Presentations:

how not to do them.

- Introduction
- Awful example
- Discussion

Presentations:

how **not** to do them.

Why so negative?

Kipling:

"There are nine and sixty ways
of constructing tribal lays,
And every single one of them is right!"

Introduction:

Learn to do good presentations:

(a) by presenting;

(b) by making mistakes;

(c) by enjoying good presentations;

(d) by enduring bad presentations.

Useful reading

- D.E. Knuth, T. Larrabee, P.M. Roberts
  Mathematical Writing, 1989, MAA

  "great prizes are ... interest and understanding; all else is secondary."

- S. Senn, "On Thinking and Learning"
  RSS News & Notes 35.4 (2007) 1-3

  "Content is of primary importance"

- E. Tufte, "Powerpoint is Evil"
  http://www.wired.com/wired/archive/11.09/ppt2.html

  "Imagine a widely used and expensive prescription drug that promised to make us beautiful but didn't."

Start of awful example

A double integral

$$\eta = \sqrt{x^2+y^2} + \sqrt{(n-x)^2+y^2}$$

$$\alpha = \tan^{-1}\frac{y}{x} + \tan^{-1}\frac{y}{n-x}$$

$$\frac{1}{2}\iint (\alpha - \sin\alpha)\exp\left(-\frac{1}{2}(\eta - n)\right)dx\,dy$$

Obviously

we can ignore

$$\int_{\pi/2}^{\pi}\int_{0}^{\infty}(\alpha-\sin\alpha)\exp\left(-\tfrac{1}{2}(q-n)\right)r\,dr\,d\theta$$

We can deal with

$$\int_0^{\pi/2}\int_0^{\frac{n}{3}\sec\theta} (\alpha - \sin\alpha)\exp(\eta - n)\, r\,dr\,d\theta$$

Theorem 7 (Asymptotic upper bound on mean perimeter length). *The mean perimeter excess length J_m is subject to the following asymptotic upper bound:*

$$J_m \quad\leq\quad O(\log m) \qquad \text{as } m \to \infty. \tag{10}$$

Proof. Without loss of generality, place the points v_1 and v_2 at $(-\frac{m}{2}, 0)$ and $(\frac{m}{2}, 0)$. The double integral in (4) possesses mirror symmetry in each of the two axes, so we can write

$$\begin{aligned} J_m &= 2\iint_{[0,\infty)^2} (\phi - \sin\phi)\exp\left(-\tfrac{1}{2}(\eta - m)\right) \mathrm{Leb}(dz) \\ &= 2\int_0^{\pi/2}\int_0^{\frac{m}{2}\sec\theta} (\phi - \sin\phi)\exp\left(-\tfrac{1}{2}(\eta - m)\right) r\,dr\,d\theta + \\ &\qquad + 2\int_{\pi/2}^{\pi}\int_0^{\infty} (\phi - \sin\phi)\exp\left(-\tfrac{1}{2}(\eta - m)\right) r\,dr\,d\theta \end{aligned} \tag{11}$$

(using polar coordinates (r, θ) about the second point v_2 located at $(\frac{m}{2}, 0)$). The integrand in the second summand is dominated by $\pi\exp\left(-\frac{r}{2}\right) r$, which is integrable over $(r, \theta) \in (0, \infty) \times (\frac{\pi}{2}, \pi)$. (In this region geometry shows that $\eta - m > r(1 - \cos\theta) \geq r$.) Thus we can apply Lebesgue's dominated convergence theorem to deduce that the second summand is $O(1)$ as $m \to \infty$, hence may be neglected.

