APTS Applied Stochastic Processes, Bristol, July 2010
Exercise Sheet for Assessment

The work here is "light touch assessment", intended to take students up to half a week to complete. Students should talk to their supervisors to find out whether or not their department requires this work as part of any formal accreditation process (APTS itself has no resources to assess or certify students). It is anticipated that departments will decide the appropriate level *of assessment locally, and may choose to drop some (or indeed all) of the parts, accordingly.*

Students are recommended to read through the relevant portion of the lecture notes before attempting each question. It may be helpful to ensure you are using a version of the notes put on the web *after* the APTS week concluded.

1 Markov chains and reversibility

Consider a population of X_t individuals such that $\{X_t : t \geq 0\}$ forms a continuous time Markov chain with transition rates

$$\begin{aligned} X &\to X+1 \quad \text{at rate } \alpha(X+1)(X_{\max}-X)\,, \\ X &\to X-1 \quad \text{at rate } \beta X\,, \end{aligned}$$

where $X_{\max} > 0$ is the maximum possible population size and α, $\beta > 0$.

(a) Use detailed balance to compute the equilibrium probability that the total population size is zero.

(b) Show that, as $X_{\max} \to \infty$ with β/α held fixed, so the distribution of $X_{\max} - X$ in equilibrium converges to a Poisson distribution.

2 Martingales

Suppose that N_1, N_2, ... are independent and identically distributed normal random variables each with mean 0 and variance $\sigma^2 > 0$. Set $S_n = N_1 + \ldots + N_n$.

(a) Show that $Y_n = \exp(S_n - \frac{n}{2}\sigma^2)$ is a martingale.

(b) Explain why the Strong Law of Large Numbers implies that $Y_n \to 0$ almost surely.

(c) Show that although $Y_n \to 0$ almost surely nevertheless $\mathrm{Var}(Y_n) \to \infty$.

3 Stopping times

Suppose that $\{X_t : t = 0, 1, 2, \ldots\}$ is a simple symmetric random walk running between 0 and n, which is stopped when it first hits the barrier 0 and which undergoes a certain kind of reflection when it hits the barrier n. To be precise, X has the transition probabilities

$$\begin{aligned} p_{x,x+1} &= 1/2 \qquad \text{for } x = 1, 2, \ldots, n-1\,; \\ p_{x,x-1} &= 1/2 \qquad \text{for } x = 1, 2, \ldots, n-1\,; \\ p_{0,0} &= 1\,; \\ p_{n,n-1} &= 1\,. \end{aligned}$$

(Consequently the reflection at n is not the same kind of reflection as occurs for reversible Markov chains.)

(a) Show that if $f(x) = x(2n - x)$ then $Y_t = f(X_t) + t$ defines a martingale up to the first time that X hits 0;

(b) Deduce that if $X_0 = x \in \{0, 1, 2, \ldots, n\}$ and $T = \inf\{t : X_t = 0\}$ then $\mathbb{E}[T|X_0 = x] = x(2n - x)$.

4 Convergence rates

Let X be a random walk on $\mathbb{R}$ with step-distribution defined as follows: if $X_n = u$ then X_{n+1} has the shifted "double-headed exponential density" $f_{u/2}(x) = f(x - u/2) = (1/2)\exp(-|x - u/2|)$.

(a) Show that X is ℓ-irreducible, where $\ell(\cdot)$ is Lebesgue (length) measure.

(b) Show that any set C of the form $C = \{x : |x| \leq c\}$, $c > 0$, is a small set of lag 1.

(c) Let $\Lambda(x) = 1 + x^2$. Using this function, establish that X is geometrically ergodic.