APTS Applied Stochastic Processes, Bristol, July 2008
Exercise Sheet for Assessment

The work here is intended to take students up to half a week to complete. Students should talk to their supervisors to find out whether or not their department requires this work as part of any formal accreditation process (APTS itself has no resources to assess or certify students). It is anticipated that departments will decide the appropriate level *of assessment locally, and may choose to drop some (or indeed all) of the parts, accordingly.*

Students are recommended to read through the relevant portion of the lecture notes before attempting each question. It may be helpful to ensure you are using a version of the notes put on the web *after* the APTS week concluded.

1: Markov chains and reversibility

Consider a queue exhibiting a form of *balking* (arrivals inhibited according to length of queue) as follows: If X_t people are in the queue system at time t then X is a continuous-time Markov chain with transition rates

$X \to X + 1$ at rate $\lambda/(X + 1)$ when $X \geq 0$;

$X \to X - 1$ at rate μ when $X > 0$.

Here λ and μ are fixed positive parameters.

1. Use detailed balance to show that in statistical equilibrium the chance of X being zero is $e^{-\lambda/\mu}$.
2. Evaluate the mean value of X in equilibrium.

2: Martingales

1. Suppose that $Y_1, Y_2, \ldots$ are independent and identically distributed random variables with a common Exponential distribution of mean 1. Show that

$$X_n \quad = \quad \exp\left(\tfrac{1}{2}(Y_1 + \ldots + Y_n) - n \log 2\right)$$

defines a martingale $X_0 = 1, X_1, X_2, \ldots$.

2. Explain why it is a consequence from martingale theory that $X_n \to 0$ almost surely as $n \to \infty$. Verify this directly by applying the strong law of large numbers to $\log X_n$, and hence identify the limit.

3: Stopping times

Suppose X is a simple asymmetric random walk which is stopped when it first hits either the barrier 0 or another fixed barrier $k > 0$, and $X_0 = x_0$ for some x_0 lying between 0 and k. Thus the transition probabilities for X are given for some fixed $\alpha \neq \frac{1}{2}$, $0 < \alpha < 1$, by

$p_{x,x+1} = \alpha$ if $0 < x < k$,

$p_{x,x-1} = 1 - \alpha$ if $0 < x < k$,

$p_{0,0} = 1$,

$p_{k,k} = 1$.

1. Show that $Y_n = \left(\frac{1-\alpha}{\alpha}\right)^{X_n}$ determines a martingale.
2. Apply the optional stopping theorem to evaluate $\mathbb{E}\left[Y_T\right]$, where $T = \inf\{n : X_n = 0 \text{ or } X_n = k\}$, the first time that X hits one or other of the barriers.
3. Hence find an expression for
$$Q(x_0) \quad = \quad \mathbb{P}\left[X \text{ hits } 0 | X_0 = x_0\right] \, .$$

4: Foster-Lyapunov criteria

Consider the discrete-time Markov chain X on the non-negative integers, with transition probabilities given by

$$p_{0,1} = 1,$$

$$p_{1,0} = p_{1,1} = p_{1,2} = \tfrac{1}{3},$$

$$p_{x,x-2} = \tfrac{1}{3} \text{ if } x > 1,$$

$$p_{x,x-1} = \tfrac{1}{3} \text{ if } x > 1,$$

$$p_{x,x+1} = \tfrac{1}{3} \text{ if } x > 1.$$

Use the Foster-Lyapunov criterion for geometric ergodicity to show that X is geometrically ergodic.
[HINT: consider $\Lambda(x) = \xi^x$ for suitable ξ.]