In fact we can also show that part of the first summand generates an $O(1)$ term: the dominated convergence theorem can be applied for any $\varepsilon \in (0, \pi/2)$ to show that

$$2\int_0^{\pi/2}\int_{\varepsilon}^{\frac{m}{2}\sec\theta} (\phi - \sin\phi)\exp\left(-\tfrac{1}{2}(\eta - m)\right) r\,dr\,d\theta \quad=\quad O(1),$$

since the integrand is dominated by $\pi\exp\left(-\frac{r}{2}(1 - \cos\theta)\right) r$ over the region $(r, \theta) \in (0, \infty) \times (\varepsilon, \frac{\pi}{2})$ (in this region geometry shows that $\eta - m > r(1 - \cos\theta) > r(1 - \cos\varepsilon)$). Thus for fixed $\varepsilon \in (0, \frac{\pi}{2})$ as $m \to \infty$ we have the asymptotic expression

$$J_m \quad=\quad 2\int_0^{\varepsilon}\int_0^{\frac{m}{2}\sec\theta} (\phi - \sin\phi)\exp\left(-\tfrac{1}{2}(\eta - m)\right) r\,dr\,d\theta + O(1).$$

Now in the region $(r, \theta) \in (0, \infty) \times (0, \varepsilon)$ we know $\phi < 2\theta < 2\varepsilon$, and moreover $\phi - \sin\phi$ is an increasing function of ϕ (so long as $\varepsilon < \frac{\pi}{2}$). Therefore there is a constant C_ε such that

$$\phi - \sin\phi \quad\leq\quad 2\theta - \sin(2\theta) \quad\leq\quad \frac{C_\varepsilon}{8}\frac{(2\theta)^3}{6} \quad\leq\quad C_\varepsilon\frac{1 - \cos\theta}{3}\sin\theta.$$

Hence

$$\begin{aligned} &2\int_0^{\varepsilon}\int_0^{\frac{m}{2}\sec\theta} (\phi - \sin\phi)\exp\left(-\tfrac{1}{2}(\eta - m)\right) r\,dr\,d\theta \\ &\quad\leq\quad \tfrac{2}{3}C_\varepsilon\int_0^{\varepsilon}\int_0^{\frac{m}{2}\sec\theta} (1 - \cos\theta)\sin\theta\exp\left(-\tfrac{r}{2}(1 - \cos\theta)\right) r\,dr\,d\theta \\ &\quad=\quad \tfrac{8}{3}C_\varepsilon\int_0^{\varepsilon}\left(\int_0^{\frac{m}{4}(\sec\theta - 1)} e^{-s}s\,ds\right)\frac{\sin\theta\,d\theta}{1 - \cos\theta} \qquad \left(\text{using } s = \tfrac{r}{2}(1 - \cos\theta)\right) \\ &\quad\leq\quad \tfrac{8}{3}C_\varepsilon\int_0^{\frac{m}{4}(\sec\varepsilon - 1)}\left(\int_0^{\infty} e^{-s}s\,ds\right)\frac{1}{1 + 4v/m}\frac{dv}{v} \qquad \left(\text{using } v = \tfrac{m}{4}(\sec\theta - 1)\right) \\ &\quad\leq\quad \tfrac{8}{3}C_\varepsilon\log\left(\tfrac{m}{4}(\sec\varepsilon - 1)\right) + O(1). \end{aligned}$$

$\square$

Actually

$$\frac{1}{2}\iint (x-\sin x)\exp\left(-\frac{1}{2}(y-n)\right)dx\,dy$$

$$= \quad \frac{8}{3}(\log n + \text{constant})$$

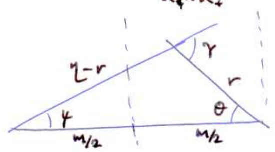
and we can do

higher-order asymptotics too.

30.1.07

Asymptotics for $J_m = 2\int\int_{\mathbb{R}_+\times\mathbb{R}_+} (\gamma - \sin\gamma)\exp(-\tfrac{1}{2}(\eta - m))\,\text{Leb}(dx)$

for large m

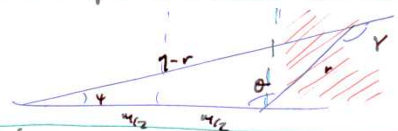


Using polar coordinates (r,θ), it is natural to split J_m into two double integrals:

$$J_m = 2\int_0^{\pi/2}\int_0^{\frac{m}{2}\sec\theta} (\gamma - \sin\gamma)\exp(-\tfrac{1}{2}(\eta - m))\, r\,dr\,d\theta$$

$$+ 2\int_{\pi/2}^{\pi}\int_0^{\infty} (\gamma - \sin\gamma)\exp(-\tfrac{1}{2}(\eta - m))\, r\,dr\,d\theta.$$

