{\perp}_k)\right) = \\ &-\frac{1}{4}\Phi^2\ \mathcal{R}\left(\hat{\boldsymbol{\rho}}\ \hat{\perp}\ \hat{\boldsymbol{\rho}}\ \hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\rho}}\ \hat{\perp}\ \hat{\perp}\ \hat{\perp}\right) \vdots \nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}} = 0 \end{aligned} \tag{G.42}$$

$$
\begin{aligned}
&-\frac{\Phi^3}{4}\,\mathcal{R}\,\Big(\,(\hat{\perp}\,\hat{\perp}:\nabla\hat{\mathbf{b}})(\hat{\boldsymbol{\rho}}\,\hat{\perp}+\hat{\perp}\,\hat{\boldsymbol{\rho}}):\nabla\hat{\mathbf{b}}\Big) = \\
&-\frac{1}{4}\Phi^3\,\mathcal{R}\left(\hat{\boldsymbol{\rho}}\,\hat{\perp}\,\hat{\perp}\,\hat{\perp}+\hat{\perp}\,\hat{\boldsymbol{\rho}}\,\hat{\perp}\,\hat{\perp}\right)\,\vdots\nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}}=0
\end{aligned}
\tag{G.43}
$$

The remaining terms in $\mathcal{R}(\hat{\perp}\cdot\nabla\widetilde{p}_1)$ have a gyroaverage equal to zero because of containing the odd number of the vectors $\hat{\perp}$, $\hat{\boldsymbol{\rho}}$

■

term 5 and term 6 This terms cancels due to the choice $\hat{\mathbf{b}}\cdot\varrho=0$ which we have made in the previous order of the perturbative expansion.

Theorem 8 *Contribution of the term 7 cancels the contribution of term 8.*

Proof 8

We start with consideration of the term 7.

term 7 The result of its gyroaveraging is given by:

$$
\begin{aligned}
&\cos\varphi\,\mathcal{R}\left(\frac{\widetilde{p}_1}{eB}\,\hat{\mathbf{b}}\cdot\nabla\,p_0\right) \\
&=\;\frac{\Phi^2}{4}\,\frac{\mathcal{A}^{\frac{3}{2}}\,B^{-\frac{1}{2}}}{\sin\varphi}\,e^{-\frac{3}{2}\varrho}\,\left(\hat{\mathbf{b}}\times\left(\frac{\nabla B}{B}-\nabla\varrho\right)\right)\cdot\left(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\right)
\end{aligned}
\tag{G.44}
$$

To obtain this result we start with the following expansion:

$$
\begin{aligned}
&\left(\frac{\widetilde{p}_1}{eB}\,\hat{\mathbf{b}}\cdot\nabla\,p_0\right)=\frac{\mathcal{A}^{\frac{3}{2}}\,B^{-\frac{1}{2}}}{\sin^2\varphi}\,e^{-\frac{3}{2}\varrho}\,\times \\
&\left(-\Phi^2\,\hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}}\;:\nabla\hat{\mathbf{b}}-\frac{1}{2}\,\hat{\boldsymbol{\rho}}\cdot\left(\frac{\nabla\,B}{B}-\nabla\varrho\right)-\frac{1}{4}\,\Phi\left(\hat{\boldsymbol{\rho}}\,\hat{\perp}+\hat{\perp}\,\hat{\boldsymbol{\rho}}\right)\;:\nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}}\right)\times \\
&\left(\frac{\hat{\mathbf{b}}\cdot\nabla B}{2B}-\frac{\hat{\mathbf{b}}\cdot\nabla\varrho}{2}+\Phi\,\hat{\perp}\,\hat{\mathbf{b}}\;:\nabla\hat{\mathbf{b}}\right)
\end{aligned}
\tag{G.45}
$$

