APTS Applied Stochastic Processes, Southampton, July 2011
Exercise Sheet for Assessment

The work here is "light touch assessment", intended to take students up to half a week to complete. Students should talk to their supervisors to find out whether or not their department requires this work as part of any formal accreditation process (APTS itself has no resources to assess or certify students). It is anticipated that departments will decide the appropriate level *of assessment locally, and may choose to drop some (or indeed all) of the parts, accordingly.*

Students are recommended to read through the relevant portion of the lecture notes before attempting each question. It may be helpful to ensure you are using a version of the notes put on the web *after* the APTS week concluded.

1 Markov chains and reversibility

Emails arrive in a researcher's inbox at rate $\alpha > 0$. The researcher deals with tasks at rate βX, if there are currently X emails in the inbox. An email which is dealt with is then immediately removed from the inbox. Interest is focussed on the proportion of time for which the researcher's inbox is empty, as this presumably corresponds to periods of undistracted productivity.

1. Model this situation using a reversible continuous-time Markov chain and produce a formula for the equilibrium probability of the researcher's inbox being empty.
2. The researcher's manager considers that the researcher is not obtaining sufficient undistracted research time. Options are:
   (a) send researcher on course "Dealing with your inbox", which will double the rate at which the researcher deals with emails;
   (b) instruct the researcher to follow the protocol, delete on arrival all emails without exception which arrive when there are at least K emails currently in the inbox.

   Under which circumstances might option (ii) be better than option (i)? (Illustrate your answer in the case in which the arrival rate of emails is seven times the rate with which the researcher deals with an individual email.)
3. Write down a general formula for the equilibrium probability with which an arriving email is immediately deleted on arrival, if option (b) is adopted.

2 Martingales

Consider a branching process Y, where $Y_0 = 1$, and Y_{n+1} is the sum $Z_{n,1} + \cdots + Z_{n,Y_n}$ of Y_n independent copies of a non-negative integer-valued family size random variable Z. Let $\mathbb{E}[Z] = \mu$ and $\mathrm{Var}[Z] = \sigma^2$.

(a) Show that Y_n/μ^n is a martingale.

(b) Show that Y_n^2 is

- a martingale if $\mu = 1$ and $\sigma^2 = 0$;
- a submartingale if $\mu \geq 1$;
- a supermartingale if $\sigma^2 \leq 1 - \mu^2$.

(c) Suppose now that $\mu = 1$ and $\sigma^2 > 0$. Find $f(n, Y_n)$ such that $Y_n^2 - f(n, Y_n)$ is a martingale.

3 Poisson processes and other counting processes

1. If N is a Poisson counting process of rate $\alpha > 0$, show that both $N(t) - \alpha t$ and $(N(t) - \alpha t)^2 - \alpha t$ determine martingales as functions of t.

2. Write down the density of a Gamma(ν, α) random variable (for $\nu > 0$, $\alpha > 0$) and confirm that the independent sum of independent Gamma(ν_1, α) and Gamma(ν_2, α) random variables is Gamma$(\nu_1 + \nu_2, \alpha)$.

3. Suppose that M is a counting process constructed as follows: incidents occur at random positive times $T_1 < T_2 < \ldots$, where T_1, $T_2 - T_1$, $\ldots$ are independent Gamma(ν, α) random variables. If $\nu = k$ is a positive integer, show that $\mathbb{P}[M(t) = r] = \mathbb{P}[N(t) \in \{kr, kr+1, \ldots k(r+1) - 1\}]$.

4 Convergence rates

Let X be a Markov chain on $\mathbb{R}$ defined by

$$X_{n+1} = \alpha X_n + W_{n+1}\,,$$

where $|\alpha| < 1$ and the random variables W_n are independent of X, with a density function f_W which is strictly positive on the whole of $\mathbb{R}$, and where $\mathbb{E}[|W|] < \infty$.

(a) Show that X is Lebesgue-irreducible.

(b) Show that any set of the form $C_d = \{x : |x| \leq d\}$ is a small set of lag 1.

(c) Show that X is geometrically ergodic. (Hint: use the Foster-Lyapunov criterion with $\Lambda(x) = |x| + 1$.)