The second double integral is concerned with red-hatched region



In general we know

$$0 < \gamma - \sin\gamma \nearrow \pi \text{ as } \gamma \nearrow \pi \quad (\text{derivative is } 1 - \cos\gamma \geq 0)$$

and in the red-hatched region

$$\eta - r > m$$

while for fixed r, θ as $m \to \infty$ so $\gamma \to \theta$, $\eta - m \to r(1 - \cos\theta)$.

Hence the second double integral has integrand dominated by that in

$$2\pi\int_{\pi/2}^{\pi}\int_0^{\infty} \exp(-\tfrac{r}{2})\, r\,dr\,d\theta \quad < \infty.$$

Therefore we can apply Lebesgue's dominated convergence theorem:

$$\underline{2\int_{\pi/2}^{\pi}\int_0^{\infty}(\gamma - \sin\gamma)\exp(-\tfrac{1}{2}(\eta - m))\, r\,dr\,d\theta}$$

$$\xrightarrow[m\to\infty]{} 2\int_{\pi/2}^{\pi}\int_0^{\infty}(\theta - \sin\theta)\exp(-\tfrac{r}{2}(1 - \cos\theta))\, r\,dr\,d\theta$$

$$= 8\int_{\pi/2}^{\pi}\frac{\theta - \sin\theta}{(1 - \cos\theta)^2}\,d\theta = \boxed{\tfrac{8}{3}(\pi + \log 2 - 1)}$$

[Mathematica calculation]

In this range $0 < \theta - \sin\theta \leq \gamma - \sin\gamma$ (using increasing nature of $\gamma - \sin\gamma$) while $0 < r(1 - \cos\theta) \leq \eta - m$.

So now analyse the error for this asymptotic: it is bounded by

$$\Big|2\int_{\pi/2}^{\pi}\int_0^{\infty}(\gamma - \sin\gamma)\exp(-\tfrac{1}{2}(\eta - m))\, r\,dr\,d\theta$$

$$- 2\int_{\pi/2}^{\pi}\int_0^{\infty}(\theta - \sin\theta)\exp(-\tfrac{r}{2}(1 - \cos\theta))\, r\,dr\,d\theta\Big|$$

$$\leq 2\int_{\pi/2}^{\pi}\int_0^{\infty}\big((\gamma - \sin\gamma) - (\theta - \sin\theta)\big)\exp(-\tfrac{r}{2}(1 - \cos\theta))\, r\,dr\,d\theta$$

$$+ 2\int_{\pi/2}^{\pi}\int_0^{\infty}(\theta - \sin\theta)\big(\exp(-\tfrac{r}{2}(1 - \cos\theta)) - \exp(-\tfrac{1}{2}(\eta - m))\big)\, r\,dr\,d\theta.$$

Consider the first of these double integrals.

While $\pi/2 \leq \gamma \leq \pi$ we have $(\gamma - \sin\gamma)' = 1 - \cos\gamma \in [1, 2)$ and $\gamma - \sin\gamma \in [\pi/2 - 1, \pi)$.

Moreover while $\psi = \gamma - \theta \in [0, \pi/2)$ we have $(\tan\psi - \psi)' = \sec^2\psi - 1 \geq 0$ so $\psi \leq \tan\psi = \frac{r\sin\theta}{m - r\cos\theta} \leq \frac{r\sin\theta}{m}$ (where here we use the fact that $\cos\theta < 0$ in this range.).

