APTS Applied Stochastic Processes, Southampton, April 2019
Exercise Sheet for Assessment

The work here is "light touch assessment", intended to take students up to half a week to complete. Students should talk to their supervisors to find out whether or not their department requires this work as part of any formal accreditation process (APTS itself has no resources to assess or certify students). It is anticipated that departments will decide the appropriate level *of assessment locally, and may choose to drop some (or indeed all) of the parts, accordingly.*

1 Continuous-time Markov chains and reversibility

(a) Model a population of bacteria by letting X_t be the number of bacteria alive at time t. Each bacterium alive independently splits into two at rate $\lambda > 0$ (increasing the population by 1), and dies at rate $\mu > 0$ (decreasing the population by 1). Write down the transition rates $q_{x,y}$ for this continuous-time Markov chain. Does the chain converge to (non-trivial) equilibrium? Justify your answer.

(b) Now suppose that additional bacteria are added to the population at rate $\alpha > 0$, independently of everything else. Show that the equilibrium equation is satisfied by

$$\pi_x = \Big(\frac{\lambda(x-1)+\alpha}{\mu x}\Big)\pi_{x-1}, \quad x \geq 1.$$

(c) Give a necessary and sufficient condition for the chain to converge to equilibrium. (You do not need to calculate the equilibrium distribution explicitly!)

2 Martingales and optional stopping

Let $S_n = \xi_1 + \cdots + \xi_n$ be a simple *asymmetric* random walk on $\mathbb{Z}$, started at zero, where $\mathbb{P}[\xi = 1] = p = 1 - \mathbb{P}[\xi = -1]$, for some $p \in (0, 1/2)$.

1. Define the function ϕ by $\phi(x) = \left(\frac{1-p}{p}\right)^x$. Show that $\phi(S_n)$ is a martingale.

2. Let $T_x := \inf\{n \geq 0 : S_n = x\}$. Prove that for any two levels $-a < 0 < b$,

$$\mathbb{P}[T_b < T_{-a}] = \frac{1-\phi(-a)}{\phi(b)-\phi(-a)}\,.$$

   (Hint: consider the stopping time $T = T_{-a} \wedge T_b$, which you know is almost surely finite!)

3. Since $p < 1/2$ we know that the random walk S is transient, and that $S_n \to -\infty$ almost surely as $n \to \infty$. Define R to be the largest value ever reached by S, i.e. $R = \max\{S_n : n \geq 0\}$: this is a finite random variable taking values in the set $\{0, 1, 2, \dots\}$. Using part 2, determine the distribution of R, and show that $\mathbb{P}[R = 0] = (1-2p)/(1-p)$.

3 Foster-Lyapunov criteria

Consider the discrete-time Markov chain X on the non-negative integers, with transition probabilities for $x \geq 2$ given by

$$p_{x,x-2} = p_{x,x+1} = \tfrac{1}{2}\,,$$

while if $X_n \in \{0, 1\}$ then

$$X_{n+1} = X_n + V_{n+1}$$

where $\{V_n\}_{n\geq 0}$ are a sequence of i.i.d. Poisson(λ) random variables, for some $\lambda \leq 1$.

1. Show that $C = \{0, 1\}$ is a small set (of lag 1).

2. Use the Foster-Lyapunov criterion for geometric ergodicity to show that X is geometrically ergodic. (Hint: try using a scale function of the form $\Lambda(x) = \beta^x$, for an appropriate β.)