We remark that the terms which will make contribution to the result of gyroaveraging are:

$$\mathcal{R}\left(\frac{\widetilde{p}_1}{eB}\,\hat{\mathbf{b}}\cdot\nabla\,p_0\right) = \frac{\mathcal{A}^{\frac{3}{2}}\;B^{-\frac{1}{2}}}{\sin^2\varphi}\;e^{-\frac{3}{2}\varrho}\quad\times \tag{G.46}$$
$$\mathcal{R}\Bigg(\underbrace{-\Phi^3\,\hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}}\,\hat{\perp}\,\hat{\mathbf{b}}\,\vdots\,\nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}}}_{bp\;0\;a}\underbrace{-\frac{\Phi}{2}\,\hat{\boldsymbol{\rho}}\cdot\left(\frac{\nabla B}{2B}-\frac{\nabla\varrho}{2}\right)\left(\hat{\perp}\,\hat{\mathbf{b}}\;:\nabla\hat{\mathbf{b}}\right)}_{bp\;0\;b}$$
$$\underbrace{-\frac{\Phi}{8}\left[\left(\hat{\boldsymbol{\rho}}\,\hat{\perp}+\hat{\perp}\,\hat{\boldsymbol{\rho}}\right)\;:\nabla\hat{\mathbf{b}}\right]\;\left[\hat{\mathbf{b}}\cdot\left(\frac{\nabla B}{2B}-\frac{\nabla\varrho}{2}\right)\right]}_{bp\;0\;c}\Bigg)$$

Now we proceed with consideration of each term in the latter expression:

- $$\mathcal{R}(bp\;0\;a) = 0 \tag{G.47}$$

  due to the property (G.11,G.12) that we have used during the calculation of the gyroaverage of the term 3

- $$\mathcal{R}(bp\;0\;b) = \frac{\Phi}{4}\left(\hat{\mathbf{b}}\times\left(\frac{\nabla B}{B}-\nabla\varrho\right)\right)\cdot\left(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\right) \tag{G.48}$$

- Finally, with (G.23) it is evident that $\mathcal{R}(\hat{\perp}\hat{\boldsymbol{\rho}}+\hat{\boldsymbol{\rho}}\,\hat{\perp}) = 0$, and we obtain that the average of the latter term is equal to zero:

  $$\mathcal{R}(bp\;0\;c) = 0 \tag{G.49}$$

So finally, we obtain the result $(G.45)$. ■
Now we proceed with gyroaveraging of the term 8

term 8

$$\sin\varphi\,\mathcal{R}\left(\frac{\widetilde{p}_1}{eB}\,\hat{\mathbf{b}}\cdot\nabla\,p_0\right) =$$
$$-\frac{\Phi^2}{4}\,\frac{\mathcal{A}^{\frac{3}{2}}\;B^{-\frac{1}{2}}}{\sin\varphi}\;e^{-\frac{3}{2}\varrho}\left(\hat{\mathbf{b}}\times\left(\frac{\nabla B}{B}-\nabla\varrho\right)\right)\cdot\left(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\right) \tag{G.50}$$

$$\left(\frac{\widetilde{p}_1}{eB}\,\hat{\perp}\cdot\nabla\,p_0\right) = \frac{\mathcal{A}^{\frac{3}{2}}\,B^{-\frac{1}{2}}}{\sin^2\varphi}\,e^{-\frac{3}{2}\varrho}\times$$
$$\left(-\Phi^2\,\hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}}\;:\nabla\hat{\mathbf{b}}-\frac{1}{2}\,\hat{\boldsymbol{\rho}}\cdot\left(\frac{\nabla\,B}{B}-\nabla\varrho\right)-\frac{1}{4}\,\Phi\left(\hat{\boldsymbol{\rho}}\,\hat{\perp}+\hat{\perp}\,\hat{\boldsymbol{\rho}}\right)\;:\nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}}\right)\times$$
$$\left(\frac{\hat{\perp}\cdot\nabla B}{2B}-\frac{\hat{\perp}\cdot\nabla\varrho}{2}+\Phi\,\hat{\perp}\,\hat{\perp}\;:\nabla\hat{\mathbf{b}}\right)$$