So in this range $(\gamma - \sin\gamma) - (\theta - \sin\theta) \leq \max_{\pi/2 \leq \alpha < \pi}(\alpha - \sin\alpha)'\,\psi$

$$\leq 2\tan\psi \leq \frac{2r\sin\theta}{m}.$$

So the first integral is bounded above by

$$\frac{4}{m}\int_{\pi/2}^{\pi}\int_0^{\infty}\sin\theta\exp(-\tfrac{r}{2}(1 - \cos\theta))\, r^2\,dr\,d\theta$$

$$\leq \frac{4}{m}\int_{\pi/2}^{\pi}\int_0^{\infty}\sin\theta\, e^{-r/2}\, r^2\,dr\,d\theta$$

$$= \frac{4}{m}\left(\int_{\pi/2}^{\pi}\sin\theta\,d\theta\right)\left(\int_0^{\infty}e^{-r/2}\, r^2\,dr\right)$$

$$= \frac{4}{m}\times 1\times \cancel{8}16 = \boxed{\frac{\cancel{32}64}{m}}$$

correction 6.7.07 checked with Mma

(In fact we could be precise here: the integral equals

$$\frac{{}^{64}\cancel{32}}{m}\int_{\pi/2}^{\pi}\frac{\sin\theta}{(1 - \cos\theta)^3}\,d\theta = \frac{\cancel{12}24}{m}\Big)$$

Use the fact, $1+a \le 1+a+a^2/4$ so $\sqrt{1+a} \le 1+a/2$ if $a \ge -1$.
Also then $(\eta - m) - r(1-\cos\theta)$

$$= \sqrt{\left((m-r\cos\theta)^2 + r^2\sin^2\theta\right)} - (m - r\cos\theta)$$

$$\le (m - r\cos\theta)\cdot\frac{1}{2}\frac{r^2\sin^2\theta}{(m-r\cos\theta)^2} = \frac{r^2(1-\cos^2\theta)}{2(m-r\cos\theta)}.$$

The second integral is bounded above by

$$2\int_{\pi/2}^{\pi}\int_0^\infty (\theta-\sin\theta)\exp\left(-\frac{r}{2}(1-\cos\theta)\right)\left(1-\exp\left(-\frac{r^2(1-\cos^2\theta)}{4(m-r\cos\theta)}\right)\right) r\,dr\,d\theta$$

(as $\cos\theta<0$ in this range!)

$$\le 2\pi\int_{\pi/2}^{\pi}\int_0^\infty \exp\left(-\frac{r}{2}(1-\cos\theta)\right)\left(1-\exp\left(-\frac{r^2\sin^2\theta}{4m}\right)\right) r\,dr\,d\theta$$

$\theta-\sin\theta\le\pi$ in this range

$$\le 2\pi\int_{\pi/2}^{\pi}\int_0^\infty \exp\left(-\frac{r}{2}(1-\cos\theta)\right)\frac{\sin^2\theta}{4m}\, r^3\,dr\,d\theta$$

$(1-e^{-u}\le u)$

$$\cancel{+\,2\pi\int_{\pi/2}^{\pi}\int_{r_m(\theta)}^\infty \exp\left(-\frac{r}{2}(1-\cos\theta)\right) r\,dr\,d\theta}$$

$$\cancel{\left(\text{taken } \frac{\sin^2\theta}{4m}r_m(\theta)^2=1 \text{ so } r_m(\theta)=\frac{2\sqrt{m}}{\sin\theta}\right)}$$

$$= \frac{\pi}{2m}\int_{\pi/2}^{\pi}\int_0^\infty \exp\left(-\frac{r}{2}(1-\cos\theta)\right) r^3\,dr\,\sin^2\theta\,d\theta$$

$$\cancel{+\,2\pi\int_{\pi/2}^{\pi}\int_{r_m(\theta)}^\infty}$$

$$= \frac{\pi}{2m}\int_{\pi/2}^{\pi}\frac{16}{(1-\cos\theta)^4}\sin^2\theta\,d\theta\int_0^\infty e^{-u}u^3\,du$$

$$= \frac{6\times 8\pi}{m}\int_{\pi/2}^{\pi}\frac{\sin^2\theta\,d\theta}{(1-\cos\theta)^4} \doteq \boxed{\frac{64\pi}{5}\,\frac{1}{m}}$$ [Mathematica calculation]

