APTS Applied Stochastic Processes, Oxford, March 2017
Exercise Sheet for Assessment

The work here is "light touch assessment", intended to take students up to half a week to complete. Students should talk to their supervisors to find out whether or not their department requires this work as part of any formal accreditation process (APTS itself has no resources to assess or certify students). It is anticipated that departments will decide the appropriate level *of assessment locally, and may choose to drop some (or indeed all) of the parts, accordingly.*

1 Detailed balance, small sets and Foster-Lyapunov

Consider a Metropolis-Hastings chain X for sampling from a target density π on $\mathbb{R}$. If $X_n = x$ then a new state y for X_{n+1} is proposed using density $q(x, y)$, and then accepted with probability

$$\alpha(x, y) = \min\left\{\frac{\pi(y)q(y, x)}{\pi(x)q(x, y)}, 1\right\} .$$

Suppose for simplicity that π and q are both continuous and strictly positive on the whole of $\mathbb{R}$. Thus transitions of X take place according to the density

$$p(x, y) = q(x, y)\alpha(x, y) , \qquad y \neq x ,$$

and with probability of remaining at the same point given by

$$\mathbb{P}\left[X_{n+1} = x \,|\, X_n = x\right] = \int_{\mathbb{R}} q(x, y)(1 - \alpha(x, y))\,\mathrm{d}\,y .$$

(a) Show that X satisfies detailed balance with respect to π.

(b) Show that X is π-irreducible.

(c) Show that any non-empty interval $C = [a, b]$ is a small set.
[HINT 1: it suffices to show that $p(x, A) \geq c\pi(A)$ for all $x \in C$ and sets $A \subseteq C$. Why?!]
[HINT 2: for a set $A \subseteq C$ and $x \in C$, write $p(x, A) = p(x, A \cap R_x(A)) + p(x, A \setminus R_x(A))$, where $R_x(A)$ is the set of states $y \in A$ for which, if a move from x to y is proposed, the acceptance probability is less than 1.]

Now suppose that $q(x, y) = q(y)$ (i.e. that X is an *independence sampler*), and that there exists a constant β with $q(y)/\pi(y) \geq \beta$ for all $y \in \mathbb{R}$.

(d) Show that the whole state space is a small set, and hence that X is uniformly ergodic.

(e) Show that the geometric Foster-Lyapunov drift condition holds with trivial scale function given by $\Lambda(x) = 1$ for all $x \in \mathbb{R}$.

2 Martingales

Suppose that $N_1, N_2, \ldots$ are independent and identically distributed normal random variables each with mean μ and variance $\sigma^2 > 0$. Set $S_n = N_1 + \ldots + N_n$.

(a) Show that $Y_n = \exp(S_n - n\mu - \frac{n}{2}\sigma^2)$ is a martingale.

(b) Explain why the Strong Law of Large Numbers implies that $Y_n \to 0$ almost surely.

(c) Show that although $Y_n \to 0$ almost surely, nevertheless $\mathrm{Var}(Y_n) \to \infty$.

3 Stopping times

Suppose that $\{X_t : t = 0, 1, 2, \ldots\}$ is a simple symmetric random walk running between 0 and n, which is stopped when it first hits the barrier 0 and which undergoes a certain kind of reflection when it hits the barrier n. To be precise, X has the transition probabilities

$$\begin{aligned} p_{x,x+1} &= 1/2 \qquad \text{for } x = 1, 2, \ldots, n-1\,; \\ p_{x,x-1} &= 1/2 \qquad \text{for } x = 1, 2, \ldots, n-1\,; \\ p_{0,0} &= 1\,; \\ p_{n,n-1} &= 1\,. \end{aligned}$$

(Consequently the reflection at n is not the same kind of reflection as occurs for reversible Markov chains.)

(a) Show that if $f(x) = x(2n - x)$ then $Y_t = f(X_t) + t$ defines a martingale up to the first time that X hits 0;

(b) Deduce that if $X_0 = x \in \{0, 1, 2, \ldots, n\}$ and $T = \inf\{t : X_t = 0\}$ then $\mathbb{E}\left[T | X_0 = x\right] = x(2n - x)$.