Here the terms that will contribute to the gyroaverage are

$$\mathcal{R}\,\left(\frac{\widetilde{p}_1}{eB}\,\hat{\perp}\cdot\nabla\,p_0\right) = \frac{\mathcal{A}^{\frac{3}{2}}\,B^{-\frac{1}{2}}}{\sin^2\varphi}\,e^{-\frac{3}{2}\varrho}\;\times \tag{G.51}$$
$$\mathcal{R}\left(\underbrace{-\Phi^2\,\left[\hat{\boldsymbol{\rho}}\,\hat{\mathbf{b}}\;:\;\nabla\hat{\mathbf{b}}\right]\left[\hat{\perp}\cdot\left(\frac{\nabla B}{2B}-\frac{\nabla\varrho}{2}\right)\right]}_{\hat{\perp}\,p\,0\,a}\underbrace{-\frac{1}{2}\,\left[\hat{\boldsymbol{\rho}}\cdot\left(\frac{\nabla B}{2B}-\frac{\nabla\varrho}{2}\right)\right]\,\left[\hat{\perp}\cdot\frac{\nabla B}{B}\right]}_{\hat{\perp}\,p\,0\,b}\right.$$
$$\left.\underbrace{+\frac{1}{2}\,\left[\hat{\boldsymbol{\rho}}\cdot\left(\frac{\nabla B}{2B}-\frac{\nabla\varrho}{2}\right)\right]\,\left[\hat{\perp}\cdot\nabla\varrho\right]}_{\hat{\perp}\,p\,0\,c}\underbrace{-\Phi^3\,\left(\hat{\boldsymbol{\rho}}\,\hat{\perp}\,\hat{\perp}\,\hat{\perp}+\hat{\perp}\,\hat{\boldsymbol{\rho}}\,\hat{\perp}\,\hat{\perp}\right)\,\vdots\,\nabla\hat{\mathbf{b}}\nabla\hat{\mathbf{b}}}_{\hat{\perp}\,p\,0\,d}\right)$$

$$\mathcal{R}(\hat{\perp}\;p\;0\;a) = \frac{\Phi^2}{4}\,\left(\left(\frac{\nabla B}{B}-\nabla\varrho\right)\times\hat{\mathbf{b}}\right)\cdot\left(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\right) \tag{G.52}$$

The contribution of the terms $\hat{\perp}\;p\;0\;b$ and $\hat{\perp}\;p\;0\;c$ is canceled:

$$\mathcal{R}(\hat{\perp}\;p\;0\;b) = \frac{1}{8B}\left(\nabla\varrho\times\hat{\mathbf{b}}\right)\cdot\frac{\nabla B}{B} \tag{G.53}$$

$$\mathcal{R}(\hat{\perp}\;p\;0\;c) = -\frac{1}{8B}\left(\nabla\varrho\times\hat{\mathbf{b}}\right)\cdot\frac{\nabla B}{B} \tag{G.54}$$

Using the property (G.35,G.36,G.37) we obtain that the gyroaverage of the latter term $\hat{\perp}\;p\;0\;d$ is equal to zero

$$\mathcal{R}(\hat{\perp}\;p\;0\;d) = 0 \tag{G.55}$$

And we have the result (G.50). We remark that the contributions of the terms 7 and 8 into the gyroaveraged equation G.4 are canceled.

■

Finally we obtain the equation G.6.

Appendix H

Equations of motion in axisymmetric magnetic geometry

In this Appendix we resume the main steps for obtaining the local particle equations of motion in a general axisymmetric magnetic geometry. In this case the magnetic surfaces represent a set of a nested curves, which possesses an analytical expression (diffeomorphism) in cylindrical coordinates. Then we consider an example of a bi-cylindrical geometry, when the magnetic surfaces represent a set of a nested concentric circles. We integrate the equations of motion in this particular case with Mathematica package in order to study trapped particle trajectories presented in 4.4.1.

H.1 General axisymmetric geometry

The transformation from the Cartesian coordinates to the general axisymmetric coordinates is given by eq. (4.75). To simplify the following expressions, here we rename the norms of the basis vectors as $|\nabla\psi|^{-1} = \sqrt{g_{\psi\psi}} \equiv \Omega_\psi(\psi,\theta)$ and $|\nabla\theta|^{-1} = \sqrt{g_{\theta\theta}} \equiv \Omega_\theta(\psi,\theta)$, $|\nabla\phi|^{-1} = \sqrt{g_{\phi\phi}} = R(\psi,\theta)$.[1].