Thus we have established

$$\boxed{J_m = 2\int_0^{\pi/2}\int_0^{\frac{m}{2}\sec\theta}(\gamma-\sin\gamma)\exp\left(-\tfrac12(\eta-m)\right)r\,dr\,d\theta + \frac{8}{3}(\pi+\log 2-1) + O\left(\frac{1}{m}\right)}$$

where in fact the $O\left(\frac1m\right)$ term is bounded in absolute value by

$$\frac{\cancel{32}\,^{24}}{m} + \frac{64\pi}{5}\frac{1}{m} = \boxed{\left(3+\frac{8}{5}\pi\right)\frac{8}{m}}.$$

corrected 6.7.07

So now we must control the second double integral

$$\int_0^{\pi/2}\int_0^{\frac{m}{2}\sec\theta} f_m(\theta,r)\,r\,dr\,d\theta \quad\text{where}\quad f_m(\theta,r) = 2(\gamma-\sin\gamma)\exp\left(-\tfrac12(\eta-m)\right).$$

First of all, consider the approximation derived by replacing $f_m(\theta,r)$ with $g(\theta,r) = \frac{2}{3}(1-\cos\theta)\sin\theta\exp\left(-\frac{r}{2}(1-\cos\theta)\right)$.

(Motivation: agrees in first factor to $O(\theta^5)$, in exponent and as limit, when θ is small and m is large!)

$$\frac{2}{3}\int_0^{\pi/2}\int_0^{\frac{m}{2}\sec\theta}(1-\cos\theta)\sin\theta\exp\left(-\frac{r}{2}(1-\cos\theta)\right)r\,dr\,d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2}\int_0^{\frac{m}{4}(\sec\theta-1)} e^{-s}s\,ds\,\frac{\sin\theta}{1-\cos\theta}\,d\theta$$

where $s = \frac{r}{2}(1-\cos\theta)$
so $ds = \frac12(1-\cos\theta)\,dr$

$$= \frac{8}{3}\int_0^\infty\int_0^v e^{-s}s\,ds\,\frac{1+4v/m}{4v/m}\,\frac{4}{m}\,\frac{dv}{(1+4v/m)^2}$$

where $v = \frac{m}{4}(\sec\theta-1)$
$\cos\theta = \frac{1}{1+4v/m}$
$1-\cos\theta = \frac{4v/m}{1+4v/m}$
$\sin\theta\,d\theta = \frac{4}{m}\frac{dv}{(1+4v/m)^2}$

$$= \frac{8}{3}\int_0^\infty\int_0^v e^{-s}s\,ds\left(\frac{1}{4v/m} - \frac{1}{1+4v/m}\right)\frac{4}{m}\,dv$$

(partial fractions)

(now integrate by parts)

$$= \frac{8}{3}\left[\int_0^v e^{-s}s\,ds\left(\log\frac{4v}{m} - \log\left(1+\frac{4v}{m}\right)\right)\right]_0^\infty$$

(first summand $\to 0$ as $v\to\infty$)

$$-\frac{8}{3}\int_0^\infty e^{-v}v\left(\log\frac{4v}{m} - \log\left(1+\frac{4v}{m}\right)\right)dv$$

use $\log\frac{4v}{m} - \log\left(1+\frac{4v}{m}\right)$
$= \log\frac{4v}{m+4v}$ so first summand vanishes
$= \log\frac{v}{1+4v/m} + \log\frac{4}{m}$

$$= \frac{8}{3}\int_0^\infty e^{-s}s\,ds\,\log\frac{m}{4}$$

$$-\frac{8}{3}\int_0^\infty e^{-v}v\log v\,dv$$

$$+\frac{8}{3}\int_0^\infty e^{-v}v\log\left(1+\frac{4v}{m}\right)dv$$

$$= \frac{8}{3}\left(\log\frac{m}{4} - 1 + \gamma\right)$$

$$-\frac{8}{3}\left[(1+v)e^{-v}\log\left(1+\frac{4v}{m}\right)\right]_0^\infty + \frac{8}{3}\int_0^\infty e^{-v}\frac{(1+v)}{1+4v/m}\frac{4}{m}\,dv$$

