APTS Applied Stochastic Processes, Warwick, July 2015
Exercise Sheet for Assessment

The work here is "light-touch assessment", intended to take students up to half a week to complete. Students should talk to their supervisors to find out whether or not their department requires this work as part of any formal accreditation process (APTS itself has no resources to assess or certify students). It is anticipated that departments will decide the appropriate level *of assessment locally, and may choose to drop some (or indeed all) of the parts, accordingly.*

Students are recommended to read through the relevant portion of the lecture notes before attempting each question. It may be helpful to ensure you are using a version of the notes put on the web *after* the APTS week concluded.

1 Markov chains and renewal processes

Consider a Markov chain X with state space $\{0, 1, 2, \ldots, m\}$ and transitions as follows:

$$p_{0,0} = 1 - p; \quad p_{0,m} = p; \quad p_{j,j-1} = 1 \text{ for } 1 \le j \le m$$

for some $p \in (0, 1)$.

1. Explain why X has an equilibrium distribution π, to which it converges, yet does not satisfy the detailed balance equations.

2. Find the equilibrium distribution π.

3. Consider the sequence of times $H_0, H_1, H_2, \ldots$ defined recursively by $H_0 = \inf\{n \ge 0 : X_n = 1\}$ and, for $k \ge 0$,
$$H_{k+1} = \inf\{n > H_k : X_n = 1\}.$$
Let $N(n) = \#\{k \ge 0 : H_k \le n\}$ and consider the renewal process $(N(n), n \ge 0)$. Find the delay distribution ν which makes this renewal process stationary.

4. Let $\tilde{Z}$ be a random variable with distribution ν. Show that the distribution of the random variable $(\tilde{Z} - m)$ can be written as a mixture of a uniform and a geometric distribution.

2 Martingales and optional stopping

1. Suppose that $Y_1, Y_2, \ldots$ are independent and identically distributed random variables with a common Exponential distribution of mean 1. Show that
$$X_n \quad = \quad \exp\left(\tfrac{1}{2}(Y_1 + \ldots + Y_n) - n \log 2\right)$$
defines a martingale $X_0 = 1, X_1, X_2, \ldots$.

2. Explain why it is a consequence from martingale theory that X_n converges almost surely as $n \to \infty$. Verify this directly by applying the strong law of large numbers to $\log X_n$, and hence identify the limit.

3 Foster-Lyapunov criteria

Let X be a random walk on $\mathbb{R}$ with step-distribution defined as follows: if $X_n = x$ then $X_{n+1} \sim N\left(\frac{x}{3}, 1\right)$.

(a) Show that X is Leb-irreducible, where $\text{Leb}(\cdot)$ is Lebesgue (length) measure.

(b) Show that any set C of the form $C = \{x : |x| \leq c\}$, $c > 0$, is a small set of lag 1.

(c) Let $\Lambda(x) = 1 + x^2$. Using this function, establish that X is geometrically ergodic.