We start with a definition of the fixed basis associated to the vector of magnetic field direction $\hat{\mathbf{b}}$ given by (4.110). Note that there is some freedom while choosing the basis vectors $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$ in a perpendicular to magnetic field line plane. This leads to the gyro-gauge dependence of dynamics (the gyro-gauge vector is defined as $\mathbf{R} = \nabla\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{b}}_2$). One of the possible issues to make dynamics gyro-gauge independent is presented in 4.5.

For example in the case of a general magnetic geometry discussed in the section 4.3, the curvature vector $\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}$ is chosen as one of the basis vectors $\hat{\mathbf{b}}_1$ while deriving the corresponding equations of motion. In an axisymmetric geometry, while the vector $\hat{\mathbf{b}}$ has only the toroidal and poloidal components, a more simple choice is

[1] The same notations are used in the Mathematica code

possible. One can simply take the vector $\hat{\mathbf{b}}_1$ equal to the third (radial) basis vector $\widehat{\nabla\psi}$.

$$\hat{\mathbf{b}} = \sin\eta\widehat{\nabla\theta} + \cos\eta\widehat{\nabla\phi},\ \hat{\mathbf{b}}_1 = -\widehat{\nabla\psi},\ \hat{\mathbf{b}}_2 = \hat{\mathbf{b}} \times \hat{\mathbf{b}}_0 = \cos\eta\widehat{\nabla\theta} - \sin\eta\widehat{\nabla\phi} \tag{H.1}$$

where we have assumed that the coordinate vectors are organized as follows: $\widehat{\nabla\theta} \times \widehat{\nabla\psi} = \widehat{\nabla\phi}$.

Note that we consider that the function $\eta = \eta(\psi)$ is only the function of the radial coordinate, so that cotan $\eta(\psi) \equiv q(\psi)$, the particular choice of the q-profile will be made just before the integration of the equations of motion.

Now we decompose the unit kinetic momentum vector in the basis $(\nabla\phi, \nabla\theta, \nabla\psi)$ as:

$$\hat{\mathbf{p}} = A_0\widehat{\nabla\phi} + B_0\widehat{\nabla\theta} + C_0\widehat{\nabla\psi} \tag{H.2}$$

with the coefficients:

$$\begin{aligned} A_0 &= \cos\eta\ \cos\varphi + \sin\eta\ \sin\varphi\ \cos\zeta && \text{(H.3)}\\ B_0 &= \sin\eta\ \cos\varphi - \cos\eta\ \sin\varphi\ \cos\zeta && \text{(H.4)}\\ C_0 &= \sin\varphi\ \sin\zeta && \text{(H.5)} \end{aligned}$$

Then we can obtain the three first equations of motion for the spatial coordinates: $\dot{\mathbf{x}} = \varepsilon\ \hat{\mathbf{p}}$ in axisymmetric geometry. Here we introduce the small parameter $\varepsilon \equiv \frac{p}{eB_0}$, which represents the ratio between the modulus of the particle kinetic momentum p and the characteristic magnitude of the magnetic field B_0 (4.116). Note that such a ratio has a length dimension. In order to obtain dimensionless small parameter, we can introduce a characteristic length scale, which can be given by the small tokamak radius a, which is equal to 1 m for Tore Supra, for example. Then we have $\mathbf{x} \to \frac{\mathbf{x}}{a}$, $\varepsilon \to \frac{1}{a}\frac{p}{eB_0}$. Note that with assumption $c = 1$ which we have made in the beginning, this operation will lead to dimensionless equations of motion.