integrating by parts again

use $\int_0^\infty e^{-v}v\log v\,dv = 1-\gamma$
for γ the Euler-Mascheroni constant

$$= \frac{8}{3}\left(\log\frac{m}{4} - 1 + \gamma\right) + \frac{32}{3m}\int_0^\infty e^{-v}\frac{(1+v)}{1+4v/m}\,dv.$$

and consequently

$$\frac{8}{3}\int_0^\infty \int_0^v \left(e^{-s} - e^{-\frac{1}{2}(\eta - m)}\right) s\,ds \;\frac{1}{1+4v/m}\;\frac{dv}{v}$$

$$= \frac{8}{3}\int_0^\infty\int_0^v \left(1 - \exp\left(-\frac{m}{2}\left(1-\frac{s}{2v}\right)\left(\sqrt{1 + \frac{(s/2v)^2}{(1-s/2v)^2}\frac{4v}{m}\left(2+\frac{4v}{m}\right)} - 1\right)\right)\right)$$

← see LH page

$$e^{-s} s\,ds \;\frac{1}{1+\frac{4v}{m}}\;\frac{dv}{v}$$

Now the integrand is dominated by the integrand in the LHS.

$$\frac{8}{3}\int_0^\infty\int_0^v \left(1 - \exp\left(-\frac{1}{4}\,\frac{(s/2v)^2}{1-s/2v}\cdot 4v\left(2+\frac{4v}{m}\right)\right)\right) e^{-s} s\,ds\;\frac{dv}{v}$$

$$\rightarrow \frac{8}{3}\int_0^\infty\int_0^v \left(1 - \exp\left(-\frac{s^2}{2v-s}\right)\right) e^{-s} s\,ds\;\frac{dv}{v}$$

$$\leq \frac{8}{3}\int_0^\infty\int_0^1 \left(1 - \exp\left(-\frac{t^2}{2-t}v\right)\right) e^{-vt}\, v\,t\,dt\,dv \qquad \boxed{(s = vt)}$$

$$= \frac{8}{3}\int_0^1\int_0^\infty \left(1 - \exp\left(-\frac{t^2}{2-t}v\right)\right) e^{-tv}\, t^2 v\,dv\;\frac{dt}{t}$$

$$= \frac{8}{3}\int_0^1\int_0^\infty \left(e^{-tv} - \exp\left(-\frac{2}{2-t}tv\right)\right) t^2 v\,dv\;\frac{dt}{t}$$

$$= \frac{8}{3}\int_0^1 \left(1 - \left(\frac{2-t}{2}\right)^2\right)\frac{dt}{t} = \frac{8}{3}\int_0^1 \left(\frac{4t-t^2}{4}\right)\frac{dt}{t}$$

$$= \frac{8}{3}\int_0^1 (1 - t/4)\,dt = \frac{8}{3}\left(1 - \frac{1}{8}\right) = \frac{7}{3} < \infty.$$

~~However the dominating integrand is also the limit as $m \to \infty$ with~~ s, v ~~(or t, v) held fixed!~~ We deduce

$$\frac{2}{3}\int_0^{\pi/2}\int_0^{m/2 \sec\theta} (1-\cos\theta)\sin\theta\left(\exp\left(-\frac{r}{2}(1-\cos\theta)\right) - \exp\left(-\frac{1}{2}(\eta - m)\right)\right) r\,dr\,d\theta$$

$$\rightarrow \boxed{7/3} \quad \text{as } m \to \infty.$$

There are prospects for establishing error bounds, using $\sqrt{1+a} \geq 1 + \frac{a}{2} - \frac{a^2}{8}$; however for now we turn to the other double integral!

Consider first $2\int_\varepsilon^{\pi/2}\int_0^{\frac{m}{2}\sec\theta}\left((\gamma - \sin\gamma) - \frac{(1-\cos\theta)\sin\theta}{3}\right)\exp\left(-\frac{1}{2}(\eta - m)\right) r\,dr\,d\theta$ for fixed $\varepsilon > 0$.

