APTS Applied Stochastic Processes, Nottingham, April 2018
Exercise Sheet for Assessment

The work here is "light touch assessment", intended to take students up to half a week to complete. Students should talk to their supervisors to find out whether or not their department requires this work as part of any formal accreditation process (APTS itself has no resources to assess or certify students). It is anticipated that departments will decide the appropriate level *of assessment locally, and may choose to drop some (or indeed all) of the parts, accordingly.*

1 Markov chains and reversibility

(a) Parasites infest a plant. Suppose that parasites arrive from elsewhere at rate $\lambda > 0$ and also individually give birth at rate λ and individually die at rate $\mu > 0$. Explain how to use the theory of detailed balance for continuous-time Markov chains to model this and to establish conditions for existence of statistical equilibrium, and compute the probability of no parasites being on the plant at a specific time when the system is in statistical equilibrium.

(b) Now suppose that it is required to take account of competition between parasites by supposing that the individual death rate is μX when the population size is X. How does your answer change?

2 Stopping times

A shuffled pack of cards contains b black and r red cards. The pack is placed face down on a table, and cards are turned over one at a time. Let B_n denote the number of black cards left *just before* the n^{th} card is turned over. (So, for example, $B_1 = b$.)

(a) Show that

$$M_n = \frac{B_n}{r + b - (n-1)},$$

the proportion of black cards left just before the n^{th} card is revealed, defines a martingale.

(b) Let T be the time at which the first black card is turned over. Compute the probability mass function of this random variable.

(c) Use part (b) and the Optional stopping theorem applied to M_{T+1} to show that

$$\sum_{k=0}^{r} \frac{\binom{r}{k}}{\binom{r+b-2}{k}} = \frac{r+b-1}{b-1}.$$

3 Convergence rates

Let X be a random walk on $\mathbb{R}$ with step-distribution defined as follows: if $X_n = u$ then X_{n+1} has the shifted "double-headed exponential density" $f_{u/2}(x) = f(x - u/2) = (1/2)\exp(-|x - u/2|)$.

(a) Show that X is ℓ-irreducible, where $\ell(\cdot)$ is Lebesgue (length) measure.

(b) Show that any set C of the form $C = \{x : |x| \leq c\}$, $c > 0$, is a small set of lag 1.

(c) Let $\Lambda(x) = 1 + x^2$. Using this function, establish that X is geometrically ergodic.