$$\begin{aligned} \dot{\phi} &= \varepsilon\frac{\widehat{\nabla\phi}}{R} \cdot \dot{\mathbf{x}} = \varepsilon\ \frac{A_0}{R} && \text{(H.6)}\\ \dot{\theta} &= \varepsilon\frac{\widehat{\nabla\theta}}{\Omega_\theta} \cdot \dot{\mathbf{x}} = \varepsilon\ \frac{B_0}{\Omega_\theta} && \text{(H.7)}\\ \dot{\psi} &= \varepsilon\frac{\widehat{\nabla\psi}}{\Omega_\psi} \cdot \dot{\mathbf{x}} = \varepsilon\ \frac{C_0}{\Omega_\psi} && \text{(H.8)} \end{aligned}$$

Two other momentum basis vectors $(\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2)$ can be defined as follows:

$$\begin{aligned} \hat{\mathbf{p}}_1 &= A_1\widehat{\nabla\phi} + B_1\widehat{\nabla\theta} + C_1\widehat{\nabla\psi} && \text{(H.9)}\\ \hat{\mathbf{p}}_2 &= -\frac{1}{\sin\varphi}\left[A_2\widehat{\nabla\phi} + B_2\widehat{\nabla\theta} + C_2\widehat{\nabla\psi}\right] && \text{(H.10)} \end{aligned}$$

with

$$A_1 = \frac{\partial A_0}{\partial \varphi} = -\cos\eta\ \sin\varphi + \sin\eta\ \cos\varphi\ \cos\zeta \tag{H.11}$$

$$B_1 = \frac{\partial B_0}{\partial \varphi} = -\cos\eta\ \cos\varphi\ \cos\zeta + \sin\eta\ \sin\varphi \tag{H.12}$$

$$C_1 = \frac{\partial C_0}{\partial \varphi} = \cos\varphi\ \sin\zeta \tag{H.13}$$

and

$$A_2 = \frac{\partial A_0}{\partial \theta} = -\sin\eta\ \sin\varphi\ \sin\zeta \tag{H.14}$$

$$B_2 = \frac{\partial B_0}{\partial \theta} = -\cos\eta\ \sin\varphi\ \sin\zeta \tag{H.15}$$

$$C_2 = \frac{\partial C_0}{\partial \theta} = \sin\varphi\ \cos\zeta \tag{H.16}$$

The fundamental object which we need to calculate in order to obtain the equations of motion for momentum part of the phase space (φ, ζ), is the vector of momentum curvature $\hat{\mathbf{p}} \cdot \nabla \hat{\mathbf{p}}$. First we decompose the scalar differential operator $\hat{\mathbf{p}} \cdot \nabla$ in the axisymmetric basis with

$$\nabla = \frac{1}{R}\widehat{\nabla\phi}\frac{\partial}{\partial\phi} + \frac{1}{\Omega_\theta}\widehat{\nabla\theta}\frac{\partial}{\partial\theta} + \frac{1}{\Omega_\psi}\widehat{\nabla\psi}\frac{\partial}{\partial\psi} \tag{H.17}$$

then

$$\hat{\mathbf{p}} \cdot \nabla = \frac{A_0}{R}\frac{\partial}{\partial\phi} + \frac{B_0}{\Omega_\theta}\frac{\partial}{\partial\theta} + \frac{C_0}{\Omega_\psi}\frac{\partial}{\partial\psi} \tag{H.18}$$

By applying this differential operator to the expression (H.2) for decomposition of the unit momentum vector $\hat{\mathbf{p}}$ in the basis $(\widehat{\nabla\phi}, \widehat{\nabla\theta}, \widehat{\nabla\psi})$, and then by using the expressions for derivatives of the basis vectors in a general axisymmetric magnetic case obtained in 4.4.1 we have:

$$\begin{aligned}
\hat{\mathbf{p}} \cdot \nabla \hat{\mathbf{p}} = \widehat{\nabla\phi}&\left[\frac{A_0}{R}\left(\frac{B_0}{\Omega_\theta}\frac{\partial R}{\partial\theta} + \frac{C_0}{\Omega_\psi}\frac{\partial R}{\partial\psi}\right) + \frac{B_0}{\Omega_\theta}\left(\frac{\partial A_0}{\partial\theta}\right) + \frac{C_0}{\Omega_\psi}\left(\frac{\partial A_0}{\partial\psi}\right)\right] \\
+\widehat{\nabla\theta}&\left[-\frac{A_0^2}{R\,\Omega_\theta}\frac{\partial R}{\partial\theta} + \frac{B_0}{\Omega_\theta}\left(\frac{\partial B_0}{\partial\theta} + \frac{C_0}{\Omega_\psi}\frac{\partial\Omega_\theta}{\partial\psi}\right) + \frac{C_0}{\Omega_\psi}\left(\frac{\partial B_0}{\partial\psi} - \frac{C_0}{\Omega_\theta}\frac{\partial\Omega_\psi}{\partial\theta}\right)\right] \\
+\widehat{\nabla\psi}&\left[-\frac{A_0^2}{R\,\Omega_\psi}\frac{\partial R}{\partial\psi} + \frac{B_0}{\Omega_\theta}\left(\frac{\partial C_0}{\partial\theta} - \frac{B_0}{\Omega_\psi}\frac{\partial\Omega_\theta}{\partial\psi}\right) + \frac{C_0}{\Omega_\psi}\left(\frac{\partial C_0}{\partial\psi} + \frac{B_0}{\Omega_\theta}\frac{\partial\Omega_\psi}{\partial\theta}\right)\right] \\
&\equiv \widehat{\nabla\phi}\,\mathcal{P}_\phi + \widehat{\nabla\theta}\,\mathcal{P}_\theta + \widehat{\nabla\psi}\,\mathcal{P}_\psi
\end{aligned} \tag{H.19}$$

Now by using the above expression for the momentum curvature and the equations (H.9),(H.10) for the axisymmetric decomposition of the momentum basis vectors

$(\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2)$, we obtain the equations of motion for pitch angle and gyrophase in a general axisymmetric geometry:

$$\begin{aligned}
\dot{\varphi} &= \varepsilon\left((\hat{\mathbf{p}}\cdot\nabla\hat{\mathbf{p}})\cdot\hat{\mathbf{p}}_1\right) = \varepsilon\left(\mathcal{P}_\phi\, A_1 + \mathcal{P}_\theta\, B_1 + \mathcal{P}_\psi\, C_1\right) && \text{(H.20)}\\
\dot{\zeta} &= \frac{1}{R\,\sin\eta} + \varepsilon\,\frac{1}{\sin\varphi}\left((\hat{\mathbf{p}}\cdot\nabla\hat{\mathbf{p}})\cdot\hat{\mathbf{p}}_2\right) = \\
&= \frac{1}{R\,\sin\eta} - \varepsilon\,\frac{1}{\sin^2\varphi}\left(\mathcal{P}_\phi\, A_2 + \mathcal{P}_\theta\, B_2 + \mathcal{P}_\psi\, C_2\right) && \text{(H.21)}
\end{aligned}$$

The next step is to chose a particular example of axisymmetric coordinates by defining the functions $R(\psi,\theta), \Omega_\psi(\psi,\theta), \Omega_\theta(\psi,\theta)$.

H.2 Bi-cylindrical coordinates

In what follows we deal with a bi-cylindrical system of coordinates, for which

$$R = R_0 + \psi\cos\theta, Z = \psi\sin\theta \tag{H.22}$$

the corresponding norms of the basis vectors

$$\begin{aligned}
\Omega_\psi &= \sqrt{\partial_\psi R^2 + \partial_\psi Z^2} = 1 && \text{(H.23)}\\
\Omega_\theta &= \sqrt{\partial_\theta R^2 + \partial_\theta Z^2} = \psi && \text{(H.24)}
\end{aligned}$$

Note that the equations of motion for space coordinates of the phase space are not affected by the calculation of the momentum curvature vector. It can be directly obtained from (H.8) by substituting the expression above for the norms of the basis vectors. We remark here that in the bi-cylindrical geometry case the dynamics is totally independent of toroidal angle ϕ, so only the equation for radial and poloidal coordinates will be necessary.

