APTS ASP Simple Exercises 2

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1. Use the martingale property to deduce that

$$\mathbb{E}\left[X_{n+k}|\mathcal{F}_n\right] = X_n\,, \quad k = 0, 1, 2, \dots .$$

2. Recall Thackeray's martingale: let $Y_1, Y_2, \dots$ be a sequence of independent random variables, with $\mathbb{P}\left[Y_1 = 1\right] = \mathbb{P}\left[Y_1 = -1\right] = 1/2$. Define the Markov chain M by

$$M_0 = 0; \qquad M_n = \begin{cases} 1 - 2^n & \text{if } Y_1 = Y_2 = \cdots = Y_n = -1, \\ 1 & \text{otherwise.} \end{cases}$$

   (a) Compute $\mathbb{E}\left[M_n\right]$ from first principles.

   (b) What should be the value of $\mathbb{E}\left[\widetilde{M}_n\right]$ if $\widetilde{M}$ is computed as for M but stopping play if M hits level $1 - 2^N$?

3. Consider a branching process Y, where $Y_0 = 1$ and Y_{n+1} is the sum $Z_{n,1} + \ldots + Z_{n,Y_n}$ of Y_n independent copies of a non-negative integer-valued family-size r.v. Z.

   (a) Suppose $\mathbb{E}\left[Z\right] = \mu < \infty$. Show that $X_n = Y_n/\mu^n$ is a martingale.

   (b) Suppose $\mathbb{E}\left[s^Z\right] = G(s)$. Let η be the smallest non-negative root of the equation $G(s) = s$. Show that η^{Y_n} defines a martingale.

   (c) Let $H_n = Y_0 + \ldots + Y_n$ be the total of all populations up to time n. Show that $s^{H_n}/(G(s)^{H_{n-1}})$ is a martingale.

   (d) How should these three expressions be altered if $Y_0 = k \geq 1$?

4. Consider asymmetric simple random walk, stopped when it first returns to 0. Show that this is a supermartingale if jumps have non-positive expectation, a submartingale if jumps have non-negative expectation (and therefore a martingale if jumps have zero expectation).

5. Suppose that a coin, with probability of heads equal to p, is repeatedly tossed: each Head earns £1 and each Tail loses £1. Let X_n denote your fortune at time n, with $X_0 = 0$. Show that

$$\left(\frac{1-p}{p}\right)^{X_n} \quad \text{defines a martingale.}$$

6. A shuffled pack of cards contains b black and r red cards. The pack is placed face down, and cards are turned over one at a time. Let B_n denote the number of black cards left *just before* the n^{th} card is turned over. Let

$$Y_n = \frac{B_n}{r + b - (n-1)}\,.$$

(So Y_n equals the proportion of black cards left just before the n^{th} card is revealed.) Show that Y is a martingale.

7. Suppose $N_1,\ N_2,\ \ldots$ are independent identically distributed normal random variables of mean 0 and variance σ^2, and put $S_n = N_1 + \ldots + N_n$.

   (a) Show that S is a martingale.

   (b) Show that $Y_n = \exp\left(S_n - \frac{n}{2}\sigma^2\right)$ is a martingale.

   (c) How should these expressions be altered if $\mathbb{E}\left[N_i\right] = \mu \neq 0$?

8. Let Y be a discrete-time birth-death process absorbed at zero:

$$p_{k,k+1} = \frac{\lambda}{\lambda + \mu}, \quad p_{k,k-1} = \frac{\mu}{\lambda + \mu}, \qquad \text{for } k > 0, \text{ with } 0 < \lambda < \mu.$$

   (a) Show that Y is a supermartingale and use the SLLN to show that $Y_n \to 0$ almost surely as $n \to \infty$.

   (b) Let $T = \inf\{n : Y_n = 0\}$ (so $T < \infty$ a.s.), and define

$$X_n = Y_{n \wedge T} + \left(\frac{\mu - \lambda}{\mu + \lambda}\right)(n \wedge T)\,.$$

   Show that X is a non-negative supermartingale, converging to

$$Z = \left(\frac{\mu - \lambda}{\mu + \lambda}\right) T\,.$$

   (c) Deduce that

$$\mathbb{E}\left[T\right] \leq \left(\frac{\mu + \lambda}{\mu - \lambda}\right) X_0\,.$$

9. Let X be a simple random walk absorbed at boundaries $a < b$.

   (a) Show that

$$f(x) = \frac{x - a}{b - a}$$

   is a bounded harmonic function.

   (b) Use the martingale convergence theorem and optional stopping theorem to show that

$$f(x) = \mathbb{P}\left[X \text{ hits } b \text{ before } a | X_0 = x\right]\,.$$