The integrand here is dominated by the integrand in

$$2\left(\frac{\pi}{2} - 1 + \frac{1}{3}\right)\int_\varepsilon^{\pi/2}\int_0^\infty \exp\left(-\frac{r}{2}(1-\cos\theta)\right) r\,dr\,d\theta$$

$$= \left(\pi - \frac{4}{3}\right)\cdot 4\int_\varepsilon^{\pi/2}\frac{d\theta}{(1-\cos\theta)^2}$$

$$= 4\left(\pi - \frac{4}{3}\right)\left(\frac{(2-\cos(\varepsilon))\sin(\varepsilon)}{3(1-\cos(\varepsilon))^2} - \frac{2}{3}\right) < \infty.$$

Hence we may apply the dominated convergence theorem: the double integrand in question converges to a limit which is approximated for small $\varepsilon > 0$ by

$$2\int_0^{\pi/2}\int_0^\infty \left((\theta - \sin\theta) - \frac{(1-\cos\theta)\sin\theta}{3}\right)\exp\left(-\frac{r}{2}(1-\cos\theta)\right) r\,dr\,d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2}\frac{3(\theta - \sin\theta) - (1-\cos\theta)\sin\theta}{(1-\cos\theta)^2}\,d\theta$$

✓ 7.7.07

$$= \boxed{\frac{8}{3}\left(\frac{4}{3} - \pi\right)}$$

[Mathematica calculation]

control established overleaf ~~control established overleaf.~~

So the final task is to establish control of

$$2\int_0^\varepsilon\int_0^{\frac{m}{2}\sec\theta}\left((\gamma - \sin\gamma) - \frac{(1-\cos\theta)\sin\theta}{3}\right)\exp\left(-\frac{1}{2}(\eta - m)\right) r\,dr\,d\theta.$$

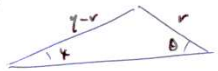
Here is a key argument.

$$\gamma - \sin\gamma = \frac{\gamma^3}{6} + O(\gamma^5) = \frac{\gamma^3}{6} + O(\theta^5) \qquad \text{as } 2\theta \geq \gamma$$

$$= \frac{\theta^3}{6}\left(1 + \frac{\psi}{\theta}\right)^3 + O(\theta^5) \qquad \text{as } \gamma = \theta + \psi$$

$$= \frac{\theta^3}{6}\left(1 + \frac{\sin\psi}{\sin\theta}\right)^3 + O(\theta^5) \qquad \text{as } \psi < \theta,\ \sin\psi = \psi + O(\psi^3),\ \sin\theta = \theta + O(\theta^3)$$

$$= \frac{\theta^3}{6}\left(1 + \frac{r}{\eta - r}\right)^3 + O(\theta^5)$$



$$\frac{\sin\theta}{\eta - r} = \frac{\sin\psi}{r}$$

$$= \frac{\theta^3}{6}\left(\frac{\eta}{\eta - r}\right)^3 + O(\theta^5)$$

$$= \frac{(1-\cos\theta)\sin\theta}{3}\left(\frac{\eta}{\eta - r}\right)^3 + O(\theta^5)_{\text{uniformity!}}$$

noting $0 < \frac{\eta}{\eta - r} \leq 1 + \frac{r}{\eta - r} \leq 1 + \frac{m\sec\varepsilon}{2(\eta - r)} \leq 1 + \sec\varepsilon$

overleaf we control uniformity of $O(\theta^5)$

so $$\frac{r}{\eta - r} = \frac{1 + 4v/m}{2v}\, s \Big/ \sqrt{\left(\left(1 - \frac{s}{2v}\right)^2 + \left(\frac{s}{2v}\right)^2 \frac{4v}{m}\left(2 + \frac{4v}{m}\right)\right)}$$

$$\to \frac{s}{2v} \Big/ \sqrt{\left(1 - \frac{s}{2v}\right)^2} = \frac{s}{2v-s} \quad \checkmark$$

$$\frac{\eta}{\eta - r} \to \frac{2v}{2v-s} \quad \checkmark$$

$$\frac{m}{\eta - r} = \frac{1}{\sqrt{\left(\left(1 - \frac{s}{2v}\right)^2 + \left(\frac{s}{2v}\right)^2 \frac{4v}{m}\left(2 + \frac{4v}{m}\right)\right)}} \to \frac{2v}{2v-s} \quad \checkmark$$