In order to obtain the equations of motion for momentum part of the phase space we substitute the expressions for $(R, \Omega_\psi, \Omega_\theta)$ in (H.19). First, we obtain momentum curvature vector in our particular case: [2]

$$\begin{aligned}
\hat{\mathbf{p}}_0\cdot\nabla\hat{\mathbf{p}}_0 = \widehat{\nabla\phi}&\left[A_0\frac{1}{R}\left(C_0\cos\theta - B_0\sin\theta\right) + C_0\frac{\partial A_0}{\partial\psi} + B_0\frac{1}{\psi}\frac{\partial A_0}{\partial\theta}\right]\\
+\widehat{\nabla\theta}&\left[A_0^2\frac{1}{R}\sin\theta + B_0\frac{1}{\psi}\left(\frac{\partial B_0}{\partial\theta} + C_0\right) + C_0\frac{\partial B_0}{\partial\psi}\right]\\
+\widehat{\nabla\psi}&\left[-A_0^2\frac{1}{R}\sin\theta + B_0\frac{1}{\psi}\left(\frac{\partial C_0}{\partial\theta} - B_0\right) + C_0\frac{\partial C_0}{\partial\psi}\right]\\
&\equiv \mathcal{P}_\phi\widehat{\nabla\phi} + \mathcal{P}_\theta\widehat{\nabla\theta} + \mathcal{P}_\psi\widehat{\nabla\psi}
\end{aligned} \tag{H.25}$$

[2]This calculation is realized analytically and then is verified by realizing the substitution of the bi-cylindrical geometry coefficients with Mathematica

Then similarly to the general axisymmetric geometry, we obtain the corresponding equations of motion for the pitch angle variable φ and the gyrophase variable ζ.

In what follows we use the q-profile defined as [3]:

$$\text{cotan}\ \eta(\psi) = q_0 + s_0\ \psi^2 \tag{H.26}$$

Then by substituting the next expression into the equations of motion by using Mathematicapackage

$$\sin\eta = \frac{1}{\sqrt{1+(q_0+s_0\ \psi^2)^2}},\quad \cos\eta = \frac{q_0+s_0\ \psi^2}{\sqrt{1+(q_0+s_0\ \psi^2)^2}} \tag{H.27}$$

we obtain:

$$\dot{\theta} = \frac{\cos\varphi - (q_0+s_0\ \psi^2)\cos\zeta\ \sin\varphi}{\psi\ \sqrt{1+(q_0+s_0\ \psi^2)^2}} \tag{H.28}$$

$$\dot{\psi} = \sin\zeta\ \sin\theta \tag{H.29}$$

$$\begin{aligned}
&\dot{\varphi} = \Big(\psi\ \big(1+(q_0+s_0\ \psi^2)^2\big)\ \ (R_0+\psi\ \cos\phi)\Big)^{-1} \times \\
&\big[\sin\varphi\,\big(R_0\ (q_0+s_0\psi^2)\ \cos\zeta\ \sin\zeta + s_0\ \psi^2\ (R_0+\psi\ \cos\theta)\ \sin(2\zeta) \\
&-\psi\ \sqrt{1+(q_0+s_0\ \psi^2)^2}\ \cos^2\zeta\sin\theta\Big) \\
&-\cos\varphi\,\big(\big(R_0+\psi\ (1+(q_0+s_0\ \psi^2)^2)\cos\theta\big)\ \sin\zeta \\
&\psi\ (q_0+s_0\ \psi^2)\ \sqrt{1+(q_0+s_0\ \psi^2)^2}\ \cos\zeta\sin\theta\Big)\Big]
\end{aligned} \tag{H.30}$$

In order to increase the integration time in numerical simulations, we implement lowest order equation of motion for gyrophase:

$$\dot{\zeta} = \frac{\sqrt{1+(q_0+s_0\ \psi^2)^2}}{R_0+\psi\ \cos\theta} \tag{H.31}$$

The full dynamical equation for gyrophase coordinate, which includes the momentum curvature contribution is also obtained.

In what follows we integrate with Mathematica the equations (H.28),(H.28),(H.30),(H.30), (H.31) in order to study trapped particle trajectories.

[3] Note that in numerical simulations we use the values of the parameters $s_0 = 1$ and $q_0 = 4$.

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