APTS Coursework for Statistical Asymptotics

The work provided here is intended to take students no more than a few hours to complete, if accuracy is maintained in the calculations. Students should talk to their supervisors to find out whether or not their department requires this work as part of any accreditation process (APTS itself has no resources to assess or certify students, but can provide sketch solutions for this coursework). It is anticipated that departments will decide on the appropriate *level* of assessment locally, and may choose to drop some (or indeed all) of the parts, accordingly.

A random variable X has the *inverse Gaussian distribution*, $IG(\mu, \lambda)$, if its probability density function is

$$f(x; \mu, \lambda) = \frac{\sqrt{\lambda}}{\sqrt{2\pi}} x^{-3/2} \exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\}, \quad x > 0,\ \lambda > 0,\ \mu > 0.$$

(i) Find the cumulant generating function of X, and hence calculate the mean and variance of X.

(ii) Suppose $X_1, \ldots, X_n$ are independent, identically distributed $IG(\mu, \lambda)$. What is the distribution of $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$?

(iii) Find the saddlepoint approximation to the density of $\bar{X}$ and comment on its exactness.

(iv) Verify that the distribution function of X has the form

$$P(X \le x) = \Phi\left\{\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} - 1\right)\right\} + \exp\left(\frac{2\lambda}{\mu}\right) \Phi\left\{-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} + 1\right)\right\},$$

in terms of the standard normal distribution function Φ.

Investigate numerically the accuracy of a Lugannani-Rice (LR) approximation to the distribution function of $\bar{X}$, for small n, and a range of values of μ, λ. Is the asymptotically equivalent form of the LR approximation more, or less, accurate?

(v) Find the forms of the maximum likelihood estimators $\hat{\mu}, \hat{\lambda}$, based on a sample $X_1, \ldots, X_n$. Using the distributional result that $\lambda(\sum X_i^{-1} - n\bar{X}^{-1})$ is distributed as χ^2_{n-1}, independently of $\bar{X}$, verify that the p^*-formula for the joint density of $(\hat{\mu}, \hat{\lambda})$ is exact.