Hence

$$2\int_0^{\varepsilon}\int_0^{\frac{m}{2}\sec\theta} \left((\gamma - \sin\gamma) - \frac{(1-\cos\theta)\sin\theta}{3} \right) \exp\left(-\tfrac{1}{2}(\eta - m)\right) r\,dr\,d\theta$$

$$\to \frac{4}{3}\int_0^{\infty}\int_0^{v} \frac{2v}{2v-s}\left(1 + \frac{2v}{2v-s} + \left(\frac{2v}{2v-s}\right)^2\right) e^{-s} \times \exp\left(-\frac{s^2}{2v-s}\right) s^2\,ds\,\frac{dv}{v^2} \quad \checkmark$$

$$= \frac{4}{3}\int_0^{\infty}\int_0^{v} \frac{2v}{2v-s}\left(1 + \frac{2v}{2v-s} + \left(\frac{2v}{2v-s}\right)^2\right) \exp\left(-\frac{2vs}{2v-s}\right) s^2\,ds\,\frac{dv}{v^2}$$

use $s = vt$ as above

$$= \frac{4}{3}\int_0^{\infty}\int_0^{1} \frac{2}{2-t}\left(1 + \frac{2}{2-t} + \left(\frac{2}{2-t}\right)^2\right) \exp\left(-\frac{2t}{2-t}v\right) v\,t^2\,dt\,dv \quad \checkmark$$

$$= \frac{4}{3}\int_0^{1}\int_0^{\infty} \exp\left(-\frac{2t}{2-t}v\right) v\,dv\; \frac{2t^2}{2-t}\left(1 + \frac{2}{2-t} + \left(\frac{2}{2-t}\right)^2\right) dt$$

$$= \frac{2}{3}\int_0^{1} (2-t)\left(1 + \frac{2}{2-t} + \left(\frac{2}{2-t}\right)^2\right) dt$$

$$= \frac{2}{3}\left(\int_0^1 (2-t)\,dt + 2\int_0^1 dt + 4\int_0^1 \frac{dt}{2-t}\right)$$

$$= \frac{2}{3}\left(2 - \frac{1}{2} + 2 + 4\left[\log\frac{1}{2-t}\right]_0^1\right)$$

$$= \frac{2}{3}\left(2 - \frac{1}{2} + 2 + 4\log 2\right) = \boxed{\frac{2}{3}\left(\frac{7}{2} + 4\log 2\right)} \quad \checkmark$$

Putting this all together we find

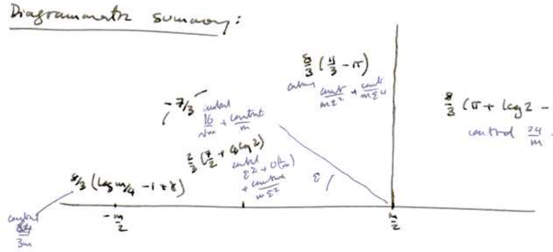
$$J_m \sim \frac{8}{3}(\pi + \log 2 - 1) + \frac{8}{3}\left(\log\frac{m}{4} - 1 + \gamma\right) - \frac{7}{3} + \frac{8}{3}\left(\frac{11}{3} - \pi\right) + \frac{2}{3}\left(\frac{7}{2} + 4\log 2\right)$$

in the slightly extended sense that we get within ε of this as an upper bound!

Hence we can approximate J_m by

$$\boxed{\frac{8}{3}\left(\log m + \gamma + \frac{5}{3}\right)} \qquad \text{for large } m$$

<u>Diagrammatic summary:</u>



$$J_m = 2\int\int_{\mathbb{R}_+ \times \mathbb{R}_+} (\gamma - \sin\gamma) \exp\left(-\tfrac{1}{2}(\eta - m)\right) \mathrm{leb}(dx)$$

End of awful example

What went wrong?

How to do better?